Killara HIGH SCHOOL

Student Number

2016/17

EXAMINATION 1

Extension 2 Mathematics

General Instructions
- Reading time – 5 minutes
- Working time - 1 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- The 2 unit / Extension 1 formulae sheet is provided with this paper.
- In Questions 5-6 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Total Marks – 40

Section I - Page 4
4 marks
- Attempt Questions 1 – 4
- Allow about 6 minutes for this section

Section II - Pages 5-8
36 marks
- Attempt Questions 5 – 7
- Allow about 54 minutes for this section

Question	Marks
1 - 4	/4
5	/14
6	/11
7	/11
Total	/40

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 15% of the Higher School Certificate Course Assessment

Section I

4 marks
Attempt Questions 1 – 4
Allow about 6 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 4 (Detach from paper)

Q1

What is the value of $\frac{3}{iz}$ if $= 1 - i$?

(A) $-3 - i$

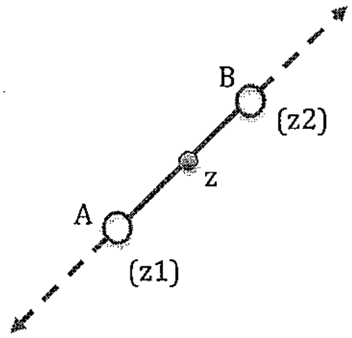
(B) $3 - 3i$

(C) $\frac{-3-3i}{2}$

(D) $\frac{3-3i}{2}$

Q2

The locus of z is the line segment AB, excluding the points A and B.



Which of the following is true?

(A) $\arg(z - z_1) - \arg(z - z_2) = 0$

(B) $\arg(z_2 - z_1) - \arg(z - z_1) = \pi$

(C) $arg\left(\frac{z-z_1}{z-z_2}\right) = \pi$

(D) $\arg(z_1 - z) = \arg(z_2 - z)$

Q3

If $n^2 \equiv an(n+1) + b(n-1) + c$, then the value of the constants a, b and c is:

(A) $a = 1, b = 1, c = 1$

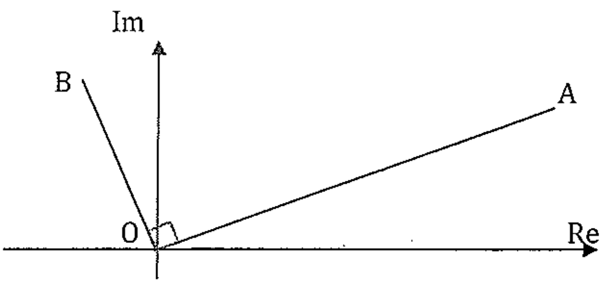
(B) $a = 1, b = -1, c = 1$

(C) $a = 1, b = 1, c = -1$

(D) $a = 1, b = -1, c = -1$

Q4

In the Argand diagram below, the points A and B represent the complex numbers z and ω respectively, where $\angle AOB = 90^\circ$.



If $|\overrightarrow{OA}| = 2|\overrightarrow{OB}|$ which of the following is correct?

(A) $\omega = 2iz$

(B) $\omega = -2iz$

(C) $\omega = -\frac{iz}{2}$

(D) $\omega = -\frac{z}{2i}$

Section II

36 marks
Attempt Questions 5-7
Allow about 54 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 4 – 5, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (14 marks) Use a SEPARATE writing booklet

(a)	The polynomial equation $x^3+3x^2-2x-1=0$ has roots α,β,γ. Find the polynomial equation whose roots are $2\alpha,2\beta,2\gamma$?	**2**
(b)	A polynomial equation $P(x)$ has remainder 6 when divided by $x+2$, and remainder 9 when divided by $x-1$. What is the remainder when $P(x)$ is divided by $(x+2)(x-1)$?	**3**
(c)	The polynomial equation $P(x)=ax^4+bx^3+cx^2+e$ has remainder -2 when divided by $x+1$ and a double root at $x=1$. Show that $4a+2c=-3$	**3**
(d)	Let $P(z)=z^4+2kz^3+2k^2z^2+2kz+1$, where k is real. Let $\beta=x+iy$, where x and y are real. Suppose that β and $i\beta$ are zeros of $P(z)$, where $\bar{\beta}\neq i\beta$.	
	i) Explain why $\bar{\beta}$ and $-i\bar{\beta}$ are zeros of $P(z)$	**1**
	ii) Show that $P(z)=z^2(z+k)^2+(kz+1)^2$	**1**
	iii) Hence show that if $P(z)$ has a real zero then $P(z)=z^2(z-1)^2+(1-z)^2$ OR $P(z)=(z^2+1)(z+1)^2$.	**2**
	iv) Show that all zeros of $P(z)$ have modulus 1.	**2**

End of Question 5

Question 6 (11 marks) Use a SEPARATE writing booklet

(a)	i) Find the square roots of $-8+6i$.	2
	ii) Solve $z^2-(3-5i)z-2-9i=0$.	2
	iii) Sketch the locus of z, if $$4\|z-2+i\| = \|z^2-(3-5i)z-2-9i\|$$	3
(b)	(i) Solve the equation $z^5+32=0$ and list down all the roots in mod-arg form.	2
	(ii) Hence express $z^4-2z^3+4z^2-8Z+16$ as the product of two quadratic factors with real coefficients.	2
	End of Question 6	

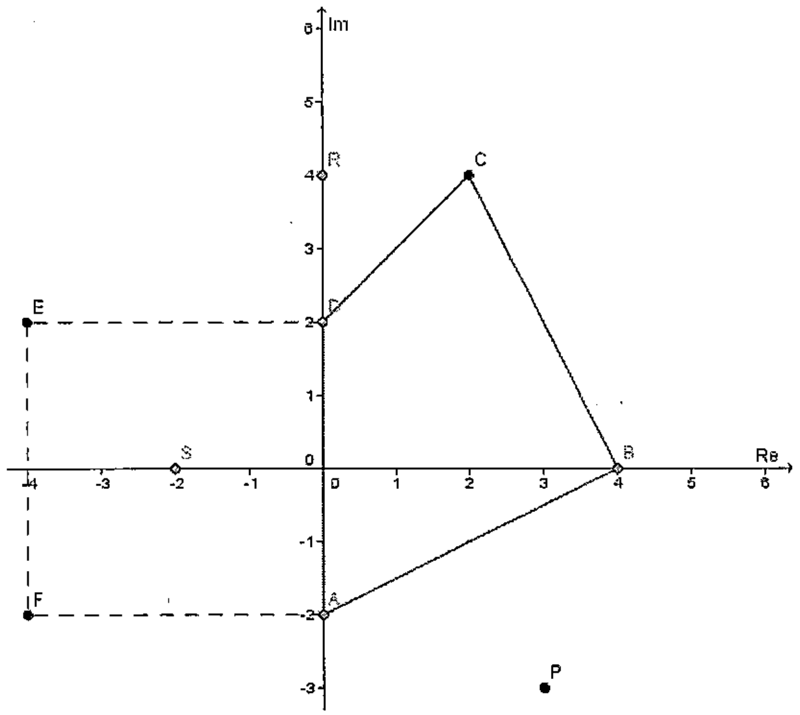
Question 7 (11 marks) Use a SEPARATE writing booklet

(a) Points A, B, C and D on an Argand diagram representing the complex numbers $-2i, 4, 2+4i$ and $2i$ respectively. ABCD form a convex quadrilateral as shown below. Squares are drawn outside the quadrilateral on each of the four sides. P, Q, R and S are the centres of the squares on AB, BC, CD and AD respectively.

i) Using vector representation of complex numbers, show that the complex number represented by Q is $5+3i$. **3**

(It is given that P, R and S represent $3-3i, 4i$ and -2 respectively. You don't need to prove this)

ii) Hence prove that PR is <u>equal and perpendicular</u> to QS, using vector representation of complex numbers. **3**



(b) If $z = cos\theta + isin\theta$,

(i) Show that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2isin(n\theta)$. **2**

(ii) Hence show that $cos^4\theta + \sin^4\theta = \frac{1}{4}(cos4\theta + 3)$ **3**

End of Question 7

End of Examination ☺

Killara HIGH SCHOOL

Student Number

Mathematics

Section I – Multiple Choice Answer Sheet

Use this multiple-choice answer sheet for questions 1 – 4. Detach this sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✖ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ✖ B ✖ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

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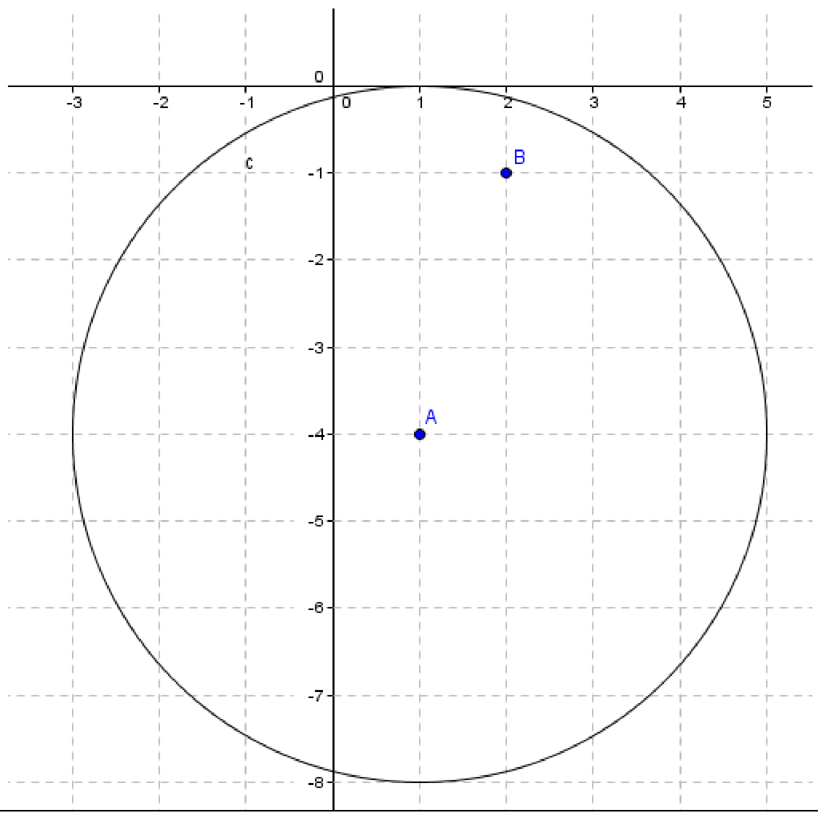
Extension 2 Task 1 2017 ANSWERS

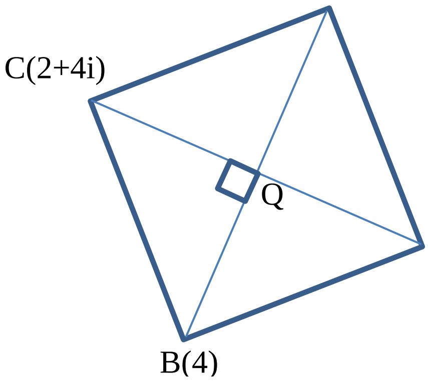
Q	Answer	Marks
1	D	1
2	C	1
3	D	1
4	D	1
5	a) Let $P(x) = x^3 + 3x^2 - 2x - 1 = 0$ have roots α, β, γ. α is a root of $P(x) \rightarrow P(\alpha) = 0$ When $2\alpha = x$, $\alpha = \frac{x}{2} \rightarrow P\left(\frac{x}{2}\right) = 0$ is the polynomial equation whose roots are $2\alpha, 2\beta, 2\gamma$ $P\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^3 + 3\left(\frac{x}{2}\right)^2 - 2\left(\frac{x}{2}\right) - 1 = 0$ i.e. $x^3 + 6x^2 - 8x - 8 = 0$ b) When $P(x)$ is divided by $(x+2)$ the remainder is $P(-2)$ by remainder theorem. $\therefore P(-2) = 6$ $Similarly\ P(1) = 9$ Since $(x+2)(x-1)$ is a quadratic polynomial, the remainder when $P(x)$ Is divided by $(x+2)(x-1)$ is a linear function $(ax+b)$ and let the quotient function be $Q(x)$. $\therefore P(x) \equiv (x+2)(x-1).Q(x) + (ax+b)$ This division transformation is an identity *ie.* the statement is true for all values of *x*. $\therefore$ when $x = -2 \quad P(-2) = -2a + b = 6 \quad (1)$ $x = 1 \quad P(1) = a + b = 9 \quad (2)$ Solving equations (1) and (2) simultaneously gives $a = 1, b = 8$ $\therefore$ the required remainder is $(x+8)$	1 1 1 1 1

Q	Answer	Marks
5	c) By remainder theorem when $P(x)$ is divided by $(x+1)$ the remainder is $P(-1)$. $P(-1)=-2 \rightarrow a(-1)^4+b(-1)^3+c(-1)^2+e=-2$ $\rightarrow a-b+c+e=-2$ (1) $P(x)$ has a double root at $x = 1$ $\rightarrow$ $P(1)=0 \rightarrow a+b+c+e=0$ (2) And as $P'(x)=4ax^3+3bx^2+2cx=0$ $P'(1)=0 \rightarrow 4a+3b+2c=0$ (3) $(2)-(1) \rightarrow 2b=2 \rightarrow b=1$ Sub $b=1$ into (3) $4a+3+2c=0$ $4a+2c=-3$	1 1 mark for using $P(1)=0$ and $P'(1)=0$ 1 mark for using the equations to get the required answer
	d) i) Complex conjugate of $\beta=\overline{\beta}$ Complex conjugate of $\overline{i\beta}=\overline{i}\overline{\beta}=-i\overline{\beta}$ $P(z)=z^4+2kz^3+2k^2z^2+2kz+1$, where k is real, and the coefficients of $P(z)$ are real $\rightarrow \beta$ is a root, implies $\overline{\beta}$ is a root of $P(z)$ by conjugate root theorem Similarly $i\beta$ is a root, implies $-i\overline{\beta}$ is a root of $P(z)$	1 mark for stating the conjugate theorem and that all coefficients are real
	ii) $P(z)=z^4+2kz^3+2k^2z^2+2kz+1$ $=(z^4+2kz^3+k^2z^2)+(k^2z^2+2kz+1)$ $=z^2(z+k)^2+(kz+1)^2$	1 mark for correct working
	iii) $P(z)=z^2(z+k)^2+(kz+1)^2$ As $P(z)$ is the sum of two complete squares, for $P(z)$ to be zero, both $z+k$ and $kz+1$ must be zero. Solving $z+k=0$ and $kz+1=0$ gives $k=\pm 1$. When $k=1, P(z)= z^2(z+1)^2+(z+1)^2$ $=(z+1)^2(z^2+1)$ ---- (1) (by factorising) When $k=-1, P(z)= z^2(z-1)^2+(1-z)^2 --- (2)$	1 mark for getting the two values of k 1 mark for getting the two required expressions for $P(z)$.
	iv) From (iii) above, when $P(z)=0$, gives $z=-1$ and $\pm i$ from (1) and $z=1$ from (2) *ie*. The roots of $P(z)=0$ are $1,-1,i$ and $-i$ The modulus of each of these roots are 1.	1 mark for each for using the the two equations to get the values of z 1 mark for stating the modulus is 1

Q	Answer	Marks
6	a) i) Students may do by a number of methods (by calculating using shorthand techniques) Go straight to $-8+6i=(1+3i)^2$ $\therefore \sqrt{-8+6i}=\pm(1+3i)$ OR long way $\sqrt{-8+6i}=x+iy$ $\therefore -8+6i=x^2-y^2+2ixy$ Equating coefficients of real and imaginary parts $-8=x^2+y^2$ (1) $6i=2ixy$ $3=xy$ $y=\frac{3}{x}$ (2) Sub. (2) into (1) $-8=x^2-\left(\frac{3}{x}\right)^2$ $-8x^2=x^4-9$ x^4+8x^2-9 $(x^2+9)(x^2-1)=0$ $\therefore x=\pm 1$ *sub.* into (2) $y=\pm 3$ $\therefore \sqrt{-8+6i}=\pm(1+3i)$	2 marks for getting the two square roots correctly by any relevant method

Q	Answer	Marks
6	a/	
	ii) $z^2 - (3-5i)z - 2 - 9i = 0$	
	By quadratic formula	
	$z = \dfrac{(3-5i) \pm \sqrt{(3-5i)^2 + 4(2+9i)}}{2}$	
	$z = \dfrac{(3-5i) \pm \sqrt{-8+6i}}{2}$	
	$z = \dfrac{(3-5i) \pm (1+3i)}{2}$ from part i)	
	$z = (2-i), (1-4i)$	2
	iii) $4\lvert z - 2 + i\rvert = \lvert z^2 - (3-5i)z - 2 - 9i\rvert$	
	$\rightarrow 4\lvert z-2+i\rvert = \lvert [z-(2-i)][z-(1-4i)]\rvert$ from part ii	
	$\rightarrow 4\lvert z-2+i\rvert = \lvert z-2+i\rvert \lvert z-1+4i\rvert$	1
	$\rightarrow \lvert z-2+i\rvert(\lvert z-1+4i\rvert - 4) = 0$	
	When $\lvert z-2+i\rvert = 0 \rightarrow$ the locus of z is just the point $(2-i)$	1
	When $\lvert z-1+4i\rvert = 4 = 0 \rightarrow$ the locus of z is the circle with centre $(1-4i)$ and radius 4.	1
	Sketch is shown below:	
		

Q	Answer	Marks
6	b) i) $z^5 = -32$ $z^5 = 2^5(-1+0i)$ $z^5 = 2^5 cis(2k-1)\pi$ where k is an integer $z = 2cis(2k-1)\frac{\pi}{5}$ (By De Moivre's Theorem) The roots of $z^5+32=0$ are: when $k=0$, $z = 2cis\left(-\frac{\pi}{5}\right) = z_1$ when $k=1$, $z = 2cis\left(\frac{\pi}{5}\right) = \overline{z_1}$ when $k=-1$, $z = 2cis\left(-\frac{3\pi}{5}\right) = z_2$ when $k=2$, $z = 2cis\left(\frac{3\pi}{5}\right) = \overline{z_2}$ when $k=-2$, $z = 2cis(\pi) = -2$ ii) $z^5+32 = (z+2)[(z-z_1)(z-\overline{z_1})][(z-z_2)(z-\overline{z_2})]$ $\frac{z^5+32}{z+2} = [z^2-(z_1+\overline{z_1})z+z_1\overline{z_1}][z^2-(z_2+\overline{z_2})z+z_2\overline{z_2}]$ $\frac{z^5+32}{z+2} = \left[z^2-4cos\frac{\pi}{5}z+4\right]\left[z^2-4cos\frac{3\pi}{5}z+4\right]$	2 for all correct roots 1 for factorising and grouping the conjugates 1 for the final answer

Q	Answer	Marks
7	a)  i) In right angled ΔBQC, $\frac{CQ}{CB} = cos\frac{\pi}{4}$ $ie. CQ = \frac{1}{\sqrt{2}} CB$ $\overrightarrow{CQ}$ is obtained by rotating $\frac{1}{\sqrt{2}}$ $\overrightarrow{CB}$ in the anti-clockwise sense by $\frac{\pi}{4}$ radians about C. $\therefore$ $\overrightarrow{CQ} = \frac{1}{\sqrt{2}} \overrightarrow{CB}\, cis\, \frac{\pi}{4}$ $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OQ}$ $= (-2-4i)+4$ $= 2-4i$ $\therefore$ $\overrightarrow{CQ} = \frac{1}{\sqrt{2}}\,(2-4i\,)cis\,\frac{\pi}{4}$ $= (1-2i)(1+i)$ $= 3-i$ $\overrightarrow{OQ} = \overrightarrow{OC} + \overrightarrow{CQ}$ $= (2+4i)+(3-i)$ $= (5+3i)$ $\therefore Q$ represents $(5+3i)$. ii) $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$ $= (-3+3i)+4i$ $= -3+7i$ $\overrightarrow{QS} = -2-(5+3i)$ $=(-7-3i)$ $i\overrightarrow{PR} = i(-3+7i)$ = (-7-3i) $= \overrightarrow{QS}$	 1 1 1 1

	Since $\overrightarrow{QS} = i\overrightarrow{PR}$, $\lvert\overrightarrow{QS}\rvert = \lvert i\overrightarrow{PR}\rvert$ and $\arg(\overrightarrow{QS}) = \arg(i\overrightarrow{PR})$. • $\lvert\overrightarrow{QS}\rvert = \lvert i\overrightarrow{PR}\rvert$ $\lvert\overrightarrow{QS}\rvert = \lvert i\rvert\lvert\overrightarrow{PR}\rvert$ $= \lvert\overrightarrow{PR}\rvert$ ***ie. the lengths of QS and PR are equal.*** • $\arg(\overrightarrow{QS}) = \arg(i\overrightarrow{PR})$ • $\arg(\overrightarrow{QS}) = \arg(i) + \arg(\overrightarrow{PR})$ $= \frac{\pi}{2} + \arg\overrightarrow{PR}$ ***ie. QS and PR are perpendicular*** ***Therefore QS and PR are both equal and perpendicular.***	1 1
7	b (i) $z = cos\theta + isin\theta$ $z^n = \cos(n\theta) + isin(n\theta)$ (1) *By De Moivre'sTheorem* $\frac{1}{z^n} = z^{-n}$ $= \cos(-n\theta) + isin(-n\theta)$ *By De Moivre'sTheorem* $= \cos(n\theta) - isin(n\theta)$ (2) (1) + (2) gives $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ (1) - (2) gives $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$	1 Mark for using De Moivre's theorem Correctly 1 mark for getting the expressions for $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$

	ii) $2\cos\theta = z + \frac{1}{z}$ $\therefore 16\cos^4\theta = \left(z + \frac{1}{z}\right)^4 \quad (1)$ $2i\sin\theta = z - \frac{1}{z}$ $\therefore 16\sin^4\theta = \left(z - \frac{1}{z}\right)^4 \quad (2)$ (1)+(2) $16(\cos^4\theta + \sin^4\theta) = \left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4$ $= 2\left(z^4 + \frac{1}{z^4}\right) + 12$ $= 2\cos 4\theta + 12$ $\therefore \cos^4\theta + \sin^4\theta = \frac{1}{8}\left[\cos 4\theta + 3\right]$	1 mark for getting expressions for $cos^4\theta$ and $sin^4\theta$ 1 mark for simplifying 1 mark for using De Moivre's theorem again to get the answer