Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

Q	Marks
1	/14
2	/15
3	/15
Total	/44

2021 /2022

Year 12

Mathematics Extension 2

Task 1

2/12/21

General Instructions:

- Reading time – 60 minutes
- Working time – 5 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- Answer all questions in the booklets provided

Total Marks: 44

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

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In Questions 1 – 3, your responses should include relevant mathematical reasoning and/or calculations.
Question 1: Answer Question 1 in a new booklet. (14 Marks)

(a) Given the complex number $z = 3 - 3i$ and $w = -\sqrt{3} + i$:

(i) Find $\arg(z)$ **1**

(ii) Find zw in the form $a + ib$ **1**

(iii) Find $\frac{z}{\overline{w}}$ in the form $a + ib$ **2**

(iv) Find $\left|\frac{z^2}{w^4}\right|$ **2**

(v) Find zw in Mod-Arg form. **2**

(vi) Hence, find the exact value of $\tan\frac{7\pi}{12}$ **1**

(vii) Sketch z, w and zw on the Argand plane supplied as a separate answer sheet. **2**

(b) Find real numbers a and b such that $z = 3 + i$ satisfies $z + \frac{a}{z} = b$. **3**

Question 2: Answer Question 2 in a new booklet. (15 Marks)

(a) (i) Find $\sqrt{6+8i}$ **3**

(ii) Hence, or otherwise, solve $z^2+\sqrt{2}iz+(-2-2i)=0$, $z \in \mathbb{C}$. **1**

(b) Give the linear factorisation of $4x^3+2x^2-14x+3$, $x \in \mathbb{R}$, given that it has one rational zero. **3**

(c) The complex numbers $z_1=2-5i$, $z_2=1+3i$ and z_3 are represented in the Argand plane by the points $P, Q$and R respectively. Find all possible values of z_3 in the form $a+ib$, such that $OPQR$ is a parallelogram. **3**

(d) (i) Sketch the region in the complex plane that satisfies the following three inequalities: **3**

$$|z-2i| \leq \sqrt{2}$$

$$Im\big((1+i)z\big) < 2$$

(ii) Hence, or otherwise, find the range of $\arg(z)$. **2**

Question 3: Answer Question 3 in a new booklet. (15 Marks)

(a) For which real values of c is $z - ci$ a factor of $P(z) = z^4 - z^3 + 9z^2 - 4z + 20$? **3**

(b) Consider the non-zero complex numbers z and w. **3**

Given that $\sqrt{wz}$ is real, show that $\frac{z}{|z|} = \frac{|w|}{w}$.

(c) (i) Expand $(x-1)(1 + x + x^2 + \cdots + x^n)$ **1**

(ii) Hence, or otherwise, show that $1 + x + x^2 + \cdots + x^n$ has no multiple roots, $n \geq 0$. **3**

(d) Consider the complex number z, with a locus that describes the horizontal line:

$$x = a$$
$$a \in \mathbb{R}$$

(i) Find the locus of $w = z^2 + z$ and describe its shape in geometric terms. **3**

(ii) Describe the locus of w when $a = -\frac{1}{2}$. **2**

End of Examination

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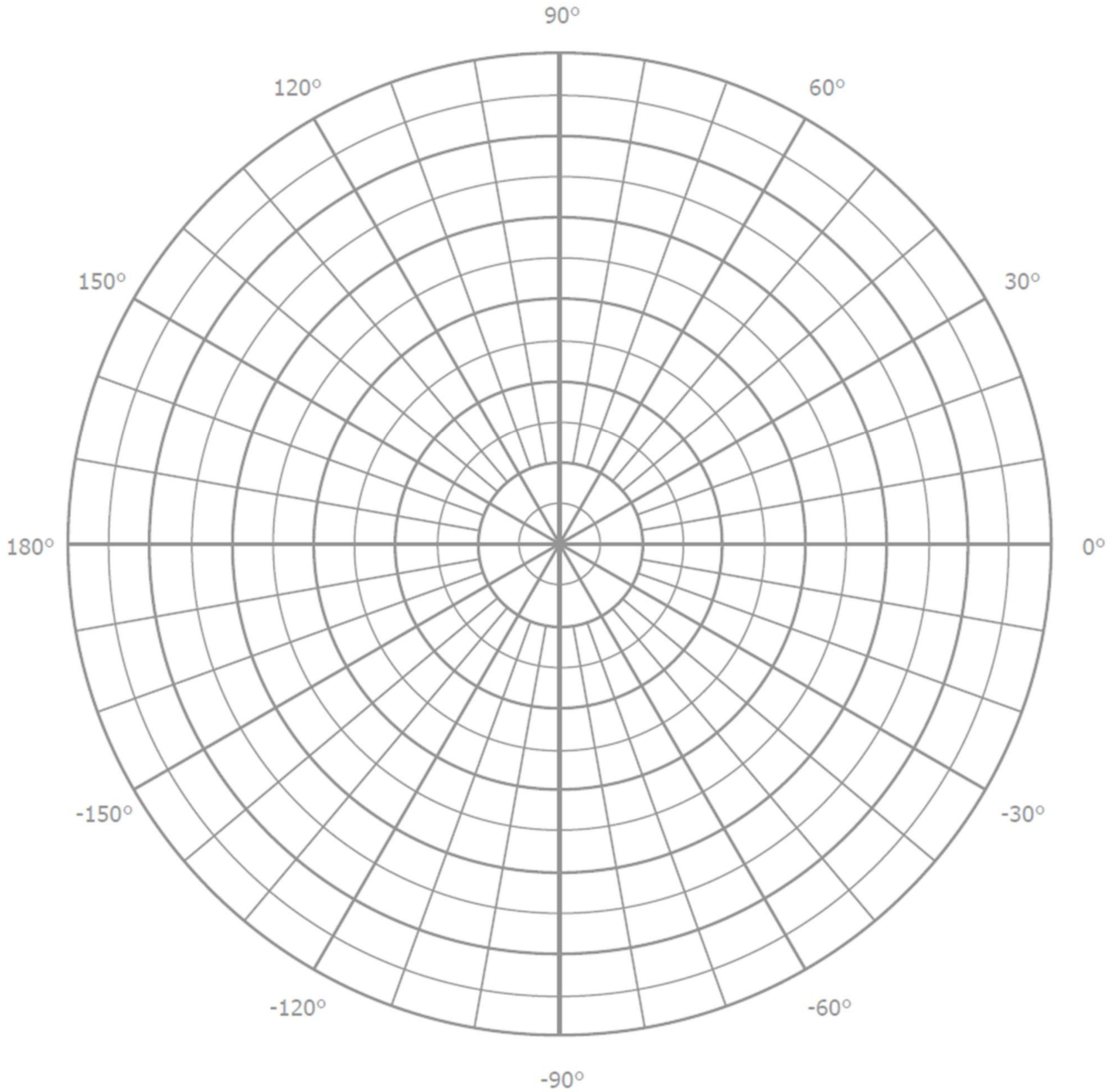
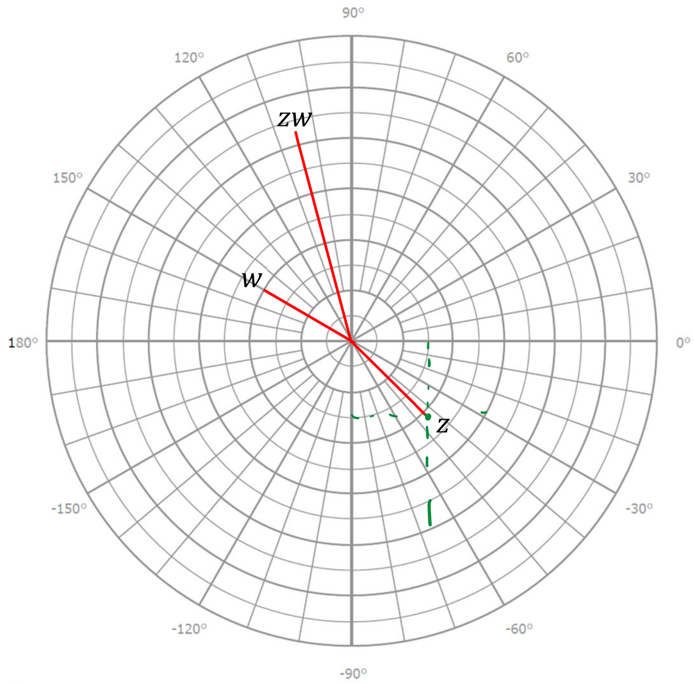
Argand plane for Question 1 Part a (vii)



2021/2022 Ext 2 Task 1 – Solutions and Feedback

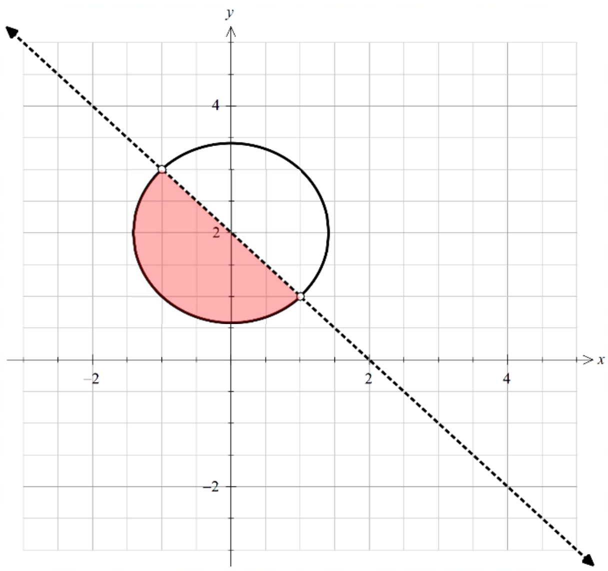
Question 1

Q	Solutions	Marking Scheme	Feedback
(ai)	$\sin(\arg(z)) < 0$ $\cos(\arg(z)) > 0$ $\therefore -\frac{\pi}{2} < \arg(z) < 0$ $\arg(z) = \tan^{-1}\frac{-3}{3}$ $= -\frac{\pi}{4}$	1 mark for solution with working	All good
(ii)	$zw = (3-3i)(-\sqrt{3}+i)$ $= -3\sqrt{3}+3+(3\sqrt{3}+3)i$	1 mark for solution with working	All good
(iii)	$\frac{z}{\overline{w}} = \frac{zw}{w\overline{w}}$ $= \frac{zw}{\|w\|^2}$ $= \frac{-3\sqrt{3}+3+(3\sqrt{3}+3)i}{(-\sqrt{3})^2+1^2}$ $= \frac{-3\sqrt{3}+3+(3\sqrt{3}+3)i}{4}$ $= \frac{-3\sqrt{3}+3}{4} + \frac{3\sqrt{3}+3}{4}i$	1 mark for multiplying by $\frac{w}{w}$ 1 mark for solution with working (a and b must be separated fractions)	Mostly well done
(iv)	$\left\|\frac{z^2}{w^4}\right\| = \frac{\|z^2\|}{\|w^4\|}$ $= \frac{\|z\|^2}{(\|w\|^2)^2}$ $= \frac{3^2+(-3)^2}{4^2}$ $= \frac{18}{16}$ $= \frac{9}{8}$	1 mark for expressing in terms of $\|z\|$ and $\|w\|$ 1 mark for solution with working	Using de Moivre's theorem was ok. Some made this harder than it needed to be.
(v)	$\|z\| = \sqrt{18}$ $= 3\sqrt{2}$ $\arg(z) = -\frac{\pi}{4}$ $z = 3\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$ $\|w\| = \sqrt{4}$ $= 2$ $\arg(w) = \tan^{-1}\frac{1}{-\sqrt{3}}\ \ (in\ QII)$ $= \frac{5\pi}{6}$ $w = 2\left(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})\right)$ $wz = 6\sqrt{2}\left(\cos(\frac{7\pi}{12}) + i\sin(\frac{7\pi}{12})\right)$	1 mark for expressing z and w in Mod-Arg form 1 mark for solution with working	Some did not show full working. Avoid Cis for calculations and results
(vi)	$\tan\left(\frac{7\pi}{12}\right) = \frac{3\sqrt{3}+3}{-3\sqrt{3}+3}$ $= \frac{\sqrt{3}+1}{-\sqrt{3}+1}$ $= -2-\sqrt{3}$	1 mark for solution with working	Mostly well done

(vii)	Note: $6\sqrt{2} \approx 8.5$ 	1 mark for any one correct 2 marks for all three correct	Poorly done. Often no use of scale.
(b)	$z + \frac{a}{z} = b$ $(3+i) + \frac{a}{3+i} = b$ $(3+i)^2 + a = b(3+i)$ $9 - 1 + a + 6i = 3b + bi$ $(8+a) + 6i = 3b + bi$ Hence, $b = 6$ equating imaginary components $8 + a = 3b$ equating real components $a = 10$ $\therefore a = 10, b = 6$	1 mark for substituting $z = 3 + i$ 1 mark for equating two complex numbers in $a + ib$ form 1 mark for equating components and finding values of a and b	Many tried to solve for z before substituting making it harder.

Question 2

Q	Solutions	Marking Scheme	Feedback
(ai)	$Let\ \sqrt{6+8i}=x+yi$ $such\ that\ 6+8i=(x+yi)^2$ $=x^2-y^2+2xyi$ $x^2-y^2=6$ (1) $xy=4$ $y=\frac{4}{x}$ (2) $x^2+y^2=\sqrt{6^2+8^2}$ $=10$ (3) (1)+(2) $2x^2=16$ $x=\pm 2\sqrt{2}$ When $x=2\sqrt{2}$ Sub in (2) $y=\frac{2}{\sqrt{2}}$ $=\sqrt{2}$ When $x=-2\sqrt{2}$ Sub in (2) $y=\frac{2}{-\sqrt{2}}$ $=-\sqrt{2}$ $\therefore \sqrt{6+8i}=\pm\left(2\sqrt{2}+\sqrt{2}i\right)$	1 mark for composing any two of the equations (1), (2) or (3) 1 mark for finding solutions for x or y 1 mark for stating both possibilities as $\pm\left(2\sqrt{2}+\sqrt{2}i\right)$	Mostly well done
(ii)	$z^2+\sqrt{2}iz+(-2-2i)=0$ $z=-\frac{\sqrt{2}i\pm\left(2\sqrt{2}+\sqrt{2}i\right)}{2}$ $=-\frac{2\sqrt{2}+2\sqrt{2}i}{2}\ or\ \frac{2\sqrt{2}}{2}$ $=-\sqrt{2}-\sqrt{2}i\ or\ \sqrt{2}$	1 mark for solutions with appropriate working	Mostly well done
(b)	Let $P(x)=4x^3+2x^2-14x+3$ By rational root theorem, possible zeroes exist for $x=\pm3,\pm1,\pm\frac{3}{2},\pm\frac{1}{2},\pm\frac{3}{4},\pm\frac{1}{4}$ By substitution, $P\left(\frac{3}{2}\right)=0$ Which gives the factor $2x-3$. $P(x)=(2x-3)(ax^2+bx+c)$ $a=2$ (coefficients of x^4) $c=-1$ (constant terms) $-3b+2c=-14$ (coefficients of x) $b=4$ $\therefore P(x)=(2x-3)(2x^2+4x-1)$ Now $2x^2+4x-1=2(x^2+2x)-1$ $=2(x^2+2x+1)-3$ $=\left(\sqrt{2}(x+1)\right)^2-\left(\sqrt{3}\right)^2$ $=(\sqrt{2}(x+1)+\sqrt{3})(\sqrt{2}(x+1)-\sqrt{3})$ $=2(x+1+\sqrt{6})(x+1-\sqrt{6})$ $P(x)=2(2x-3)\ (x+1+\sqrt{6})(x+1-\sqrt{6})$ $P(x)=4(x-\frac{3}{2})(x+1+\sqrt{6})(x+1-\sqrt{6})$	1 mark – finds rational zero 1 mark – factorises P(x) into linear and quadratic factors 1 mark – linear factorisation (monic linear factors)	Many were lost on this question, they could not find the rational root $x=\frac{3}{2}$ As well, some did not go all the way to linear factors.

(c)	For a parallelogram, $z_3 = z_1 + z_2$ or $z_3 = z_1 - z_2$ or $z_3 = z_2 - z_1$ <u>Case 1</u> $z_3 = z_1 + z_2$ $= 3 - 2i$ <u>Case 2</u> $z_3 = z_1 - z_2$ $= 1 - 8i$ <u>Case 3</u> $z_3 = z_2 - z_1$ $= -1 + 8i$	1 mark for each solution with appropriate working	Many missed at least one solution
(di)		1 mark for circle 1 mark for linear relationship 1 mark for area with excluded values	Many missed the open circles on the circle.
(ii)	$\frac{\pi}{4} < \arg(z) \leq \frac{3\pi}{4}$	1 mark for maximum 1 mark for minimum (with correct inequality)	Mostly ok when attempted.

Question 3

Q	Solutions	Marking Scheme	Feedback
(a)	$x - ci$ is a factor implies: $P(ci) = 0$ $(ci)^4 - (ci)^3 + 9(ci)^2 - 4(ci) + 20 = 0$ $c^4 + c^3 i - 9c^2 - 4ci + 20 = 0$ $(c^4 - 9c^2 + 20) + (c^3 - 4c)i = 0 + 0i$ $c^4 - 9c^2 + 20 = 0$ (1) and $c^3 - 4c = 0$ (2) From (2) $c(c^2 - 4) = 0$ $c(c+2)(c-2) = 0$ $c = 0, \pm 2$ From (1) $c^4 - 9c^2 + 20 = 0$ $(c^2 - 4)(c^2 - 5) = 0$ $c = \pm 2, \pm\sqrt{5}$ To satisfy both (1) and (2) $c = \pm 2$	1 mark for substituting ci into $P(x)$ 1 mark for equating real and imaginary components 1 mark for solutions with appropriate working	- Mostly well done - Some tried conjugate root theorem, although factor theorem was much more relevant here - Testing for integer solutions was not a valid method as it assumed an integer solution, when this was not given in the question. - Some students equated real components, and unfortunately ignored imaginary components
(b)	$\sqrt{wz} \in \mathbb{R}, w \neq 0, z \neq 0$ implies $wz \in \mathbb{R}^+$ Hence, $\arg(z) + \arg(w) = 0$ $\arg(z) = -\arg(w)$ $\arg(z) = \arg(\overline{w})$ Now $\frac{z}{\|z\|} = \cos(\arg(z)) + i\sin(\arg(z))$ $= \cos(\arg(\overline{w})) + i\sin(\arg(\overline{w}))$ $= \frac{\overline{w}}{\|\overline{w}\|}$ $= \frac{\overline{w}}{\|w\|}$ $= \frac{\overline{w}\|w\|}{\|w\|^2}$ $= \frac{\overline{w}\|w\|}{w\overline{w}}$ $= \frac{\|w\|}{w}$	1 mark for $wz \in \mathbb{R}^+$ 1 mark for $\arg(z) = \arg(\overline{w})$ 1 mark for converting between $\frac{\|w\|}{w}$ and $\frac{\overline{w}}{\|w\|}$ Or equivalent	- This was a difficult conceptual question. - Many went straight into the algebra without considering the consequences of the given condition $\sqrt{wz} \in \mathbb{R}$ - Some assumed that $w = \overline{z}$, which was not the case, as the moduli were different. However, dividing by the moduli would have led to the result (c.f. unit vectors)
(b)	**Alternatively,** Let $\alpha = \sqrt{wz}, \alpha \in \mathbb{R}$ $\alpha^2 = wz$ $= \sqrt{wz}\left(\overline{\sqrt{wz}}\right)$ $= \sqrt{wz\overline{wz}}$ $= \sqrt{w\overline{w}z\overline{z}}$ $= \sqrt{\|w\|^2\|z\|^2}$ $= \|w\|\|z\|$ $\therefore wz = \|w\|\|z\|$ Hence, $\frac{z}{\|z\|} = \frac{\|w\|}{w}$	1 mark for considering $\sqrt{wz}\left(\overline{\sqrt{wz}}\right)$ 1 mark for manipulating to $\sqrt{w\overline{w}z\overline{z}}$ 1 mark for considering $wz = \|w\|\|z\|$ and hence justifying the given statement	- Noting that $wz \in \mathbb{R}^+$ could lead to $wz = \|wz\|$, which could have led quickly to the result

(ci)	$(x-1)(1+x+x^2+\cdots+x^n)$ $=x+x^2+x^3+\cdots+x^n+x^{n+1}$ $-1-x-x^2-x^3-\cdots-x^n$ $=x^{n+1}-1$	1 mark	Mostly well done. This would be a good place to using two rows for your working of the expansion
(ii)	Roots of $1+x+x^2+\cdots+x^n$ are also roots of $x^{n+1}-1$ Let $P(x)=x^{n+1}-1$ $P'(x)=(n+1)x^n$ $P'(x)=0$ only when $x=0$ (for $n\geq 0$) But $P(0)=0-1$ $=-1$ $\therefore x=0$ is not a root of $P(x)$ and $P(x)$ has no multiple roots and thus $1+x+x^2+\cdots+x^n$ has no multiple roots.	1 mark for comparing roots of $1+x+x^2+\cdots+x^n$ and roots of $x^{n+1}-1$ 1 mark for taking derivative and letting $P(x)=0$ 1 mark for stating contradiction and logically proving result (Or equivalent method).	- Many missed the connection between part (i) and part (ii). **Explaining** this connection was required for the first mark and not many achieved this. - Students should **define** $\boldsymbol{P(x)}$. Many took the derivative without stating what $P(x)$ was. Were you using $P(x)=x^{n+1}-1$ OR $P(x)=1+x+x^2+\cdots+x^n$? - Dividing $\frac{x^{n+1}-1}{x-1}$ let to a discontinuity in the function, which changed the nature of the question. - A few students started finding roots of n, although the polynomial is in x.
(di)	Let $z=a+iy$ $w=z^2+z$ $=(a+iy)^2+(a+iy)$ $=a^2+a-y^2+(2ay+y)i$ Consider $w=X+iY$, where $X=a^2+a-y^2$ (1) and $Y=2ay+y$ (2) $y=\frac{Y}{2a+1}, a\neq -\frac{1}{2}$ from (2) Sub in (1) $X=a^2+a-\left(\frac{Y}{2a+1}\right)^2$ $=a^2+a-\frac{Y^2}{(2a+1)^2}$ $Y^2=-(2a+1)^2\big(X-(a^2+a)\big)$ This describes a concave left parabola, for $a\neq -\frac{1}{2}$. (vertex at $(a^2+a,0)$)	1 mark for establishing $z=a+iy$ 1 mark for relating X to Y 1 mark for describing shape as parabola, based on sufficient working/justification	- Many achieved the first mark by establishing $z=x+iy$ and substituting $x=a$ - Not many took the next step of observing y as a parametric - Stated the shape as a parabola without justification did not achieve the mark
(ii)	When $a=-\frac{1}{2}$ $X=-\frac{1}{4}-y^2$ from (1) $Y=0$ from (2) This describes the X-axis for $X\leq -\frac{1}{4}$.	1 mark for shape 1 mark for condition $X\leq -\frac{1}{4}$	- Disappointingly, many did not attempt this question. However, this could have been achieved relatively easily, after the straightforward algebraic step of the first mark of part (i).