Killara HIGH SCHOOL

MATHEMATICS DEPARTMENT

Centre Number

Student Number

Q	Marks
1	/13
2	/13
Total	/26

2022 Year 12

Mathematics Extension 2

Task 3 – Validation Test

29/6/22

General Instructions:

- Reading time – 5 minutes
- Working time – 45 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- Answer all questions in the booklets provided

Total Marks: 26

This question paper must not be removed from the examination room.

This examination constitutes 60% of the task 3 weighting. The total assessment task constitutes 15% of the course.

Question 1: Answer Question 1 in a new booklet. (13 Marks)

(a) Find the Cartesian equation of the sphere with centre $C(2, -3, 5)$ that touches the xz-plane. **2**

(b) (i) Find the direction angles, α, β and γ, that the vector $\underset{\sim}{v} = -6\underset{\sim}{i} - 4\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k}$ makes with the x, y and z axes respectively, to the nearest minute where necessary. **2**

(ii) Given that two direction angles of a vector are $\frac{2\pi}{3}$, what is the size of the third direction angle? **2**

(c) The points A, B and C are defined by the position vectors **4**

$$\underset{\sim}{a} = \begin{pmatrix} 3 \\ -2 \\ -\sqrt{5} \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \\ -\sqrt{5} \end{pmatrix} \text{ and } \underset{\sim}{c} = \begin{pmatrix} \lambda \\ -2 \\ 0 \end{pmatrix}$$

respectively, for some real constant λ.

Find the value of λ, $g$iven that the magnitude of angle ABC is $\frac{\pi}{3}$.

(d) The position vectors of two moving particles are given by: **3**

$$\underset{\sim}{p_1} = 2t^2\underset{\sim}{i} + 8\sqrt{5}t\underset{\sim}{j}$$

and

$$\underset{\sim}{p_2} = (15 - t^2)\underset{\sim}{i} + (2t^2 + 30)\underset{\sim}{j}$$

where $t \geq 0$.

Find the coordinates of the point/s where the particles collide.

Question 2: Answer Question 2 in a new booklet. (13 Marks)

(a) Consider the points $A = (0, 0, -2), B = (-1, 0, -1)$, and $C = (2, -4, 2)$.

(i) Show that the vector equation of the line passing through B and C can be given as: **1**

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

for some real constant λ.

(ii) Show that A does not lie on the line BC. **2**

(iii) Find the shortest distance between A and the line BC. **2**

Consider another point D that does not lie on BC. Furthermore, let A' and D' be the points corresponding to the reflection of A and D across the line BC respectively.

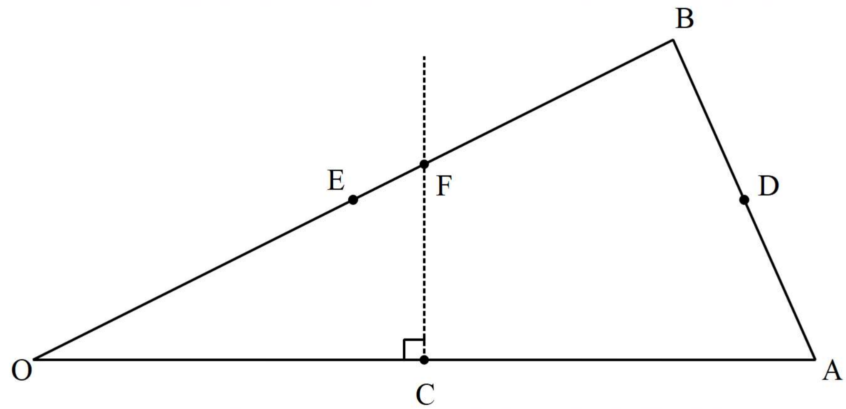
(iv) Find the coordinates of the point A'. **1**

(v) Name the two possible shapes formed by the points A, D, A' and D' and state the conditions under which they are formed. **2**

Question 2 continues on the next page

(b) Consider a circle in the 3D plane that has a centre at $P(3,-1,-2)$ with a radius of 3 that is parallel to the yz-plane. State a parametric vector equation that forms the circle. **2**

(c) In ΔOAB shown below, C, D and E are the midpoints of OA, AB and OB respectively. F lies on OB such that $\angle OCF = 90°$. CF is known as the perpendicular bisector of OA. **3**



Using vectors, show that the perpendicular bisectors of the sides of a triangle are concurrent.

End of Examination

2022 KHS Ext 2 Task 3 – VT Solutions, Marking Criteria and Marker's Feedback

Q	Worked Solutions	Marking Criteria	Marker's Feedback
1a	xz-plane occurs at $y=0$. Sphere touches xz-plane at $(2,0,5)$ and hence has a radius of 3. Equation of sphere $(x-2)^2+(y+3)^2+(z-5)^2=9$	1 mark finds radius 1 mark – gives equation with correct centre	Some confusion between xy and xz planes. Some students did a fair amount of unnecessary working for this question.
bi	$\left\|\underset{\sim}{v}\right\| = \sqrt{(-6)^2+(-4)^2+\left(2\sqrt{3}\right)^2}$ $=\sqrt{64}$ $=8$ $\underset{\sim}{\hat{v}} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$ $\cos\alpha = -\frac{3}{4}$, $\alpha = 138°35'$ | $\cos\beta = -\frac{1}{2}$, $\beta = 120°$ | $\cos\gamma = \frac{\sqrt{3}}{4}$, $\gamma = 64°20'$	1 mark for finding $\left\|\underset{\sim}{v}\right\|$ 1 mark for 3 correct angles	
ii	$\cos\frac{2\pi}{3} = -\frac{1}{2}$ $\cos^2\frac{2\pi}{3} = \frac{1}{4}$ Now as $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ Let $\alpha=\frac{2\pi}{3}$ and $\beta=\frac{2\pi}{3}$ $\cos^2\frac{2\pi}{3}+\cos^2\frac{2\pi}{3}+\cos^2\gamma=1$ $\frac{1}{4}+\frac{1}{4}+\cos^2\gamma=1$ $\cos^2\gamma=\frac{1}{2}$ $\cos\gamma=\pm\frac{1}{\sqrt{2}}$ $\gamma=\frac{\pi}{4}$ or $\frac{3\pi}{4}$	1 mark for recalling $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ 1 mark for 2 correct solutions for γ	A surprising number lost marks for not taking the $\pm$ when reversing a squared term

c	$\overrightarrow{BA} = \begin{pmatrix} 3-2 \\ -2--1 \\ -\sqrt{5}--\sqrt{5} \end{pmatrix}$ $= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} \lambda-2 \\ -2--1 \\ 0--\sqrt{5} \end{pmatrix}$ $= \begin{pmatrix} \lambda-2 \\ -1 \\ \sqrt{5} \end{pmatrix}$ $\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda-2 \\ -1 \\ \sqrt{5} \end{pmatrix}$ $= \lambda - 2 + 1$ $= \lambda - 1$ $\left\|\overrightarrow{BA}\right\| = \sqrt{1^2 + (-1)^2}$ $= \sqrt{2}$ $\left\|\overrightarrow{BC}\right\| = \sqrt{(\lambda-2)^2 + (-1)2 + \left(\sqrt{5}\right)^2}$ $= \sqrt{(\lambda-2)^2 + 6}$ $= \sqrt{\lambda^2 - 4\lambda + 10}$ $\cos\frac{\pi}{3} = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left\|\overrightarrow{BA}\right\| \left\|\overrightarrow{BC}\right\|}$ $\frac{1}{2} = \frac{\lambda - 1}{\sqrt{2}\sqrt{\lambda^2 - 4\lambda + 10}}$ $\sqrt{2}\sqrt{\lambda^2 - 4\lambda + 10} = 2(\lambda - 1)\ (**)$ $2(\lambda^2 - 4\lambda + 10) = 4(\lambda - 1)^2$ $\lambda^2 - 4\lambda + 10 = 2(\lambda^2 - 2\lambda + 1)$ $\lambda^2 - 4\lambda + 10 = 2\lambda^2 - 4\lambda + 2$ $\lambda^2 = 8$ $\lambda = \pm 2\sqrt{2}$ However, consider $(**)$: $LHS \geq 0$ (Square roots) $\therefore 2(\lambda - 1) \geq 0$ $\lambda - 1 \geq 0$ $\lambda \geq 1$ So $\lambda = -2\sqrt{2}$ is rejected as a possible solution and hence, $\lambda = 2\sqrt{2}$	1 mark for $\overrightarrow{BA}$ and $\overrightarrow{BC}$ 1 mark for substituting moduli, dot product, and angle 1 mark for solution for λ 1 mark for disregarding negative solution with reasoning	Some marks lost to fiddly algebra. This is the sort of question you want to double check. Do not just assume that negative solutions are rejected. Some rejected without reason, some rejected stating that $\lambda \geq 0$. Find the point in the working where an inequality must be formed to restrict the domain of possible solutions.

d	$$\underset{\sim}{p_1} = \begin{pmatrix} 2t^2 \\ 8\sqrt{5}t \end{pmatrix}$$ $$\underset{\sim}{p_2} = \begin{pmatrix} 15 - t^2 \\ 2t^2 + 30 \end{pmatrix}$$ For point of intersection, equation x and y components $$2t^2 = 15 - t^2 \ (1)$$ $$8\sqrt{5}t = 2t^2 + 30 \ (2)$$ From (1) $$3t^2 = 15$$ $$t^2 = 5$$ $$t = \pm\sqrt{5}$$ From (2) $$2t^2 - 8\sqrt{5}t + 30 = 0$$ $$t^2 - 4\sqrt{5}t + 15 = 0$$ $$t = \sqrt{5} \text{ or } t = 3\sqrt{5}$$ To satisfy both (1) and (2) $$t = \sqrt{5}$$ To find coordinates, substitute into $\underset{\sim}{p_1}$ or $\underset{\sim}{p_2}$ In $\underset{\sim}{p_1}$ $$x = 2 \times \left(\sqrt{5}\right)^2$$ $$= 10$$ $$y = 8\sqrt{5} \times \sqrt{5}$$ $$= 40$$ ∴Intersection at $(10, 40)$	1 mark for equating x or y components 1 mark for recognising $t = \sqrt{5}$ as the only common solution 1 mark for finding coordinates of intersection	Some students took $t = \sqrt{5}$ as the solution without checking that it was also a solution for the y components. If the question asks for coordinates, give coordinates. A lot of students lost a mark for giving the position vector of the point, not the coordinates of the point. This mark was only lost once, and students were not penalised if they made the same mistake in question 2aiv.

<table>
<tr><td>2ai</td><td>Direction vector $\overrightarrow{BC} = \begin{pmatrix} 2 - -1 \\ -4 - 0 \\ 2 - -1 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$
As C is on the line, the position vector is given as $\overrightarrow{OC} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$

Hence, the equation of the line can be given as $\underset{\sim}{r} = \overrightarrow{OC} + \lambda \overrightarrow{BC}$
$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$</td><td>1 mark for calculating direction vector and showing substitution into line equation</td><td></td></tr>
<tr><td>ii</td><td>For A to lie on the line, λ must be consistent for x, y and z components of $\underset{\sim}{r}$

$x = 0 = 2 + 3\lambda \ (1)$
$y = 0 = -4 - 4\lambda \ (2)$
$z = -2 = 2 + 3\lambda \ (3)$

Consider equations (1) and (3)
From (1)
$2 + 3\lambda = 0$
$\lambda = -\frac{2}{3}$
Sub in (3)
$RHS = 2 + 3\lambda$
$= 2 - 2$
$= 0$
$LHS = -2 \neq 0$
Hence, A does not lie on the line BC</td><td>1 mark for equating x, y or z components

1 mark for showing inconsistency with reasoning</td><td></td></tr>
</table>

iii	$\overrightarrow{BA} = \begin{pmatrix} 0 - -1 \\ 0 - 0 \\ -2 - -1 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$ $\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$ $= 3 - 3$ $= 0$ Hence, $BA \perp BC$ and minimum distance between A and the line is $\left\|\overrightarrow{BA}\right\| = \sqrt{1^2 + (-1)^2}$ $= \sqrt{2}$	1 mark for correctly applying dot product or projection 2 marks for stating minimum distance	A variety of methods were attempted here. Some were very long. Students should learn which method is best/quickest based on the question. If $\text{proj}_{\overrightarrow{CB}} \overrightarrow{CA} = \overrightarrow{CB}$, then $BA \perp CB$ Many found $\text{proj}_{\overrightarrow{CB}} \overrightarrow{CA} = \overrightarrow{CB}$ or found that $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$ without realising that this meant $BA \perp CB$. This could have saved them a bit of working. A lot of diagrams were wrong as a result (although not penalised).
iv	A' is the point A translated by the vector $2\overrightarrow{AB} = 2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ $A = (0, 0, -2)$ $A' = (-2, 0, 0)$	1 mark for the coordinates of A'	Students who clearly labelled their vectors in earlier parts completed this with very minimal working. Careful not to confuse coordinates of points with position vectors.
v	Generally speaking, the points form a tetrahedron in 3D space. A trapezium is formed if A, D, A' and D' are in the same plane (coplanar), i.e., A, B, C and D are coplanar.	1 mark for either of the shapes 2 marks given if both are stated with conditions distinguishing them	Many wrong diagrams showing the midpoint of D and D’ was the midpoint of A and A’. Barely anybody considered that this was a 3D problem. Answers of rectangle and square were not considered, as these are forms of trapezium, and the most general shape formed is foremost the tetrahedron.

b	$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$ $$Radius = 3$$ Consider the following perpendicular unit vectors in the yz-plane: $\underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ Hence, equation of the circle can be given as: $$\underset{\sim}{r} = \overrightarrow{OP} + 3\underset{\sim}{j}\cos\theta + 3\underset{\sim}{k}\sin\theta$$ $$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\sin\theta$$	2 marks for correct equation 1 mark off for each error	Some tried to delete the third component and make this a 2D circle in 2D space. This resulted in an odd combination of 2D and 3D vectors in the same question. Some mixed up the axes and defaulted to $\underset{\sim}{i}$ and $\underset{\sim}{j}$. The correct solution was given in a variety of valid forms.
c	Let X be the intersection point of the perpendicular bisectors from OA and OB. $$\overrightarrow{OA} \cdot \overrightarrow{CX} = 0$$ $$\underset{\sim}{a} \cdot \left(\underset{\sim}{x} - \underset{\sim}{c}\right) = 0$$ $$\underset{\sim}{a} \cdot \underset{\sim}{x} - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$$ $$\underset{\sim}{a} \cdot \underset{\sim}{x} = \underset{\sim}{a} \cdot \underset{\sim}{c}$$ $$\underset{\sim}{a} \cdot \underset{\sim}{x} = \underset{\sim}{a} \cdot \frac{1}{2}\underset{\sim}{a}$$ $$\underset{\sim}{a} \cdot \underset{\sim}{x} = \frac{1}{2}\left\lvert\underset{\sim}{a}\right\rvert^2$$ Similarly, $$\underset{\sim}{b} \cdot \underset{\sim}{x} = \frac{1}{2}\left\lvert\underset{\sim}{b}\right\rvert^2$$ Now consider the line segment XD $$\overrightarrow{XD} \cdot \overrightarrow{AB} = (\underset{\sim}{d} - \underset{\sim}{x}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$$ $$= \underset{\sim}{d} \cdot \underset{\sim}{b} - \underset{\sim}{d} \cdot \underset{\sim}{a} - \underset{\sim}{x} \cdot \underset{\sim}{b} + \underset{\sim}{x} \cdot \underset{\sim}{a}$$ $$= \underset{\sim}{d} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) - \left(\underset{\sim}{x} \cdot \underset{\sim}{b} - \underset{\sim}{x} \cdot \underset{\sim}{a}\right)$$ $$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) - \left(\frac{1}{2}\left\lvert\underset{\sim}{b}\right\rvert^2 - \frac{1}{2}\left\lvert\underset{\sim}{a}\right\rvert^2\right)$$ $$= \frac{1}{2}\left(\left\lvert\underset{\sim}{b}\right\rvert^2 - \left\lvert\underset{\sim}{a}\right\rvert^2\right) - \frac{1}{2}\left(\left\lvert\underset{\sim}{b}\right\rvert^2 - \left\lvert\underset{\sim}{a}\right\rvert^2\right)$$ $$= 0$$ Hence, $XD \perp AB$, and thus, the perpendicular bisectors intersect at point X and are concurrent.	1 mark for applying perpendicularity and dot product 1 mark for applying midpoint 1 mark for giving sufficient logic to prove concurrency	The central idea to the question was perpendicular bisectors. This implied that two characteristics would be useful in the proof: 1- Bisection i.e. midpoint -> vectors divided in two 2- Perpendicularity i.e. dot product=0 Most students used the first characteristic, but many did not attempt to use the second. A number of students achieved partial marks at the beginning of their proof by introducing clearly defined points/lines and stating a logical RTP statement. The RTP statement was an underused strategy that some students should practice.