Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

Q	Marks
MC	/2
3	/14
4	/15
5	/14
6	/13
Total	/58

2021

Year 12

Mathematics Extension 2

Task 2
18/3/21

General Instructions:

- Reading time – 75 minutes
- Working time – 5 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- Answer all questions in the booklets provided

Total Marks: 58

Section I – 2 marks
- Allow about 3 minutes for this section

Section II – 58 marks
- Allow about 72 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 20% of the course.

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Section I

2 marks
Allow about 3 minutes for this section

Use the multiple-choice sheet for Question 1–2

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1 Consider the statement,

"There is no integer which is the largest integer"

Which of the following statements is equivalent to this statement?

(A) $\exists y\colon\ (\forall x,\ \ y > x)$

(B) $\forall x,\ \ (\exists y\colon\ \ y > x)$

(C) $\exists x\colon\ (\forall y,\ \ y > x)$

(D) $\forall y,\ \ (\exists x\colon\ \ y > x)$

2 If z_1 and z_2 are two non $-$ zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ equals

(A) 0

(B) $-\dfrac{\pi}{2}$

€ $\dfrac{\pi}{2}$

(D) $-\pi$

End of Section I

Section II

In Questions 3 – 6, your response should include relevant mathematical reasoning and/or calculations.
Question 3: Answer Question 3 in a new booklet. (14 Marks)

(a) Let $z = 5 + i$ and $\omega = 3 + 2i$.

(i) Find $z - \overline{\omega}$ in $a + ib$ form. **2**

(ii) Find $\dfrac{10\omega}{z - 2}$ in $a + ib$ form. **2**

(iii) Find $\left|\dfrac{z}{\omega^8}\right|$ **2**

(iv) Find $\arg(z\omega)$ **2**

(v) Use the polar form of $z\omega$, and hence, find $|z\omega|^8$ **2**

(b) Consider the complex number polynomial $p(z) = z^4 - 3z^3 + 6z^2 - 12z + 8$.

(i) Given that $p(2i) = 0$, factorise p into linear and quadratic factors with real coefficients. **3**

(ii) Find all roots of p. **1**

Question 4: Answer Question 4 in a new booklet. (15 Marks)

(a) $z^2 = -3 + 4i$.

(i) Express z in component form. **3**

(ii) Hence, or otherwise, solve the quadratic equation **2**

$$z^2 - 4z + (7 - 4i) = 0$$

(b) (i) On an Argand diagram, shade the region where $|z - 1| \leq 1$ and $0 \leq \arg z \leq \frac{\pi}{6}$. **2**

(ii) Find the perimeter of the shaded region **2**

(c) Consider the factorisation

$$z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$$

(i) Show that the roots of the equation $z^6 + z^3 + 1 = 0$ are: **2**

$e^{\pm\frac{i2\pi}{9}}$, $e^{\pm\frac{i4\pi}{9}}$ and $e^{\pm\frac{i8\pi}{9}}$.

(ii) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \frac{1}{z}$ **2**

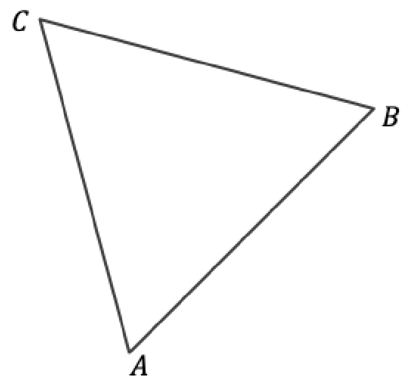
Find the cubic equation satisfied by x.

(iii) Deduce that **2**

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$$

Question 5:Answer Question 5 in a new booklet. (14 Marks)

(a) On the Argand diagram, the points A, B and C represent the complex numbers α, β and γ respectively. ΔABC is equilateral, named with its vertices taken in anti-clockwise direction.



(i) Show that **2**

$$\gamma - \alpha = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\ (\beta - \alpha).$$

(ii) Show that **2**

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha$$

(b) Consider the statement:

If n^3 is even, then n is even, $n \in \mathbb{Z}$

(i) Write the contrapositive of this statement. **1**

(ii) Let p: n^3 is even, and q: n is even **3**

Prove the contrapositive.

(iii) Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational. **4**

(c) Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. **2**

Alice says: 'We are both knaves.'

Using method of Direct proof, determine what Alice and Bob are.

Question 6: Answer Question 6 in a new booklet. (13 Marks)

(a) In the Tower of Hanoi puzzle, you have three pegs and a collection of n discs of different sizes. Initially the discs are stacked in size order on the left-hand peg.

- Discs can be moved one at a time from one peg to any other peg
- A larger disc can never rest on a smaller one.
- The third peg can be temporarily used.

The aim of the puzzle is to transfer all the discs to the second peg using the smallest possible number of moves.

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi of n discs.

(i) Prove that the number of moves needed for a Tower of Hanoi of $n+1$ discs, a_{n+1} is given by **2**

$$a_{n+1} = 2a_n + 1$$

(ii) Use the equation $a_{n+1} = 2\,a_n + 1$, **3**

to prove by mathematical induction that:

$$a_n = 2^n - 1, \forall n \in \mathbb{N}.$$

(b) (i) If $a > 0$, $b > 0$ show that $\dfrac{a+b}{2} > \sqrt{ab}$. **2**

(ii) If $a > 0$, $b > 0$ and $c > 0$, show that $a^2 + b^2 + c^2 \geq ab + bc + ca$ **1**

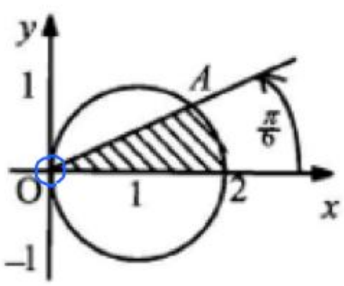
(iii) Hence, show that $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$. **2**

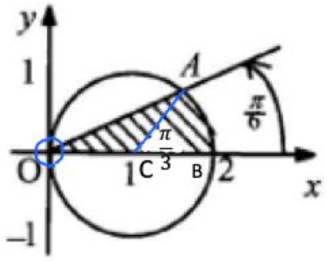
(iv) Also, prove that $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$ **3**

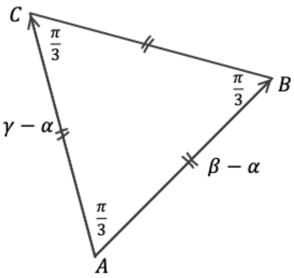
End of Examination

Year 12 Mathematics Extension 2 Marking Scheme Task 2 2021

Question		Marks	
1.	B		
2.	A Equality holds only when z_1 and z_2 are collinear, $z_2 = kz_1, k > 0$ Hence, $\arg z_1 - \arg z_2 = 0$		
3. (a) (i)	Let $z - \overline{\omega} = 5 + \mathrm{i} - (3 - 2\mathrm{i}) = 2 + 3\mathrm{i}$		
(ii)	$\frac{10\omega}{z-2} = \frac{10(3+2i)}{5+i-2}$ $= \frac{10(3+2i)}{3+i}$ $= \frac{10(3+2i)(3-i)}{10}$ $= (3+2i)(3-i)$ $= 9 - 3i + 6i + 2$ $= 11 + 3i$	2 marks: correct solution with correct working 1 mark: rationalises the denominator, or a minor error	
(iii)	$\left\|\frac{z}{\omega^8}\right\| = \frac{\|z\|}{\|\omega\|^8}$ $\|z\| = \sqrt{5^2+1^2} = \sqrt{26}$ $\|\omega\| = \sqrt{3^2+2^2} = \sqrt{13}$ $\frac{\|z\|}{\|\omega\|^8} =$ $= \frac{\sqrt{5^2+1^2}}{\left(\sqrt{3^2+2^2}\right)^8} = \frac{\sqrt{26}}{13^4}$	2 marks: correct answer from correct working 1 mark: calculates $\|z\|$ and $\|\omega\|$	
(iv)	$zw = (5+i)(3+2i)$ $= 15 + 10i + 3i - 2$ $= 13 + 13i = 13(1+i)$ $\arg(zw) = \tan^{-1} 1 = \frac{\pi}{4}$	1 mark: correctly calculates zw 1 mark: correctly calculates the argument	
(v)	$zw = \left(13, \frac{\pi}{4}\right)$ $zw = 13 \times 2\, e^{\frac{i\pi}{4}}$ $(zw)^8 = 13^8 \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^8$ $= 13^8 \times 16 \times e^{2\pi i}$ $= 13^8 \times 16$	1 mark: expresses zw in polar form 1 mark: applies De-Moivre theorem and gives the correct answer	

(b) (i)	$p(z) = z^4 - 3z^3 + 6z^2 - 12z + 8.$ Since $z = 2i$ is a root of a polynomial with real coefficients, then $z = -2i$ is also a root. Hence, $(z - 2i)(z + 2i) = z^2 + 4$ is a factor. $z^4 - 3z^3 + 6z^2 - 12z + 8 =$ $(z^2 + 4)(z^2 + az + 2)$ Comparing the coefficients of z, $-12 = 4a \quad \rightarrow \quad a = 3$ $z^2 + 3z + 2$ is a factor. $\therefore \quad p(z) = (z^2 + 4)(z^2 + 3z + 2)$ $= (z^2 + 4)(z - 2)(z - 1)$	1 mark: states that -2i is also a root with reason. 1 mark: states that $z^2 + 4$ is a factor 1 mark: correctly divides, or otherwise finds the other factor $(z^2 + 3z + 2)$ 1 mark: expresses $p(z)$ as factors with real coefficients	
(b) (ii)	$p(z) = 0$ Thus the four roots are $\pm 2i, \quad 1, \quad 2$	1 mark: finds the correct roots	
4. (a) (i)	$z^2 = -3 + 4i = (a + ib)^2$ $a^2 - b^2 = -3; \quad 2ab = 4 \quad \therefore ab = 2$ Hence, $a = 1, \quad b = 2$ $-3 + 4i = (1 + 2i)^2$ $z = \sqrt{-3 + 4i}$ $= \sqrt{(1 + 2i)^2}$ $= \pm(1 + 2i)$	3 marks: correctly solves the equation and finds the square root. 2 mark: 1 mark:	
(ii)	$z^2 - 4z + (7 - 4i) = 0$ $z = \dfrac{4 \pm \sqrt{16 - 4(7 - 4i)}}{2}$ $z = 2 \pm \sqrt{4 - (7 - 4i)}$ $= 2 \pm \sqrt{-3 + 4i} \qquad (1)$ $z = 2 \pm (1 + 2i)$ $z = 3 + 2i, \quad 1 - 2i$	1 mark: applies the quadratic formula and gives (1) 1 mark: uses result from (a), to find z.	
(b) (i)		*draws the circle with centre (1,0) and radius 1 unit ** draws the line OA clearly marking $\frac{\pi}{6}$ as the angle subtended at the origin. *** shades the inside of the circle between OX and OA **** Excludes the origin 2 marks: All aspects are correct 1 mark: at least two correct	

(ii)	 $\angle ACB = 2\angle AOB = \frac{\pi}{3}$ Hence $\angle AOC = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ OA = $\sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\frac{2\pi}{3}} = \sqrt{3}$ Arc BA = $r\theta = 1 \times \frac{\pi}{3} = \frac{\pi}{3}$ Perimeter = $2 + \sqrt{3} + \frac{\pi}{3}$ units	1 mark: Calculates the length of chord OA showing all working with reasoning 1 mark: Calculates length of arc AB and calculates perimeter	
(c) (i)	$z^9 - 1 = 0$ $z^9 = 1 + 0i = e^{i2k\pi}$, where $k = 0, \pm1, \pm2, \pm3, \pm4$ Hence, $z = e^{\frac{i2k\pi}{9}}$, where k = $0, \pm1, \pm2, \pm3, \pm4$ That is the 9 roots, staring from 1 are located on a unit circle at an angle of $\frac{2\pi}{9}$ between them. However, $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$ Using the same argument, the three cube roots of unity (solutions of $z^3 - 1 = 0$ are 1, $e^{\pm\frac{i2\pi}{3}} = e^{\pm\frac{i6\pi}{9}}, that\ is\ when\ k = \pm3$ Hence, the remaining roots of $z^9 - 1 = 0$ are roots of $z^6 + z^3 + 1 = 0$ are $z = e^{\frac{i2k\pi}{9}}$, where k = $\pm1, \pm2, \pm4$ Hence, the roots are $e^{\pm\frac{i2\pi}{9}}$, $e^{\pm\frac{i4\pi}{9}}$ and $e^{\pm\frac{i8\pi}{9}}$.	1 mark: Finds the 9 roots of $z^9 - 1 = 0$ 1 mark: explains how the solutions of $z^3 - 1 = 0$ are got, and hence gives the roots of $z^6 + z^3 + 1 = 0$	
(ii)	Dividing $z^6 + z^3 + 1 = 0$ $z^3(z^3 + 1 + \frac{1}{z^3}) = 0$ $z \neq 0, \quad hence\ z^3 + 1 + \frac{1}{z^3} = 0$ That is, $z^3 + \frac{1}{z^3} = -1 \qquad 1\ mark$ We have, $x^3 = \left(z + \frac{1}{z}\right)^3$ $x^3 = z^3 + \frac{1}{z^3} + 3z + \frac{3}{z}$ Hence, $x^3 = -1 + 3\left(z + \frac{1}{z}\right)$ $\therefore x^3 = -1 + 3x$	1 mark: divides $z^6 + z^3 + 1 = 0$ by z^3 and establishes $z^3 + \frac{1}{z^3} = -1$ (* must note z$\neq$ 0) 1 mark: reorganises the equation and finds the polynomial that satisfies x	

	$x^3 - 3x + 1 = 0$ satisfies $x = z + \frac{1}{z}$		
(iii)	Let $z_1 = e^{\frac{i2\pi}{9}}, z_2 = e^{\frac{i4\pi}{9}}, z_3 = e^{\frac{i8\pi}{9}}$ Hence $z_1 + \frac{1}{z_1} = 2\cos\frac{2\pi}{9}$, $z_2 + \frac{1}{z_2} = 2\cos\frac{4\pi}{9}$, $z_3 + \frac{1}{z_3} = 2\cos\frac{8\pi}{9}$ These are the three roots of the polynomial $x^3 - 3x + 1 = 0$. Sum of roots, $$2\left(\cos\frac{2\pi}{9} + cos\frac{4\pi}{9} + cos\frac{8\pi}{9}\right) = 0$$ Hence, $\cos\frac{2\pi}{9} + cos\frac{4\pi}{9} + cos\frac{8\pi}{9} = 0$ However, $\cos\frac{8\pi}{9} = \cos\left(\pi - \frac{\pi}{9}\right) = -\cos\frac{\pi}{9}$ $$\cos\frac{2\pi}{9} + cos\frac{4\pi}{9} - cos\frac{\pi}{9} = 0$$ $$\cos\frac{2\pi}{9} + cos\frac{4\pi}{9} = cos\frac{\pi}{9}$$	2 marks: Correct answer from correct working 1 mark: shows that $z + \frac{1}{z} = 2\cos\frac{2k\pi}{9}$ and attempts to find sum of roots	
5. (a) (i)	 ΔABC is equilateral, hence $AC = AB$ and the angle between these sides is $\frac{\pi}{3}$. $\therefore \overrightarrow{AC}$ is the rotation of $\overrightarrow{AB}$ in the anticlockwise direction by $\frac{\pi}{3}$. $\gamma - \alpha = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\ (\beta - \alpha)$.	1 mark: relates the complex numbers $\gamma - \alpha$ and $\beta - \alpha$ to the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ 1 mark: explains properties of equilateral triangle and vector relationships to deduce the result	
(ii)	Similarly, $\overrightarrow{CB}$ is the rotation of $\overrightarrow{CA}$ in the anticlockwise direction by $\frac{\pi}{3}$. $\therefore \beta - \gamma = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\ (\alpha - \gamma)$. Hence, $\frac{\gamma-\alpha}{\beta-\alpha} = \frac{\beta-\gamma}{\alpha-\gamma} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ (using result from (a) $(\gamma - \alpha)(\alpha - \gamma) = (\beta - \alpha)(\beta - \gamma)$ $-\alpha^2 - \gamma^2 + 2\gamma\alpha = \beta^2 - \alpha\beta + \beta\gamma + \gamma\alpha$ $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta\gamma + \beta\gamma + \gamma\alpha$	1 mark: writes a similar result to (a), considering a second pair of triangle sides as vectors 1 mark: equates the two results to obtain the required result	
(b) (i)	If n^3 is even, then n is even, $n \in \mathbb{Z}$ Contrapositive: *	1 mark: for either statement	

	If n is not even, then n^3 is not even. Or If n is odd, then n^3 is odd.	the statement cannot begin with 'if n^3 is even)	
(ii)	p: n^3 is even, hence, $\neg p$: n^3 is odd and q: n is even, hence, $\neg q$: n is odd ** $n\ is\ odd, n = 2k+1\ , k \in \mathbb{Z}$ Then, $n^3 = (2k+1)^3$ $= 8k^3 + 12k^2 + 6k + 1$ $= 2(4k^3 + 6k^2 + 3k) + 1$ $= 2\,M + 1, \quad M \in \mathbb{Z}.$ *** Hence, n^3 is odd, $\neg q$ is true. Ie. $\neg q \Rightarrow \neg p$ *** The contrapositive is true, hence, the Proposition $p \Rightarrow q$ is true.	1 mark: uses mathematical notations and correctly expresses n as an odd number. 1 mark: proves $n^3$is odd, by expressing all statements using correct notations. 1 mark: clearly states the equivalence, the proposition and its contrapositive.	
(iii)	Proof by contradiction: Let p: $\sqrt[3]{6}$ is irrational We need to prove that $\neg p$ is false. $\neg p$: $\sqrt[3]{6}$ is rational Assume $\neg p$ is true. * $\therefore\ \sqrt[3]{6} = \frac{p}{q}$ where $p, q \in Z$ and $p \neq kq, k \in Z$ (p and q have no common factors) $\sqrt[3]{6} = \frac{p}{q}$ $p^3 = 6q^3 \qquad (1) \qquad **$ $\Rightarrow p^3$is divisible by 2 $\Rightarrow p$ is divisible by 2 $(using\ the\ result\ (b))$ $\Rightarrow p = 2k, k \in \mathbb{Z}$ $\Rightarrow (2k)^3 = 6q^3 \quad (substituting\ in\ (1))$ $\Rightarrow 8k^3 = 6q^3$ $\Rightarrow 4k^3 = 3q^3$ $\Rightarrow q^3$ is divisible by 2 $\Rightarrow q$ is divisible by 2 $(using\ result\ (b))$ So, p and q are both divisible by 2, which contradicts the fact that p and q have no common factors.	1 mark: writes the $\neg p$: statement as assumes it is true. 1 mark: establishes $p^3 = 6q^3$ giving reasons and with correct notations 2 marks: Correctly proves the contradiction 1 mark: Minor error in the contradiction proof	

(c)	Case1: Suppose Alice is a knight. ⇒ Alice is telling the truth. ⇒ Alice and Bob are both knaves. ⇒ Alice is a knave and a knight. This is impossible. Case 2: Suppose Alice is a knave ⇒ Alice is not telling the truth. ⇒ Alice and Bob are not both knaves. ⇒ Bob is a knight.		
6. (a) (i)	Suppose there are n+1 discs on the left-hand peg. If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other n discs to the centre peg. This takes a minimum of a_n moves. It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the n discs on the centre peg to the right-hand peg. This takes a minimum of a_n moves. Hence the minimum number of moves required to transfer all the discs is $a_{n+1} = a_n + a_n + 1$ $= 2a_n + 1$	1 mark: clearly explains the process and proves the result 1 mark: some explanation is given to prove the result	
(ii)	For each natural number n, let $P(n)$ be the proposition: $a_n = 2^n - 1$ Step 1: The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true. * Step 2: Let k be any natural number, and assume $P(k)$ is true. That is, $a_k = 2^k - 1$ ** We now wish to prove that $P(k + 1)$ is true, that is, $a_{k+1} = 2^{k+1} - 1$ We have LHS of $P(k+1) = a_{k+1}$ $= 2a_k + 1$***	1 mark: verifies the result for n =1 2 mark: writes the assumption statement, proves the induction and gives the conclusion. (ALL STEPS) Award 1 mark: correct assumption, attempts to prove for n = k+1 and gives the conclusion. (*, **, ***, *****)	

	$= 2(2^k - 1) + 1$ using $P(k)$ $= 2^{k+1} - 2 + 1$ **** $= 2^{k+1} - 1$ $= RHS$ of $P(k+1)$ We have proved that if $P(k)$ is true, then $P(k+1)$ is true, $\forall n \in \mathbb{N}$ Conclusion: By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \text{N}$. Hence, we have shown that $a_n = 2^n - 1,\ \forall\, n \in N$. *****		
(b) (i)	It is clear that $(a+b)^2 = (a-b)^2 + 4ab \Rightarrow (a+b)^2 > 4ab$ (equality iff $a = b$), since $(a-b)^2 \geq 0$. Then, if $a > 0$, $b > 0$, we get $\frac{a+b}{2} > \sqrt{ab}$ (equality iff $a = b$).		
(ii)	It is clear that $a^2 + b^2 \geq 2ab,\ b^2 + c^2 \geq 2bc,$ $a^2 + c^2 \geq 2ac$. By addition $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$. Hence $a^2 + b^2 + c^2 \geq ab + bc + ca$ with equality iff $a = b = c$.		
(iii)	$a^3 + b^3 + c^3 - 3abc$ $= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$ Since $a + b + c$ is positive, and $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$, the right-hand side of identity is not negative. Hence $a^3 + b^3 + c^3 \geq 3abc$ (equality iff $a = b = c$). The substitution $a^3 \to a,\ b^3 \to b,\ c^3 \to c$ gives $a + b + c \geq 3a^{1/3}b^{1/3}c^{1/3}$. Hence $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ (equality iff $a = b = c$).		
(iv)	From (b), $ab + bc + ca \leq a^2 + b^2 + c^2$ (equality iff $a = b = c$). Exchanging in (2) a by a^2, b by b^2 and c by c^2 yields $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$. 1 mark Also, from (b), we have		

	$ab \le \frac{a^2+b^2}{2}$, $bc \le \frac{b^2+c^2}{2}$, $ca \le \frac{c^2+a^2}{2}$. 1 mark Hence, $abc(a+b+c) = a^2bc + b^2ac + c^2ab \le a^2\left(\frac{b^2+c^2}{2}\right) + b^2\left(\frac{a^2+c^2}{2}\right) + c^2\left(\frac{a^2+b^2}{2}\right) =$ $a^2b^2 + b^2c^2 + a^2c^2$ (equality iff $a=b=c$). Hence $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2 \ge abc(a+b+c)$. 1 mark		