Killara HIGH SCHOOL

Student Number

2016

Mathematics Extension 2

HSC Course Assessment Task 3
2nd June 2016

General Instructions

- Reading time – 5 minutes
- Working time 1 hour
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 6 to 8, show relevant mathematical reasoning and/or calculations
- Answer each question in a separate writing booklet
- This paper must not be removed from the examination room
- A reference sheet is provided at the back of this paper
- Diagrams are NOT to scale

Total Marks – 42

Section I - Pages 2 - 3
5 marks

- Attempt Questions 1 to 5
- Allow about 7 minutes for this section

Section II - Pages 4 - 7
37 marks

- Attempt Questions 6 to 8
- Allow about 53 minutes for this section

Question	Marks
1-5	/5
6	/15
7	/14
8	/8
Total	/42

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 20% of the HSC *Course Assessment*

Section I

5 marks
Attempt Questions 1 to 5
Allow about 7 minutes for this section
Use the multiple-choice answer sheet for questions 1-5 (Detach from paper)

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1 A suitable substitution to evaluate $\int \sqrt{9-x^2}\ dx$ is

(A) $x = 3sin\theta$

(B) $x = 3tan\theta$

(C) $x = 3cosec\theta$

(D) $x = 3sec\theta$

2 $\int \frac{x^2+6}{x^2+4}\ dx$ is equal to

(A) $x + \frac{1}{2}\ln(x^2+4) + c$

(B) $x + \ln(x^2+4) + c$

(C) $x + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

(D) $x + \tan^{-1}\frac{x}{2} + c$

3 Using the substitution $t = tan\frac{\theta}{2}$, $\int \frac{dx}{1-\cos x}$ is equal to

(A) $\int \frac{2}{t^2}\ dt$

(B) $\int \frac{1+t^2}{t^2}\ dt$

(C) $\int \frac{1}{t^2}\ dt$

(D) $\int \frac{2(1+t^2)}{t^2}\ dt$

4 The region bounded by the curve $y = x - \frac{x^2}{12}$ above the x-axis, to the right of the line $x = 6$, is rotated about the line $x = 6$. The integral evaluated to obtain this volume is:

(A) $V = \pi \int_0^6 (y-6)^2\, dy$

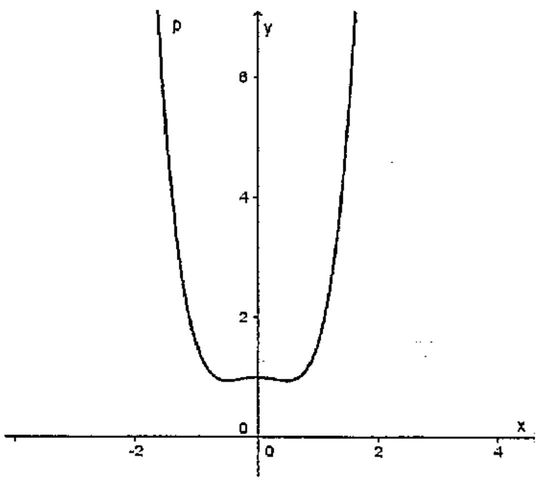
(B) $V = \pi \int_6^{12} (36-12y)\, dy$

(C) $V = \pi \int_0^3 (y-6)^2\, dy$

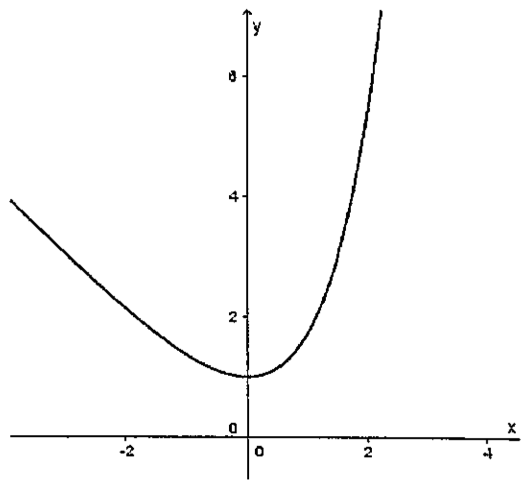
(D) $V = \pi \int_0^3 (36-12y)\, dy$

5 Which curve is the graph of $y = e^x - x$?

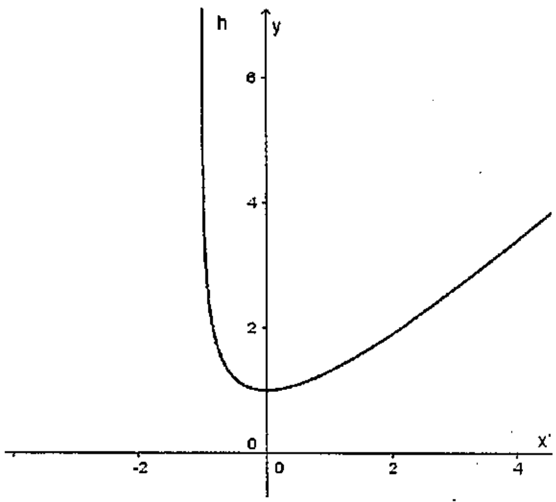
(A)



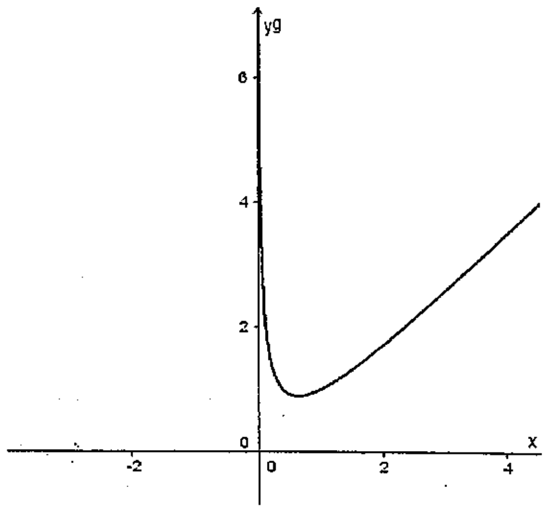
(B)



(C)



(D)



Section II
37 marks
Attempt Questions 6 to 8
Allow about 53 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 6 to 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Use a SEPARATE writing booklet

(a) Find $\int \frac{x}{\sqrt{9x^2-16}}\ dx$ **2**

(b) Find $\int_4^5 \frac{x^2+5x-20}{x-3}\ dx$ **3**

(c) Find $\int sin^5\theta\ \ d\theta$ **3**

(d) (i) Find real numbers A, B and C such that **2**

$$\frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

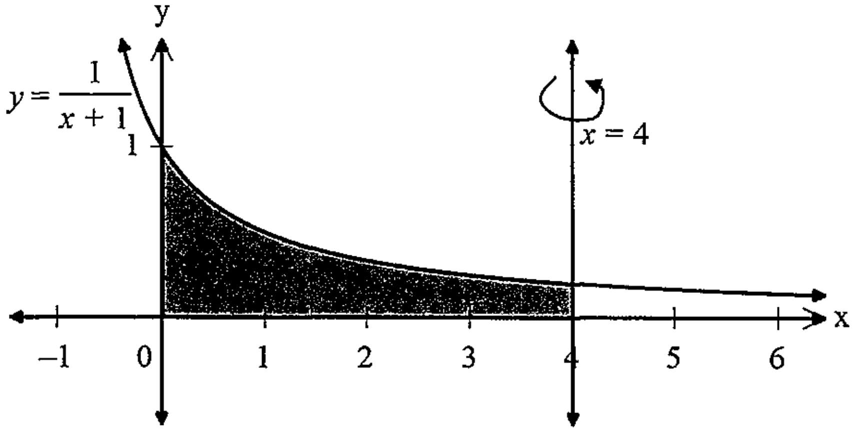
(ii) Hence, find $\int \frac{4}{x^2(x-2)}\ dx$ **2**

(e) Find $\int cos\,(\ln x)\, dx$ **3**

End of Question 6

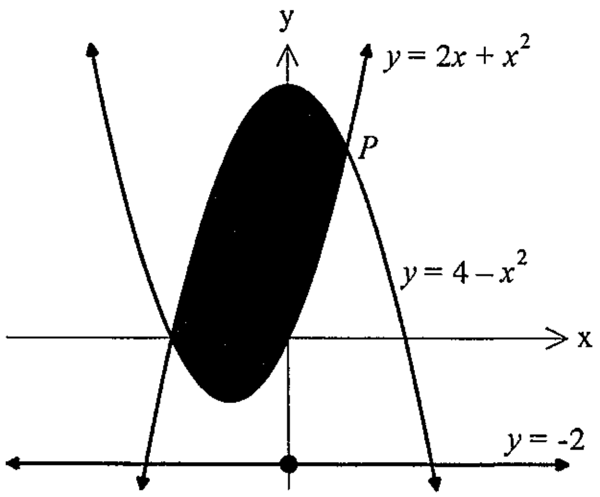
Question 7 (14 marks) Use a SEPARATE writing booklet

(a) The diagram shows the graph of $y = \dfrac{1}{x+1}$. The shaded region between the curve and the x-axis for $0 \le x \le 4$ is rotated about the line $x = 4$ to form a solid.



Use the method of cylindrical shells to find the volume of the solid. **4**

(b)



The shaded region bounded by $y = 4 - x^2$ and $y = 2x + x^2$ is rotated about the line $y = -2$. The point P is the intersection of $y = 4 - x^2$ and $y = 2x + x^2$ in the first quadrant.

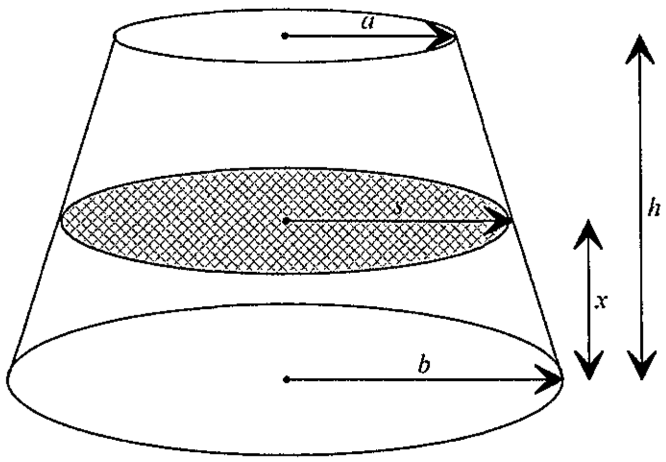
(i) Find the x co-ordinate of P. **1**

(ii) Use the method of slicing to express the volume of the resulting solid of revolution as an integral. **3**

(iii) Evaluate the integral in part (ii). **2**

Question 7 continues on page 6

(c) The diagram shows the frustum of a right cone. (A frustum of a cone is a cone with its top cut off.) The height of the frustum is h metres. Its base is a circle of radius b metres, and its top is a circle of radius a metres. (with $b > a > 0$)



A horizontal cross-section of the frustum, taken at height x metres, is a circle of radius s metres, shown shaded in the diagram.

(i) Show that $s = b - \frac{(b-a)}{h}x$. **2**

(ii) By integration, find the volume of the frustum. **2**

End of Question 7

Question 8 (8 marks) Use a SEPARATE writing booklet

(a) A curve is defined implicitly by $x^3 + y^3 = -1$.

(i) Use implicit differentiation to find $\frac{dy}{dx}$. **2**

(ii) Find the equations of the tangents to the curve $x^3 + y^3 = -1$ at the x and y intercepts and at the point where the curve meets the line $y = x$. **2**

(iii) Show that $x^3 + y^3 = -1$ can be rewritten as $y = -x\left(\sqrt[3]{1 + \frac{1}{x^3}}\right)$ and hence find the equation of the asymptote to the curve $x^3 + y^3 = -1$. **2**

(iv) Sketch the curve $x^3 + y^3 = -1$. **2**

End of Question 8

End of Examination

2016 Mathematics Extension 2
HSC Assessment Task 3
2nd June 2016
Sample Solutions.

Section 1 - Multiple Choice

1.) $\int \sqrt{9-x^2}\,dx$

Let $x = 3\sin\theta$
$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$dx = 3\cos\theta\,d\theta$

$= \int \sqrt{9-9\sin^2\theta} \times 3\cos\theta\,d\theta$

$= \int 3\cos\theta \times 3\cos\theta\,d\theta$

$= 9\int \cos^2\theta\,d\theta$

etc.

A

2.) $\int \frac{x^2+6}{x^2+4}\,dx$

$= \int \frac{x^2+4+2}{x^2+4}\,dx$

$= \int \left(1 + \frac{2}{x^2+4}\right) dx$

$= x + 2 \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

$= x + \tan^{-1}\frac{x}{2} + C$

D

②

3.) $\displaystyle\int \frac{d\theta}{1-\cos\theta}$

$$= \int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

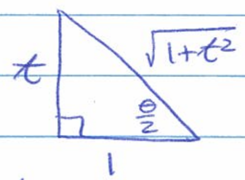
$$= \int \frac{1+t^2}{1+t^2-(1-t^2)} \times \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2}{2t^2}\, dt$$

$$= \int \frac{1}{t^2}\, dt$$

Let

$t = \tan\frac{\theta}{2}$



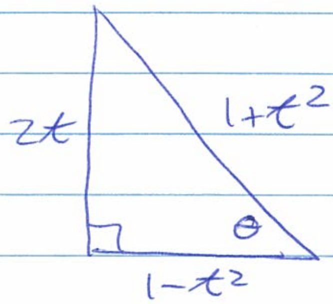
$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$

$$= \frac{1}{2}\left(\sqrt{1+t^2}\right)^2$$

$$= \frac{1+t^2}{2}$$

$$\frac{d\theta}{dt} = \frac{2}{1+t^2}$$

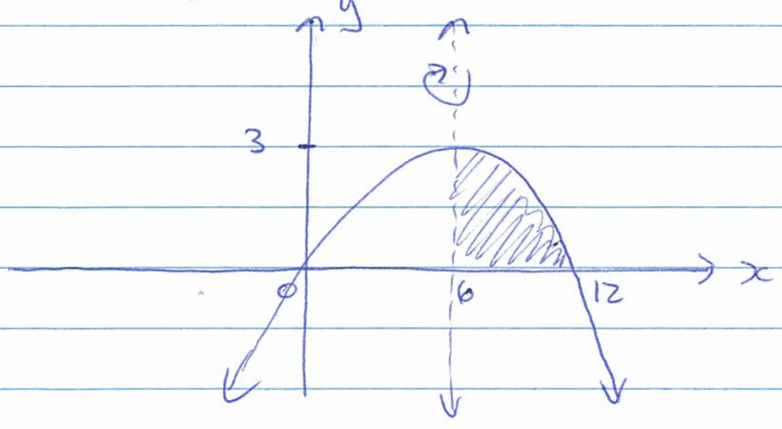
$$d\theta = \frac{2}{1+t^2}\, dt$$



C

4.) $y = x\left(1 - \frac{x}{12}\right)$

when $y=0$, $x=0$, $x=12$.



when $x=6$, $y=3$

$$\delta V = \pi r^2 \delta y$$

$$= \pi (x-6)^2 \delta y$$

$$= \pi (x^2 - 12x + 36)\, \delta y$$

$$V = \pi \int_0^3 (x^2 - 12x + 36)\, dy$$

$$= \pi \int_0^3 (-12y + 36)\, dy$$

$$= \pi \int_0^3 (36 - 12y)\, dy$$

D

5.) As $x \to -\infty$, $y = e^{x} - x \to y = -x$

$\therefore$ $y = -x$ is an asymptote as $x \to -\infty$

B

6.)

a.) $\int \frac{x}{\sqrt{9x^2-16}} \, dx$

$= \int x(9x^2-16)^{-\frac{1}{2}} \, dx$

$= 2(9x^2-16)^{\frac{1}{2}} \times \frac{1}{18} + C$

$= \frac{1}{9}(9x^2-16)^{\frac{1}{2}} + C$

$= \frac{1}{9}\sqrt{9x^2-16} + C$

b.) $\int_4^5 \frac{x^2+5x-20}{x-3} \, dx$

$= \int_4^5 \left(x+8+\frac{4}{x-3}\right) dx$

$= \left[\frac{1}{2}x^2+8x+4\ln(x-3)\right]_4^5$

$= \frac{25}{2}+40+4\ln 2-(8+32+4\ln 1)$

$= \frac{25}{2}+4\ln 2$

$$\begin{array}{r} x+8 \\ x-3 \,\big)\, x^2+5x-20 \\ \underline{x^2-3x} \\ 8x-20 \\ \underline{8x-24} \\ 4 \end{array}$$

c.) $\int \sin^5\theta \, d\theta$

$= \int \sin^4\theta \sin\theta \, d\theta$

$= \int (1-\cos^2\theta)^2 \sin\theta \, d\theta$

$= \int (1-2\cos^2\theta+\cos^4\theta)\sin\theta \, d\theta$

$= \int \sin\theta \, d\theta - 2\int \cos^2\theta\sin\theta \, d\theta + \int \cos^4\theta \sin\theta \, d\theta$

$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

6d.) (i) $\dfrac{4}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$

$= \dfrac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$

$4 = Ax(x-2) + B(x-2) + Cx^2$

when $x = 2$,

$4 = 4C$

$C = 1$

when $x = 0$,

$4 = -2B$

$B = -2$

coefficient of x^2,

$A + C = 0$

$A + 1 = 0$

$A = -1$

$\therefore A = -1, \quad B = -2, \quad C = 1$

(ii) $\displaystyle\int \frac{4}{x^2(x-2)}\,dx = \int \frac{-1}{x}\,dx + \int \frac{-2}{x^2}\,dx + \int \frac{1}{x-2}\,dx$

$= -\ln|x| + 2x^{-1} + \ln|x-2| + C$

$= \ln\left|\dfrac{x-2}{x}\right| + \dfrac{2}{x} + C$ (2)

- Some left out the absolute signs in 'ln'
- The answer must be simplified as in (2)

6e.) $\int \cos(\ln x)\, dx$

Let $u = \cos(\ln x)$

$\therefore du = -\sin(\ln x) \times \frac{1}{x}\, dx$

$dv = dx$

$\therefore v = x$

$$I = \int \cos(\ln x)\, dx$$

$$= x\cos(\ln x) - \int -\sin(\ln x)\, dx$$

$$= x\cos(\ln x) + \int \sin(\ln x)\, dx$$

Let $u = \sin(\ln x)$

$du = \frac{1}{x}\cos(\ln x)\, dx$

$dv = dx$

$v = x$

$$I = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\, dx$$

$$= x\cos(\ln x) + x\sin(\ln x) - I$$

$$2I = x\cos(\ln x) + x\sin(\ln x)$$

$$I = \frac{x}{2}\cos(\ln|x|) + \frac{x}{2}\sin(\ln|x|) + C$$

Most students used the substitution $u = e^x$ which was not the most efficient method.

7a.)

$$\delta V = \pi\left[(r+\delta r)^2 - r^2\right]h$$

$$= \pi\left\{2r\delta r + (\delta r)^2\right\}h$$

$$\delta V = 2\pi r h\,\delta r \qquad \text{since } (\delta r)^2 \text{ is effectively zero.}$$

$$\delta V = 2\pi(4-x)\frac{1}{x+1}\,\delta x \qquad \begin{array}{|l} r = 4-x \\ h = y \\ \delta r = \delta x \\ y = \frac{1}{x+1} \end{array}$$

$$V = 2\pi\int_0^4 (4-x)\frac{1}{x+1}\,dx$$

$$= 2\pi\int_0^4 \frac{4-x}{x+1}\,dx$$

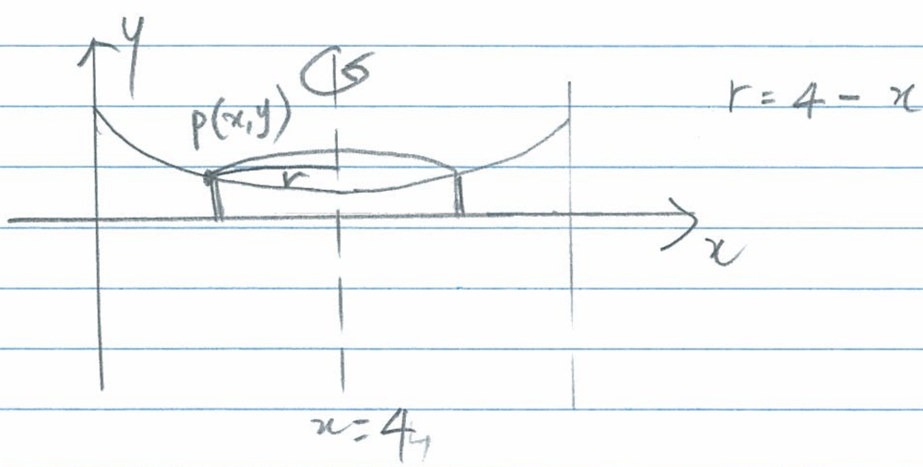
$$= -2\pi\int_0^4 \frac{x+1-5}{x+1}\,dx$$

$$= -2\pi\int_0^4 1 - \frac{5}{x+1}\,dx$$

$$= -2\pi\left[x - 5\ln(x+1)\right]_0^4$$

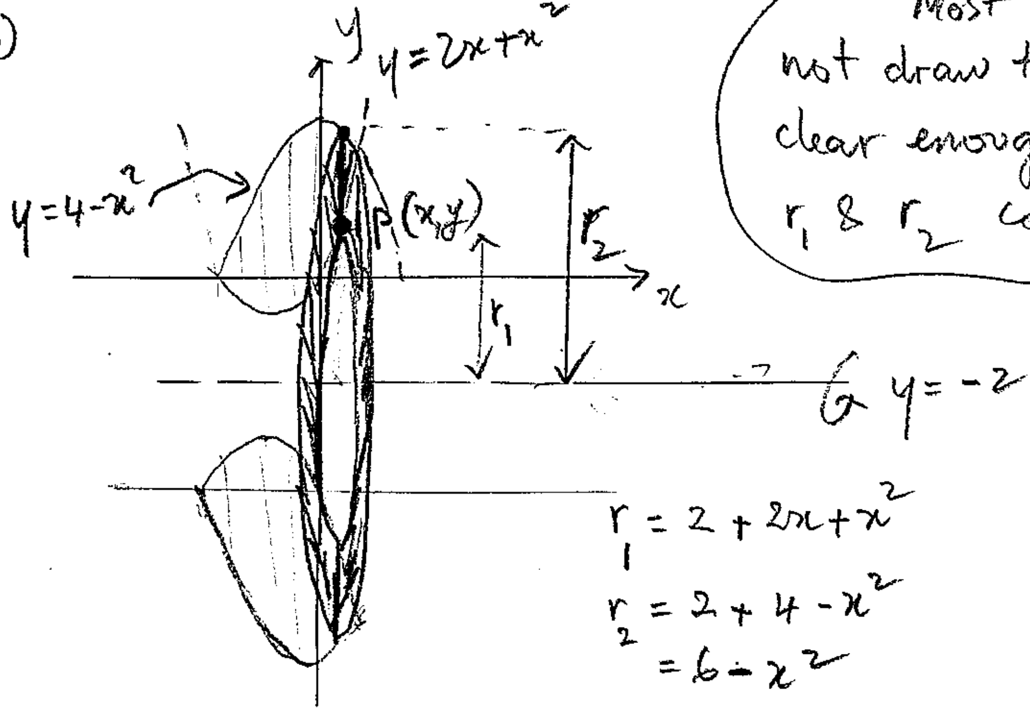
$$= -2\pi\left[4 - 5\ln 5 - (0 - 5\ln 1)\right]$$

$$= 2\pi\left\{5\ln 5 - 4\right\} \text{ units}^3$$

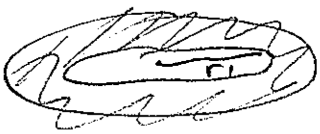


Q7 (b)

Most students did not draw the diagram clear enough to find r_1 & r_2 corectly.



$$r_1 = 2 + 2x + x^2$$
$$r_2 = 2 + 4 - x^2$$
$$= 6 - x^2$$



$$\partial V = \pi\left(r_2^2 - r_1^2\right) \partial x$$

Where $r_2^2 - r_1^2 = (r_2 - r_1)(r_2 + r_1)$

$$= (6 - x^2 - 2 - 2x - x^2)(2 + 2x + x^2 + 6 - x^2)$$
$$= (4 - 2x - 2x^2)(8 + 2x) = 4(2 - x - x^2)(4 + x)$$

$$\therefore V = \lim_{x \to 0} \sum_{-2}^{1} 4\pi\left(8 + 2x - 4x - x^2 - 4x^2 - x^3\right)$$

$$= 4\pi \int_{-2}^{1} 8 - 2x - 5x^2 - x^3 \, dx$$

$$= 4\pi \left[8x - x^2 - \frac{5}{3}x^3 - \frac{x^4}{4}\right]_{-2}^{1}$$

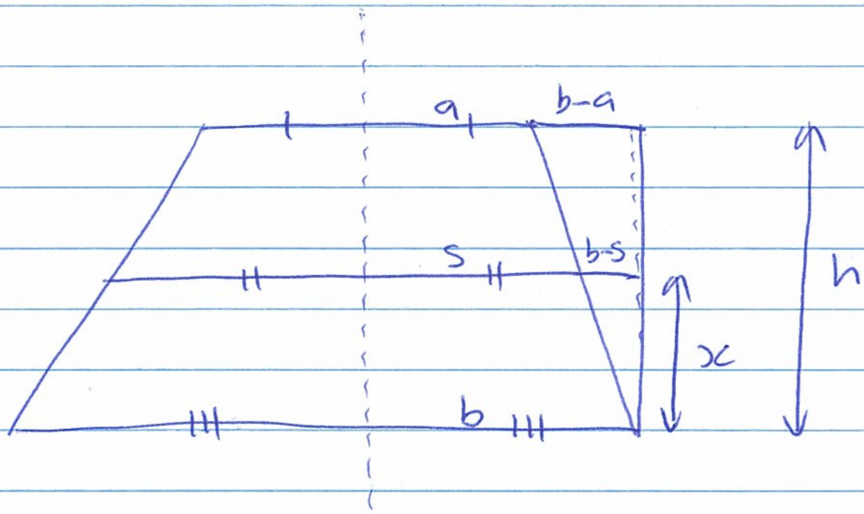
$$= 4\pi\left(8 - 1 - \frac{5}{3} - \frac{1}{4} + 16 + 4 - \frac{40}{3} + 4\right)$$

$$= 4\pi\left(\frac{63}{4}\right)$$

$$= 63\pi \text{ units}^3.$$

7c.)

(i)



By similar triangles,

$$\frac{b-s}{b-a} = \frac{x}{h}$$

$$b-s = \frac{b-a}{h}x$$

$$s = b - \frac{b-a}{h}x$$

(ii)

$$\delta V = \pi r^2 \, \delta x$$

For shaded circle,

$$\delta V = \pi s^2 \, \delta x$$

$$V = \pi \int_0^h s^2 \, dx$$

$$= \pi \int_0^h \left(b - \frac{b-a}{h}x\right)^2 dx$$

7c.) (ii)

$$V = \pi \int_0^h \left(b - \frac{(b-a)}{h} x\right)^2 dx$$

$$= \pi \int_0^h \left(b^2 - \frac{2b(b-a)}{h} x + \frac{(b-a)^2}{h^2} x^2\right) dx$$

$$= \pi \left\{ b^2 x - \frac{b(b-a)}{h} x^2 + \frac{(b-a)^2}{3h^2} x^3 \right\}_0^h$$

$$= \pi \left\{ b^2 h - \frac{h^2 b(b-a)}{h} + \frac{h^3 (b-a)^2}{3h^2} - 0 \right\}$$

$$= \pi \left\{ b^2 h - bh(b-a) + \tfrac{1}{3} h (b-a)^2 \right\}$$

$$= \frac{\pi h}{3} \left\{ 3ab + b^2 - 2ab + a^2 \right)$$

$$= \frac{\pi h}{3} \left(a^2 + ab + b^2 \right)$$

8a.) (i) $x^3 + y^3 = -1$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

(ii) x-intercept, $y = 0$

$$\frac{dy}{dx} = \frac{x^2}{0} = \text{undefined}$$

$$x^3 + 0 = -1$$

$$x = -1$$

tangent at x-intercept has equation $x = -1$

y-intercept, $x = 0$

$$\frac{dy}{dx} = \frac{0}{y^2} = 0$$

$$0 + y^3 = -1$$

$$y = -1$$

tangent at y-intercept has equation $y = -1$.

where curve meets line $y = x$

$$\frac{dy}{dx} = \frac{-x^2}{x^2} \text{ or } = \frac{-y^2}{y^2} = -1$$

$$x^3 + x^3 = -1$$

$$2x^3 = -1$$

$$x^3 = -\frac{1}{2}$$

$$x^3 = -2^{-1}$$

$$x = -2^{-\frac{1}{3}}$$

8a.) (ii) (cont.)

at $x = -2^{-\frac{1}{3}}$,

$$x^3 + y^3 = -1$$

$$\left(-2^{-\frac{1}{3}}\right)^3 + y^3 = -1$$

$$-\frac{1}{2} + y^3 = -1$$

$$y^3 = -\frac{1}{2}$$

$$y = -2^{-\frac{1}{3}}$$

equation of tangent at $\left(-2^{-\frac{1}{3}}, -2^{-\frac{1}{3}}\right)$

$$y - y_1 = m(x - x_1)$$

$$y + 2^{-\frac{1}{3}} = -\left(x + 2^{-\frac{1}{3}}\right)$$

$$y = -x - 2 \times 2^{-\frac{1}{3}}$$

$$y = -x - 2^{\frac{2}{3}}$$

8a.) (iii)

$$x^3 + y^3 = -1$$

$$y^3 = -1 - x^3$$
$$= -x^3\left(1 + \frac{1}{x^3}\right)$$

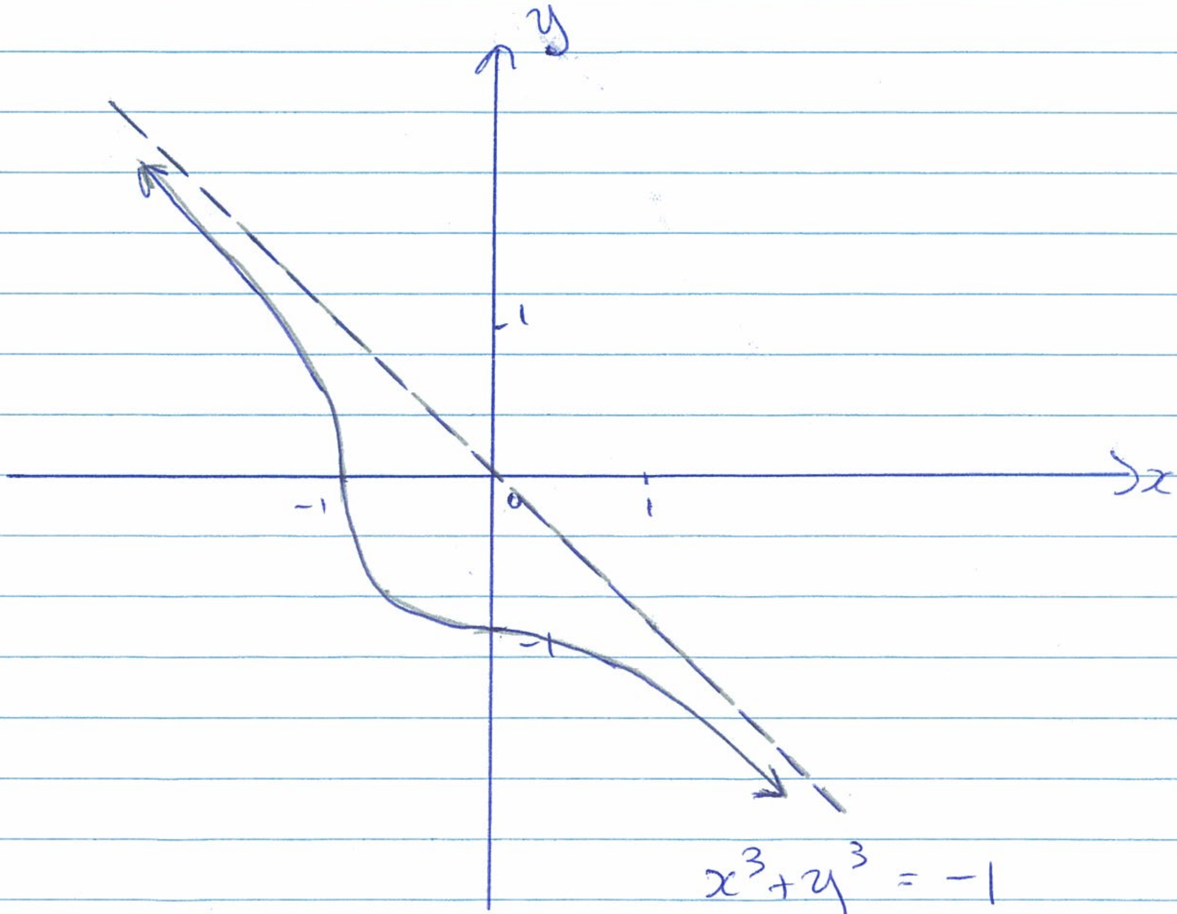
$$y = -x\sqrt[3]{1 + \frac{1}{x^3}}$$

as $x \to \infty$, $\frac{1}{x^3} \to 0$

$\therefore y \to -x$

$\therefore$ oblique asymptote is $y = -x$

8a.) (iv)



$x^3 + y^3 = -1$