Killara HIGH SCHOOL

Student Number

2016
Mathematics Extension 2

HSC Course Assessment Task 2

Date: 15/03/16

General Instructions

- Reading time – 5 minutes
- Working time 3 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 11 – 16 show relevant mathematical reasoning and/or calculations
- Answer each question in a separate writing booklet
- This paper must not be removed from the examination room
- A reference sheet is provided at the back of this paper
- Diagrams are NOT to scale

Total Marks – 100

Section I - Pages 2 - 4

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II - Pages 5 - 10

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

QUESTION 1-10 Multiple Choice	**/10**
QUESTION 11	**/15**
QUESTION 12	**/15**
QUESTION 13	**/15**
QUESTION 14	**/15**
QUESTION 15	**/15**
QUESTION 16	**/15**

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 30% of the Preliminary / HSC *Course Assessment*

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1 – 10 (Detach from paper)

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1 Consider the curve: $\frac{x^2}{a} + \frac{y^2}{b} = 1$, which of the following statement is true?

(A) Every line passing through the origin is a line of symmetry

(B) The x-intercepts are $(a, 0)$ and $(-a, 0)$

(C) The y-intercepts are $(0, \sqrt{b})$ and $(0, -\sqrt{b})$

(D) The curve is an ellipse centre at (a, b)

2 Find all solutions of $z^3 = -1$

(A) $z = -1$

(B) $z = -1, \frac{1 \pm \sqrt{3}i}{2}$

(C) $z = -1, \frac{1 \pm \sqrt{3}}{2}$

(D) $z = \pm 1,$

3 When a plane intersects a cone at an angle that is parallel to the slant edge of the cone, what shape is formed?

(A) Parabola

(B) Hyperbola

(C) Circle

(D) Ellipse

4 Evaluate $\frac{i}{5-i}$

(A) $\frac{-1+5i}{26}$

(B) $\frac{5-i}{26}$

(C) $-\frac{1}{5}+i$

(D) None of the above

5 Which of the following is an equation of an ellipse with center $(2,-3)$, horizontal major axis of length 10 and minor axis of length 6.

(A) $\frac{(x-2)^2}{25}+\frac{(y+3)^2}{9}=1$

(B) $\frac{(x+2)^2}{5}+\frac{(y-3)^2}{3}=1$

(C) $\frac{(x-2)^2}{10}+\frac{(y+3)^2}{6}=1$

(D) $\frac{(x-2)^2}{10^2}+\frac{(y+3)^2}{6^2}=1$

6 Consider the following equation: $|x+1|+|x-1|=2$

(A) It has only 1 solution

(B) It has 2 solutions

(C) It has no solutions

(D) None of the above

7 The equation $x^3+2x-1=0$ has roots α, β and γ. Find the monic equation with roots α^2, β^2 and γ^2.

(A) $x^3+2x^2+2x-3=0$

(B) $x^3+4x^2+2x-1=0$

(C) $x^3+4x^2+4x-1=0$

(D) $x^3+4x^2+4x-3=0$

8 Let $z = \sqrt{48} - 4i$. What is the value $\arg(z^7)$?

(A) $\frac{-2\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $\frac{5\pi}{6}$

(D) $\frac{-5\pi}{6}$

9 Given the graph of $y = f(x)$, how would you sketch the graph of $y = f(2 - x)$.

(A) Shift $y = f(x)$ to the right by 2 units and then reflect on y −axis

(B) Shift $y = f(x)$ to the left by 2 units and then reflect on x −axis

(C) Reflect on y −axis and then shift to the left by 2 units

(D) Reflect on y −axis and then shift to the right by 2 units

10 The polynomial $P(x) = x^5 + ax^4 + x^3 + bx^2 - 8x - 4$ has a double root at $x = -1$. What are the values of a and b ?

(A) $a = -2$ and $b = -4$

(B) $a = 2$ and $b = -4$

(C) $a = 2$ and $b = 4$

(D) $a = -2$ and $b = 4$

End of Section I

Section II

90 marks
Attempt Questions 11 – 15
Allow about 2 hours and 15 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Let $z = 2i + 1$ and $w = 5 - 3i$, evaluate the following and give your answer in the form of $a + ib$.

(i) $z + 2w$ **1**

(ii) $\frac{z}{w}$ **2**

(iii) $z\overline{w} + \bar{z}w$ **2**

(b) Show that $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$ is a real number. **2**

(c) Let ω be a non real cube root of unity.

(i) Show that $\omega^2 + \omega + 1 = 0$ **2**

(ii) Hence show that: $(b + c)(b + c\omega)(b + c\omega^2) = b^3 + c^3$ **2**

(d) (i) Express $z = \frac{i-1}{\sqrt{3}+i}$ in mod-arg form. **2**

(ii) Hence evaluate $\cos\left(\frac{7\pi}{12}\right)$ in surd form. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) A cubic equation: $x^3 - 2x^2 + 13x - 12 = 0$ has roots α, β and γ. Given that:

$$\alpha^2 + \beta^2 + \gamma^2 = -22$$

Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) The roots of $x^3 + x + 1 = 0$ are α, β, γ. Find the cubic equation whose roots are: $\frac{1}{1-\alpha}, \frac{1}{1-\beta}, \frac{1}{1-\gamma}$ **2**

(c) (i) Use De Moivre's theorem to express $\sin(3\theta)$ in term of $\sin(\theta)$. **2**

(ii) Use the results above to solve: $8x^2 - 6x + \sqrt{3} = 0$, give answer to 3 significant figures. **2**

(iii) Hence deduce that: $\sin\left(\frac{\pi}{9}\right) + \sin\left(\frac{7\pi}{9}\right) = \sin\left(\frac{5\pi}{9}\right)$ **1**

(d) The polynomial $P(x) = 3x^4 + 2x^3 - 6x^2 - 6x + a$ has a root of multiplicity 2.

(i) Find the possible values of a. **3**

(ii) If a is positive, state the repeated zero, and hence factorise $P(x)$ into linear factors. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are points on $xy = c^2$ which moves such that P, Q and $S(c\sqrt{2}, c\sqrt{2})$ are always collinear.

(i) Show that the tangent to $xy = c^2$ at any point $T\left(ct, \frac{c}{t}\right)$ is given by: $x + t^2 y = 2ct$ **2**

(ii) Show that the tangents at P and Q intersect at point R: $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ **1**

(iii)Show that $p + q = \sqrt{2}(1 + pq)$ **1**

(iv)Find the locus of R **1**

(b) The tangent at $P(x, y)$ to the hyperbola $x^2 - y^2 = a^2$ in the first quadrant meets the x-axis at acute angle α. The line OP, where O is the origin, meets the x-axis at an acute angle β.

(i) Prove that the product of the gradients of the line OP and tangent at P is 1. **1**

(ii) Show that $\alpha + \beta = \frac{\pi}{2}$ **2**

(c) (i) Show that the equation of the chord joining the points $P(cp, c/p)$ and $Q(cq, c/q)$ is $x + pqy = c(p + q)$. **2**

(ii) Hence determine the gradient of the tangent to the rectangular hyperbola $xy = c^2$ at the point $P(cp, c/p)$. **1**

(iii)The chord AB of the hyperbola $xy = c^2$ subtends a right angle at a point P on the curve.

(α) Draw a diagram to show the information **1**

(β) Prove that AB is parallel to the normal at P. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) $\bar{z}$ is the complex conjugate of z. If $z\bar{z} + 2iz = 12 + 6i$, find the possible solutions for z. **2**

(b) (i) If $z = r(cos\theta + isin\theta)$, show, without applying De Moivre's theorem, that $z^2 = r^2(cos2\theta + isin2\theta)$ **1**

(ii) Hence, find the square roots of $\left(2\sqrt{3} - 2i\right)$. **2**

(c) In an Argand diagram, the points P and Q represent the complex numbers $a + ib$ and $-i(a + ib)$ respectively, where a and b are real and $b > a > 0$.

(i) On a single diagram, sketch the loci of z defined by the following equations: **3**

$$|z - a - ib| = |z - b + ia|, \arg(z - a - ib) = \arg(b - ia)$$

Indicate clearly on your diagram, the point of intersection R of the two loci.

(ii) Hence find, in terms of a and b, the complex number represented by R. **2**

(d) (i) If $z = cos\theta + isin\theta$, show that **2**

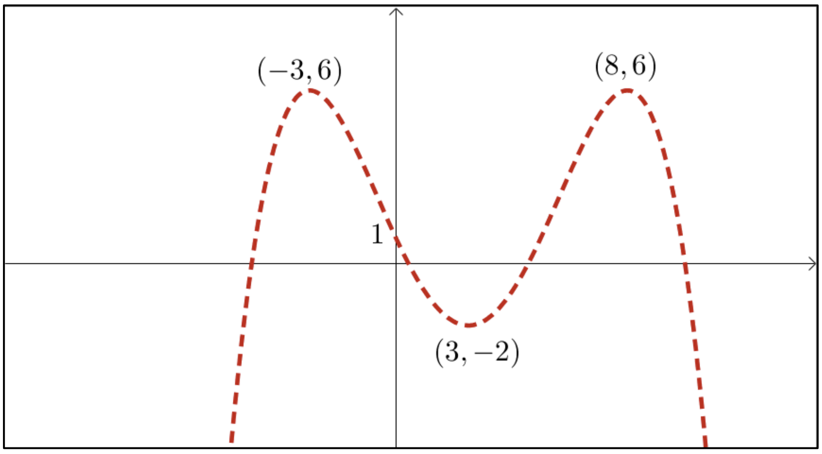
$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \text{ and } z^n - \frac{1}{z^n} = 2i\sin(n\theta)$$

(ii) Use the result to deduce that $cos^4\theta + sin^4\theta = \frac{1}{4}(cos4\theta + 3)$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a)



Given $y = f(x)$ in the diagram above, sketch the following graphs with $y = f(x)$ on the same axis.

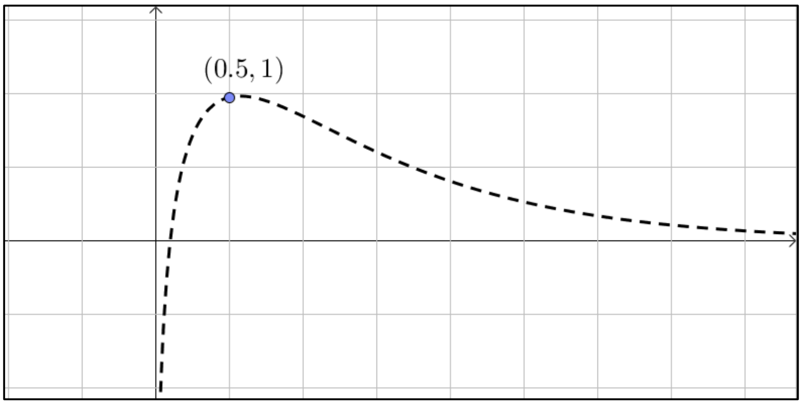
(a) $y = f'(x)$ **1**

(b) $|y| = f(|x|)$ **2**

(c) $y = \sqrt{f(x)}$ **2**

(d) $y = f(|x|)$ **2**

(b)



Given $y = f(x)$ in the diagram above, sketch the following graphs with $y = f(x)$ on the same axis.

(a) $y = 3 - f(x + 1)$ **2**

(b) $y = \frac{1}{f(x)}$ **2**

(c) $y = 1 - |f(x)|$ **2**

(d) $y = f(x)^2$ **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that if $y = mx + k$ is a tangent to the hyperbola: **2**

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m^2a^2 - b^2 = k^2$.

(ii) Hence find the equations of the tangents from the point $(1, 3)$ to the hyperbola, $\frac{x^2}{4} - \frac{y^2}{15} = 1$ and the co-ordinates of the points of contact. **2**

(b) (i) Show that the parametric coordinate of the point $P\left(\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **2**

(ii) Show that the equation of the tangent at P in part (i) is given by **3**

$$\left(\frac{t^2+1}{a}\right)x - \left(\frac{t^2-1}{b}\right)y = 2t$$

(iii) If S is a focus and if the tangent at P, the line through S perpendicular to the x-axis and one of the asymptotes are concurrent. Prove that the line joining P to the focus is parallel to the other asymptote. **3**

End of Question 16

End of paper

2016 Killara Ext 2 Task 3 Solutions

Multiple Choice:

Q.1) C	Q.6) D
Q.2) B	Q.7) C
Q.3) A	Q.8) C
Q.4) A	Q.9) D
Q.5) A	Q.10) B

Question 11:

(a) $z = 2i + 1\,, w = 5 - 3i$

(i) $z + 2w$

$= 2i + 1 + 2(5 - 3i)$

$= 2i + 1 + 10 - 6i$

$= 11 - 4i$ **1 mark:** correct answer

(ii) $\frac{z}{w} = \frac{2i+1}{5-3i} \times \frac{5+3i}{5+3i}$ **1 mark:** multiply conjugate

$= \frac{10i-6+5+3i}{25+9}$

$= \frac{-1+13i}{34}$

$= -\frac{1}{34} + \frac{13}{34}i$ **1 mark:** correct answer

(iii) $z\overline{w} + \overline{z}w$ **1 mark:** correct substitution

$= (2i + 1)(5 + 3i) + (-2i + 1)(5 - 3i)$

$= 10i - 6 + 5 + 3i - 10i - 6 + 5 - 3i$

$= -2$ **1 mark:** correct answer

(b) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

$= \frac{5i+10i^2+6-3i-8i+4i^2}{15i+20}$ **1 mark:** common denominator

$= \frac{-6i-8}{15i+20}$

$= \frac{-2(3i+4)}{5(3i+4)}$

$= -\frac{2}{5}$ **1 mark:** correct answer

(c) $w^3 = 1$

1 mark: using root of unity
1 mark: reasonable explanation

(i) $w^3 - 1 = 0$

$(w - 1)(w^2 + w + 1) = 0$

since w is a non-real root, hence $w \neq 1$

Therefore $(w^2 + w + 1) = 0$

(ii) $LHS = (b + c)(b + cw)(b + cw^2)$

$= (b + c)(b^2 + bcw^2 + bcw + c^2w^3)$

$= (b + c)(b^2 + bc(w^2 + w) + c^2w^3)$

$= (b + c)(b^2 - bc + c^2)$

$= b^3 - b^2c + bc^2 + b^2c - bc^2 + c^3$

$= b^3 + c^3 = RHS$

1 mark: attempt to expand
1 mark: correct proof

(d) $z = \frac{i-1}{\sqrt{3}+i}$

(i) $i - 1 = \sqrt{2}Cis\left(\frac{3\pi}{4}\right)$ **1 mark:** convert to mod-arg form

$\sqrt{3} + i = 2Cis\left(\frac{\pi}{6}\right)$ **1 mark:** correct answer

$\frac{i-1}{\sqrt{3}+i} = \frac{\sqrt{2}}{2}Cis\left(\frac{3\pi}{4} - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}Cis\left(\frac{7\pi}{12}\right)$

$= \frac{\sqrt{2}}{2}\left[Cos\left(\frac{7\pi}{12}\right) + isin\left(\frac{7\pi}{12}\right)\right]$

(ii) $\frac{i-1}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{\sqrt{3}i-\sqrt{3}+1+i}{3+1} = \frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4}i$

Equating real parts: **1 mark:** attempt to equate real parts

$\frac{\sqrt{2}}{2}Cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{4}$ **1 mark:** correct answer

$Cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$

Q12 $\quad x^3 - 2x^2 + 13x - 12 = 0 \quad$ sub. α, β, γ

$$\alpha^3 = 2\alpha^2 - 13\alpha + 12$$
$$\beta^3 = 2\beta^2 - 13\beta + 12$$
$$\gamma^3 = 2\gamma^2 - 13\gamma + 12$$

$$\alpha^3+\beta^3+\gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) - 13(\alpha+\beta+\gamma) + 36$$
$$= 2\times -22 - 13\times 2 + 36$$
$$= -34$$

b) let $x = \frac{1}{1-\alpha} \Rightarrow 1-\alpha = \frac{1}{x} \Rightarrow \alpha = 1 - \frac{1}{x} = \frac{x-1}{x}$ sub into $P(x)$

$$\left(\frac{x-1}{x}\right)^3 + \frac{x-1}{x} + 1 = 0$$
$$x^3 - 3x^2 + 3x - 1 + x^3 - x^2 + x^3 = 0$$
$$3x^3 - 4x^2 + 3x - 1 = 0 \quad \text{is the required equation.}$$

(c) let $z = \cos\theta + i\sin\theta$ ①

$$z^3 = \cos 3\theta + i\sin 3\theta$$
$$= (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta - i\sin^3\theta$$
$$= \cos^3\theta - 3\cos\theta\sin^2\theta + \left[3(1-\sin^2\theta)\sin\theta - \sin^3\theta\right]i \quad ②$$

equate im. parts of ① & ②

$$\Rightarrow \sin 3\theta = 3\sin\theta - 4\sin^3\theta \quad ③$$

$$\boxed{\begin{array}{l}\theta = 0.342, \\ 0.643, \\ -0.985\end{array}}$$

NOW $\quad 8x^3 - 6x = -\sqrt{3} \quad$ let $x = \sin\theta$

$$\Rightarrow 8\sin^3\theta - 6\sin\theta = -\sqrt{3} \quad \div -2$$
$$-4\sin^3\theta + 3\sin\theta = \frac{\sqrt{3}}{2}$$

ie $\quad \sin 3\theta = \frac{\sqrt{3}}{2} \Rightarrow 3\theta = \frac{\pi}{3} + 2\pi k$

$$\theta = \frac{\pi}{9} + \frac{2\pi k}{3} \Rightarrow \theta = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9}$$

$$\Sigma \text{roots} = \frac{\pi}{9} + \frac{7\pi}{9} + \frac{13\pi}{9} = 0 \quad \text{but } \sin\frac{13\pi}{9} = \sin\left(-\frac{5\pi}{9}\right)$$

$$\therefore \frac{\pi}{9} + \frac{7\pi}{9} = \frac{5\pi}{9}$$

Q12(d) (i)

$P(x) = 3x^4 + 2x^3 - 6x^2 - 6x + a$

$P(x)$ has a root of multiplicity 2 (at say $x=\alpha$)

$\Rightarrow P(\alpha)=0$ and $P'(\alpha)=0$

$\Rightarrow 3\alpha^4 + 2\alpha^3 - 6\alpha^2 - 6\alpha + a = 0$ — (1)

and $12\alpha^3 + 6\alpha^2 - 12\alpha - 6 = 0$ — (2)

1 mark

(2) $\Rightarrow$

$2\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$

$\alpha^2(2\alpha+1) - (2\alpha+1) = 0$

1 mark for solving for α

$\alpha = 1$; $\alpha = -1$; $\alpha = -\frac{1}{2}$

$a = 7$; $a = -1$; $a = -\frac{25}{16}$

1 mark for all correct values of a

(ii) $a > 0 \Rightarrow a = 7$

when $a=7$, the repeated zero is 1 (part (i)) → 1 mark

$\Rightarrow (x-1)^2$ is a factor of $P(x)$

$\Rightarrow P(x) = 3x^4 + 2x^3 - 6x^2 - 6x + 7$ → 1 mark

$= (x-1)^2(3x^2 + 8x + 7)$

$= 3(x-1)^2\left[x - \left(\frac{4}{3} - \frac{\sqrt{5}}{3}i\right)\right]\left[x - \left(\frac{4}{3} + \frac{\sqrt{5}}{3}i\right)\right]$ → 1 mark

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Question 13:

(a) Given $xy = c^2$,

(i) $x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

1 mark:
correctly differentiate

At $T\left(ct, \frac{c}{t}\right)$, gradient is $-\frac{1}{t^2}$

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

1 mark:
correct answer

(ii) Tangent at P: $x + p^2y = 2cp$

Tangent at Q: $x + q^2y = 2cq$

The intersection R:

$2cp - p^2y = 2cq - q^2y$

$2c(p - q) = (p^2 - q^2)y$

$y = \frac{2c}{(p+q)}$

1 mark:
reasonable proof

$x = 2cp - \frac{2c}{p+q} = \frac{2cpq}{p+q}$

(iii) Since P, Q and S are collinear:

$M_{PQ} = M_{QS}$

$\frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\frac{c}{q} - c\sqrt{2}}{cq - c\sqrt{2}}$

$-\frac{q - \sqrt{2}}{pq} = \frac{1}{q} - \sqrt{2}$

$-q + \sqrt{2} = p - pq\sqrt{2}$

$\sqrt{2} + pq\sqrt{2} = p + q$

$\sqrt{2}(1 + pq) = p + q$

1 mark:
reasonable proof

(iv) $x = \frac{2cpq}{p+q} \quad y = \frac{2c}{p+q}$

$x = \frac{2c(pq)y}{2c}$

$x = y\left(\frac{p+q}{\sqrt{2}} - 1\right)$

$x = y\left(\frac{2c}{y\sqrt{2}} - 1\right)$

$x = \frac{2c}{\sqrt{2}} - y$

$x = \sqrt{2}c - y$

$y = \sqrt{2}c - x$

1 mark:
correctly eliminate the parameters

(b) Given $x^2 - y^2 = a^2$

(i) $2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

Slope at P: $\frac{x}{y}$

1 mark:
reasonable proof

Slope OP: $\frac{y-0}{x-0} = \frac{y}{x}$

$M_p \times M_{OP} = \frac{x}{y} \times \frac{y}{x} = 1$

(ii) Since $\tan(\alpha) = \frac{x}{y}$ and $\tan(\beta) = \frac{y}{x}$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - 1} \quad (undefined)$$

Therefore $\quad \alpha + \beta = \frac{\pi}{2}$

1 mark:
decent attempt

1mark:
reasonable explaination

Q13 (c)(i) $P(cp, c/p), Q(cq, c/q)$

gradient of $PQ = \dfrac{c/q - c/p}{cq - cp}$

$= \dfrac{p-q}{pq(q-p)}$

$= -\dfrac{1}{pq}$ → 1 mark for for getting the gradient

∴ Eqⁿ of PQ is

$y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

$\Rightarrow x + pqy = c(p+q)$ → 1 mark

(ii) as $Q \to P$, chord $PQ \to$ tangent at P

$\Rightarrow q = p$

$\Rightarrow$ Eqⁿ of tangent at P is,

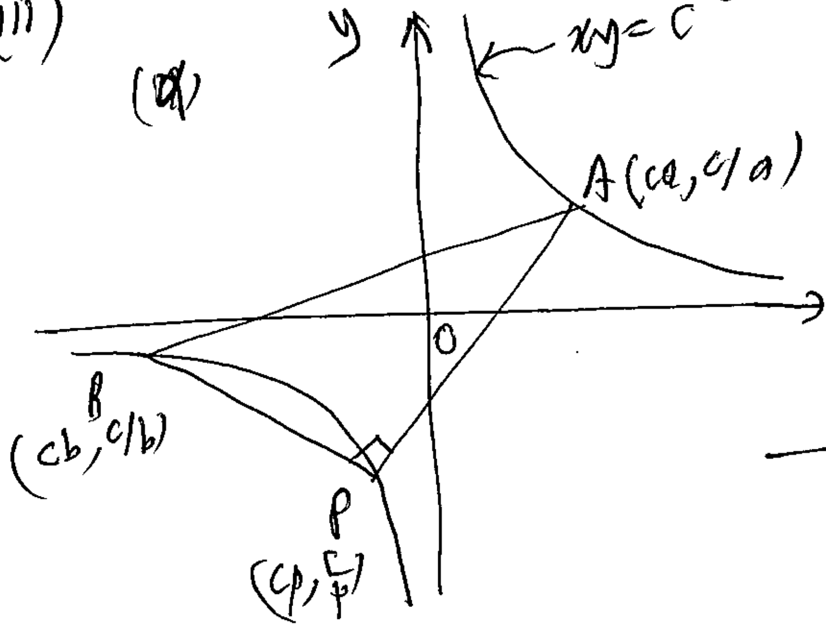
$x + p^2y = c(p+p)$

(ie) $x + p^2y - 2cp = 0$ → 1 mark

~~(iii)~~

Q 13 (c) (iii)

(α)



→ 1 mark for A and B on diff branch of the hyperbola

(β) $\angle APB = 90^0$

$\therefore m_{AP} \times m_{BP} = -1$ → 1 mark

$\Rightarrow -\frac{1}{ap} \times -\frac{1}{bp} = -1$

$\Rightarrow p^2 = -\frac{1}{ab}$ → 1 mark

gradient of the normal at P is p^2 from c(ii) above

gradient of the chord AB is $-\frac{1}{ab}$ from c(i) above → 1 mark

$\Rightarrow$ normal at P $\parallel$ chord AB

Q14. (a) $z\bar{z} + 2iz = 12 + 6i$

let $z = a + ib$

$\Rightarrow (a^2 + b^2 - 2b) + 2ia = 12 + 6i$

Matching the real parts

$a^2 + b^2 - 2b = 12$

and matching the imaginary parts

$2a = 6$

$\Rightarrow \left.\begin{matrix} a=3 \\ b=3 \end{matrix}\right\} \quad \left.\begin{matrix} a=3 \\ b=-1 \end{matrix}\right\}$ → 1 mark for the 2 equations

$\therefore z = \underline{\underline{3+3i}}$ or $z = \underline{\underline{3-i}}$ → 1 mark for correct answer

(b) (i) $z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos\theta + i\sin\theta)^2$

$= r^2\left[(\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta\right]$ } 1 mark

$= \underline{\underline{r^2\left[\cos 2\theta + i\sin 2\theta\right]}}$

(ii) $(2\sqrt{3} - 2i) = 4\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$

$= 4\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= 4\left[\cos\left(2\left(-\frac{\pi}{12}\right)\right) + i\sin\left(2\left(-\frac{\pi}{12}\right)\right)\right]$ } 1 mark

$\therefore \sqrt{2\sqrt{3} - 2i} = \pm 2\left[\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right]$

→ 1 mark

14 (C)

P represents $a+ib$, $b>a>0$

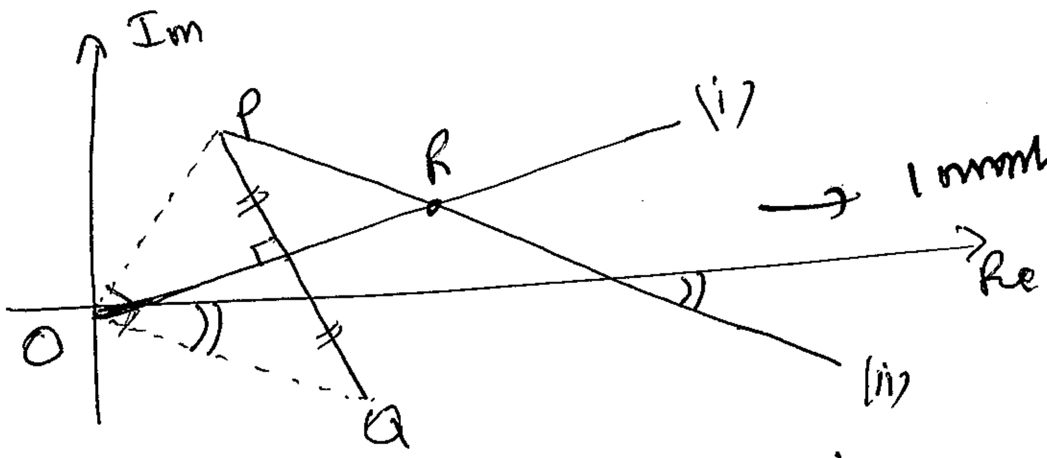
Q represents $-i(a+ib) = b-ia$

(i) $|z-a-ib| = |z-b+ia|$

$\equiv$ perpendicular bisector of line segment joining P and Q $\longrightarrow$ 1 mark

~~(ii)~~ $\arg(z-a-ib) = \arg(b-ia)$

$\equiv$ half line starting from P in the direction parallel to OQ $\longrightarrow$ 1 mark



$\longrightarrow$ 1 mark

(ii) . Let p, q, r = complex numbers representing by points P, Q, R $\longrightarrow$ 1 mark

Deduce that $OPRQ$ is a square

$\Rightarrow r = p+q = (a+ib)+(b-ia)$ $\longrightarrow$ 1 mark

$= (a+b)+i(b-a)$ (Answer)

14) (d)

$$z = \cos\theta + i\sin\theta \implies z^n = \cos(n\theta) + i\sin(n\theta) \quad \text{—— (1)}$$

De Moivre's Theorem

1 mark

and $\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$ —— (2)

(1) + (2) $\quad z^n + \frac{1}{z^n} = 2\cos(n\theta)$

(1) − (2) $\quad z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

→ 1 mark

$2\cos\theta = z + \frac{1}{z}$

$\therefore 16\cos^4\theta = (z + \frac{1}{z})^4$ —— (3)

$2i\sin\theta = z - \frac{1}{z}$

$\therefore 16\sin^4\theta = (z - \frac{1}{z})^4$ —— (4)

1 mark

(3) + (4) $\quad 16(\cos^4\theta + \sin^4\theta) = (z+\frac{1}{z})^4 + (z-\frac{1}{z})^4$

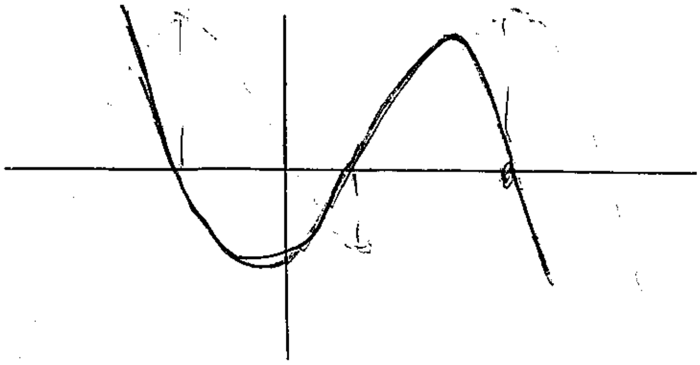
$= 2(z^4 + \frac{1}{z^4}) + 12$ → 1 mark

$16(\cos^4\theta + \sin^4\theta) = 2[2\cos 4\theta] + 12$

1 mark for simplifying

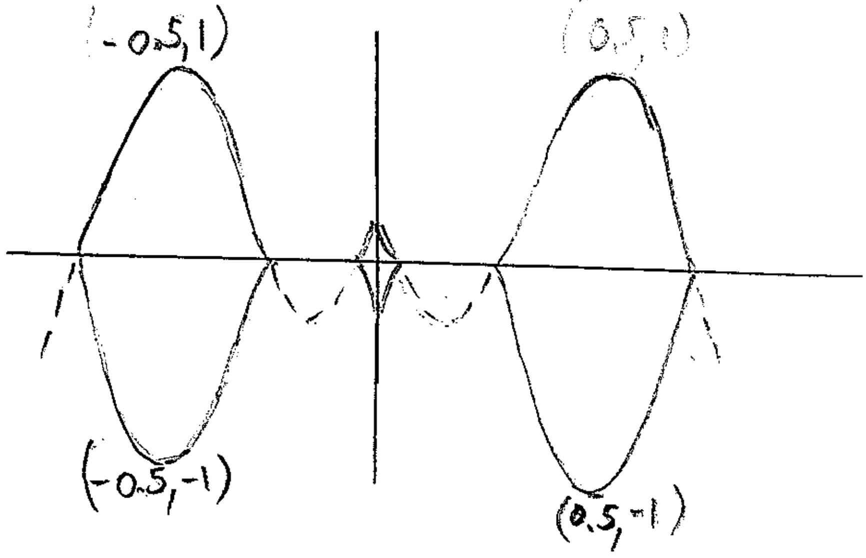
$\therefore \cos^4\theta + \sin^4\theta = \frac{1}{4}[\cos 4\theta + 3]$ Ans

(a)



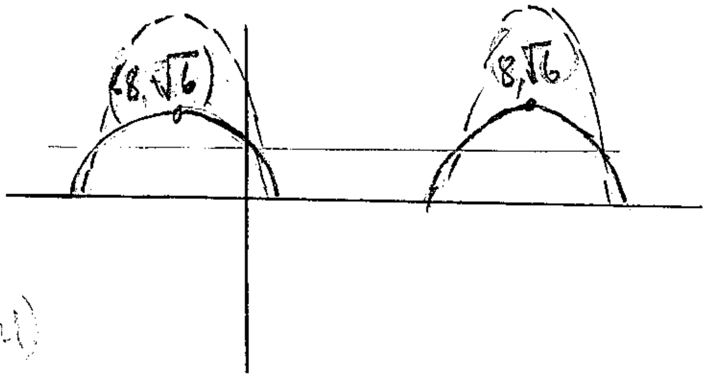
(b)

$|y| = f(|x|)$

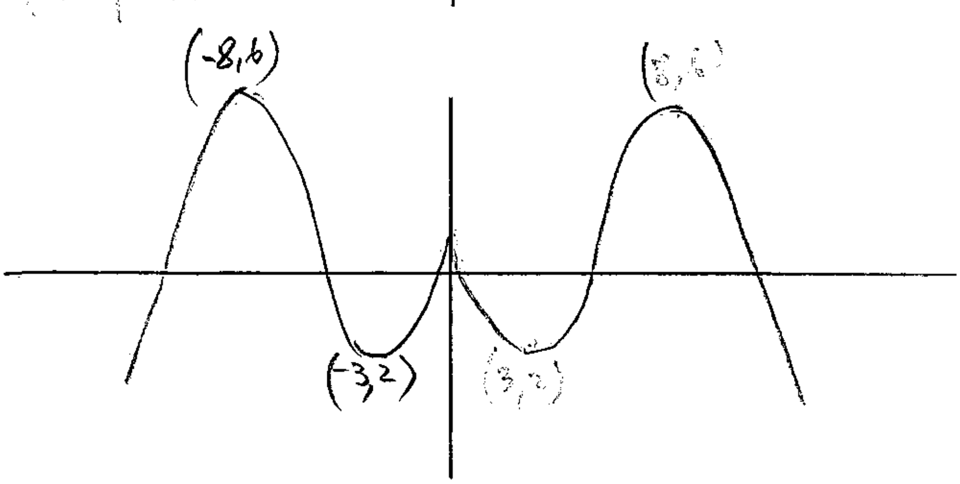


(c)

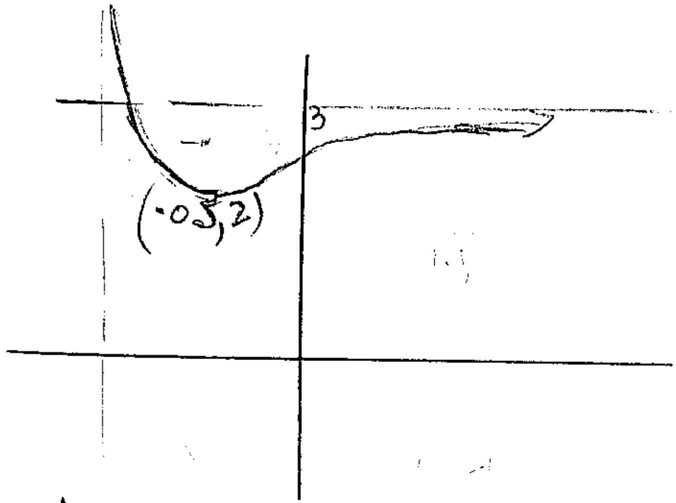
$y = \sqrt{x}$



(d) $y = f(|x|)$



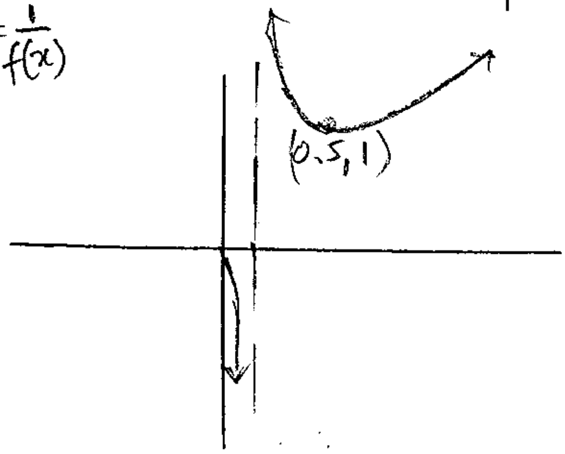
a)



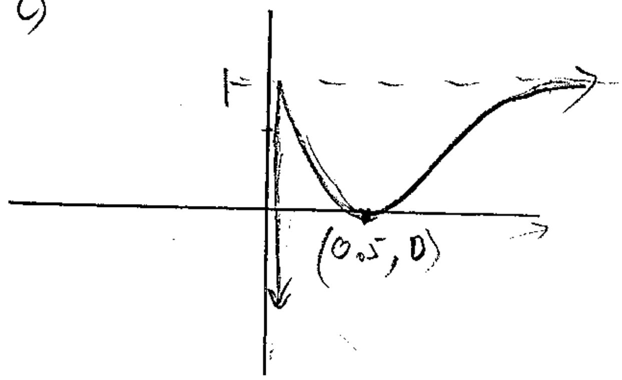
$y = 3 - f(x+1)$

$y = 1 - |f(x)|$

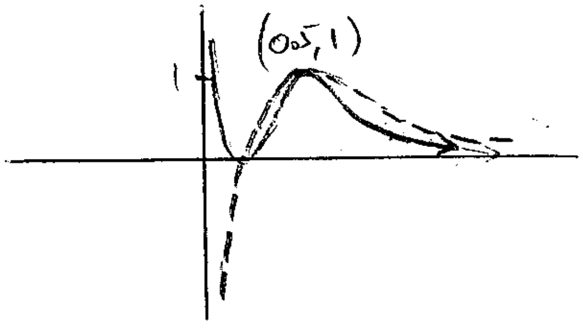
b) $y = \frac{1}{f(x)}$



c)



d)



$y = [f(x)]^2$

Q16 (a) Solve simultaneously $\begin{cases} y = mx+k \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{cases}$ for pts. of contact.

$$\frac{x^2}{a^2} - \frac{(mx+k)^2}{b^2} = 1 \Rightarrow x^2(b^2-a^2m^2) - 2mka^2x - a^2k^2 + a^2b^2 = 0 \quad (1)$$

Since $y = mx+k$ is the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, there is only 1 pt. of intersection $\therefore \Delta = 0$

$$\Delta = 4m^2k^2a^4 + 4(b^2-a^2m^2)(a^2k^2+a^2b^2) = 0$$

$$4a^2m^2b^2 = 4a^2b^2k^2 + 4a^2b^2$$

$$\Rightarrow a^2m^2 = k^2 + b^2 \Rightarrow a^2m^2 - b^2 = k^2$$

(ii) The equation of the tangents are $y = mx \pm \sqrt{a^2m^2 - b}$

Since $(1,3)$ lies on the tangents, sub. $(1,3)$, $a^2 = 4$, b^2

$$\Rightarrow (3-m)^2 = 4m^2 - 15 \Rightarrow 9 - 6m + m^2 = 4m^2 - 15$$

$$(m+4)(m-2) = 0$$

$\therefore m = -4$ & 2

when $m = -4$

Sub. into $k = \sqrt{a^2m^2 - b^2}$

$\Rightarrow k = = \sqrt{4 \times 16 - 15}$

$= 7$

$m = 2$

$k = \sqrt{4 \times 16 - 15}$

$= 1$

$\therefore$ Equations of the 2 tangents are $y = -4x + 7$ or $y = 2x + 1$

(iii) The points of contact are found by solving (1) for x.

Since $\Delta = 0 \Rightarrow x = \frac{b}{2a}$

ie $x = \frac{2mka^2}{2(b^2 - a^2m^2)}$

when $\left.\begin{matrix} m = -4 \\ k = 7 \end{matrix}\right\} \Rightarrow x = \frac{-2 \times 7 \times 4 \times 4}{2(15 - 4 \times 16)}$,

$= \frac{16}{7}$

$y = -\frac{15}{7}$

$\therefore \left(\frac{16}{7}, -\frac{15}{7}\right)$ is pt. of contact of $y = -4x + 7$

When $\begin{cases} m = 2, \\ k = 1 \end{cases} \Rightarrow x = \frac{2 \times 2 \times 1 \times 4}{2(15 - 4 \times 4)}$

$= -8$

$y = -15$

$(-8, -15)$ is pt. of contact of $y = 2x + 1$

b) Sub $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ into $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$L.H.S = \frac{\frac{\cancel{a^2}}{4}\left(t^2+2+\frac{1}{t^2}\right)}{\cancel{a^2}} - \frac{\frac{\cancel{b^2}}{4}\left(t^2-2+\frac{1}{t^2}\right)}{\cancel{b^2}}$$ ✓

$$= \frac{4}{4} = 1 = R.H.S$$ ✓

∴ P lies on the hyperbola

(ii) gradient of the tangent :

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

at $\left(\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right) \Rightarrow m = \frac{\frac{b^2a}{2}\left(t+\frac{1}{t}\right)}{\frac{a^2b}{2}\left(t-\frac{1}{t}\right)} = \frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)$ ✓

∴ equation of the tangent is :

$$y - \frac{b}{2}\left(t-\frac{1}{t}\right) = \frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)\left(x - \frac{a}{2}\left(t+\frac{1}{t}\right)\right)$$ ✓

$$\left(\frac{t^2+1}{a}\right)x - \left(\frac{t^2-1}{b}\right)y = -\frac{\cancel{b}}{2}\left(1-\frac{1}{t}\right)\cdot\frac{(t^2-1)}{\cancel{b}} + \frac{\cancel{a}}{2}\cdot\frac{t^2+1}{\cancel{a}}\left(t+\frac{1}{t}\right)$$ ✓

$$= -\frac{(t^2-1)^2}{2t} + \frac{(t^2+1)^2}{2t}$$

$$= \frac{-t^4+2t^2-1-t^4+2t^4+1}{2t} = 2t$$ ✓

(iii) Sub $x = ae$, into $y = \frac{bx}{a}$

$\Rightarrow y = \frac{bae}{a} = be$ ie the concurrent point $P(ae, be)$

P lies on the tangent, Sub (ae, be)

$\Rightarrow \left(\frac{t^2+1}{a}\right)x - \left(\frac{t^2-1}{b}\right)y = 2t \Rightarrow (t^2+1)e - (t^2-1)e = 2t$ ✓

$\Rightarrow 2e = 2t$ ∴ $\underline{\underline{e = t}}$

hence P has coordinates $\left(\frac{a}{2}\left(e+\frac{1}{e}\right), \frac{b}{2}\left(e-\frac{1}{e}\right)\right)$ ✓

$$m_{PS} = \frac{\frac{b}{2}\left(e-\frac{1}{e}\right)}{\frac{a}{2}\left(e+\frac{1}{e}\right)-ae} = \frac{-b(e^2-1)}{a(e^2-1)} = -\frac{b}{a}$$ ✓

The gradient of the other asymptote is $-\frac{b}{a}$

∴ $m_{PS} \parallel m_1 = -\frac{b}{a}$