Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Number: ____________________

2024 Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 65 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show mathematical reasoning and/or calculations

Total Marks: 42

Section I – 3 marks (Pages 1–2)

- **Attempt Questions 1–3**
- **Allow about 5 minutes for this section**

Section II – 39 marks (Pages 3–5)

- **Attempt Questions 4–6**
- **Allow about 60 minutes for this section**

Question:	1	2	3	4	5	6	Total
Marks:	1	1	1	14	14	11	42
Awarded:							

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Section I

3 marks

Attempt Questions 1–3

Allow about 5 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–3.

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1. Let $z = 3 - i$. What is the value of $\overline{iz}$ **1**

   A. $-1 - 3i$

   B. $-1 + 3i$

   C. $1 - 3i$

   D. $1 + 3i$

2. Consider the following statement: **1**

   If a person has eaten the devil fruit, then they gain amazing powers and the sea turns against them.

   The contrapositive of this statement is:

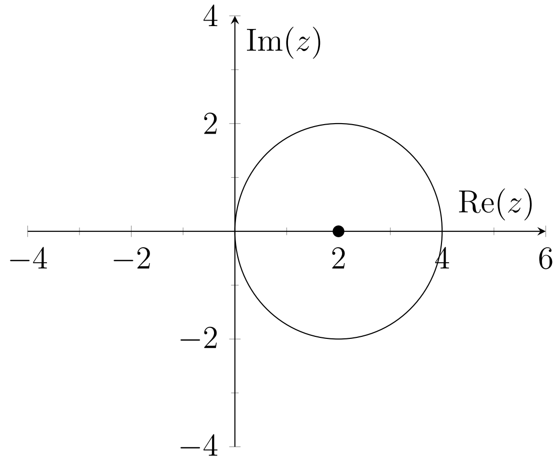
   A. If a person has gained amazing powers or the sea turns against them, then they have eaten the devil fruit.

   B. If a person has not eaten the devil fruit, then they will not gain amazing powers and the sea won't turn against them.

   C. If a person has not gained amazing powers or eaten the devil fruit, then the sea has not turned against them.

   D. If a person has not gained amazing powers or the sea has not turned against them, then they have not eaten the devil fruit.

3. Which of the following is the equation of the circle shown below? **1**



A. $(z+2)(\bar{z}+2)=4$

B. $(z-2)(\bar{z}-2)=4$

C. $(z+2i)(\bar{z}-2i)=4$

D. $(z+2)(\bar{z}-2)=4$

Section II

39 marks
Attempt Questions 4–6
Allow about 60 minutes for this section.
Answer each question in the appropriate writing booklet.
All necessary working should be shown in every question.

Question 4 (14 Marks) Use a SEPARATE writing booklet.

(a) Let $z_1 = 3 - 4i$ and $z_2 = -3 + 2i$. Find:

(i) Find $\operatorname{Arg} z_1$. **1**

(ii) $z_1 - \overline{z_2}$ **1**

(iii) $\dfrac{z_1}{z_2}$ **2**

(b) Evaluate $i + 2i^2 + 3i^3 + \ldots + 16i^{16}$. **2**

(c) Express $(1 - i)^5$ in the form $x + iy$. **2**

(d) Find all complex numbers z such that $|z - 1| = |z + 3| = |z - i|$. **2**

(e) Given $(1 + 2i)$ is a root of the equation $x^3 - 4x^2 + 9x - 10 = 0$, find the other two roots. **2**

(f) Prove that every square number has a remainder 0 or 1 when divided by 4. **2**

End of Question 4

Question 5 (14 Marks) Use a SEPARATE writing booklet.

(a) (i) Factorise $z^5 + 1$ over the real field. **1**

(ii) List the roots of $z^5 + 1 = 0$ in $r(\cos\theta + i\sin\theta)$ form. **1**

(iii) Deduce that $2\cos\dfrac{\pi}{5} + 2\cos\dfrac{3\pi}{5} - 1 = 0$. **2**

(b) Find the two complex numbers z that satisfy: $z\bar{z} = 37$ and $\dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$. **3**

(c) (i) On **one** Argand diagram, shade the region satisfying the following inequalities, labelling each clearly. **5**

$$|z + 1 - 3i| \leqslant 2$$

$$\frac{2\pi}{3} \leqslant \arg(z - 2i) \leqslant \frac{3\pi}{4}$$

$$\text{and } |z| \geqslant |z + 2|$$

(ii) Express z, satisfying the above inequalities, in the form $a + ib$ when $\operatorname{Re}(z)$ takes its lowest value. **2**

End of Question 5

Question 6 (11 Marks) Use a SEPARATE writing booklet.

(a) (i) Find the modulus and argument of each of the complex numbers z and w, where $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$. **2**

(ii) Plot the points representing z, w and $z+w$ on an Argand diagram. Deduce from the diagram that $\tan\dfrac{3\pi}{8} = \sqrt{2}+1$. **3**

(b) Let z be a complex number. Find the minimum value of **3**

$$|z-2| + |z-(1+2i)\,| + |z-(-1+i)\,| + |z-(-1-i)\,|$$

(c) For how many positive integers n less than or equal to 1000 is **3**

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

END OF PAPER

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- **Attempt Questions 1–3**
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Section II – 39 marks (Pages 4–14)

- **Attempt Questions 4–6**
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Section I

3 marks

Attempt Questions 1–3

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    A. $-1 - 3i$

    B. $-1 + 3i$

    C. $1 - 3i$

    D. $1 + 3i$

Solution:

C

2. Consider the following statement: **1**

    If a person has eaten the devil fruit, then they gain amazing powers and the sea turns against them.

    The contrapositive of this statement is:

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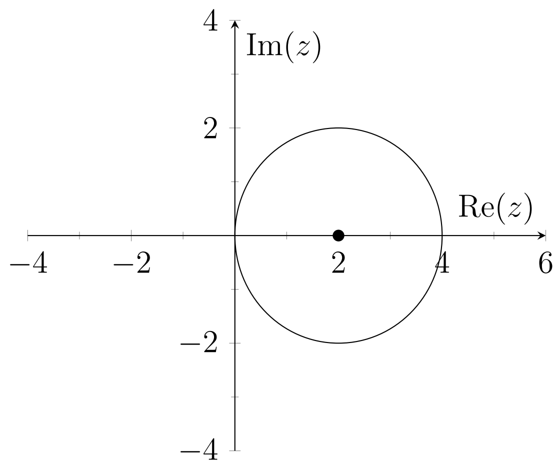
D. If a person has not gained amazing powers or the sea has not turned against them, then they have not eaten the devil fruit.

Solution:

D

3. Which of the following is the equation of the circle shown below? **1**



A. $(z+2)(\bar{z}+2)=4$

B. $(z-2)(\bar{z}-2)=4$

C. $(z+2i)(\bar{z}-2i)=4$

D. $(z+2)(\bar{z}-2)=4$

Solution:

B

Section II

39 marks
Attempt Questions 4–6
Allow about 60 minutes for this section.
Answer each question in the appropriate writing booklet.
All necessary working should be shown in every question.

Question 4 (14 Marks) Use a SEPARATE writing booklet.

(a) Let $z_1 = 3 - 4i$ and $z_2 = -3 + 2i$. Find:

(i) Find $\operatorname{Arg} z_1$. **1**

Solution:

$$\operatorname{Arg}(z_1) = -\tan^{-1}\left(\frac{4}{3}\right)$$

(ii) $z_1 - \overline{z_2}$ **1**

Solution:

$$\begin{aligned} z_1 - \overline{z_2} &= 3 + 4i - (-3 - 2i) \\ &= 6 - 2i \end{aligned}$$

(iii) $\dfrac{z_1}{z_2}$ **2**

Solution:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3 - 4i}{-3 + 2i} \times \frac{-3 - 2i}{-3 - 2i} \\ &= \frac{-16 + 6i}{13} \end{aligned}$$

(b) Evaluate $i + 2i^2 + 3i^3 + \ldots + 16i^{16}$. **2**

Solution:

$$i + 2i^2 + 3i^3 + 4i^4 = 2 - 2i \qquad \textbf{taking the first 4 terms}$$

$$5i^5 + 6i^6 + 7i^7 + 8i^8 = 2 - 2i \qquad \textbf{taking the second 4 terms}$$

This pattern repeats, and so we get $4 \times (2 - 2i) = 8 - 8i$ **as our solution.**

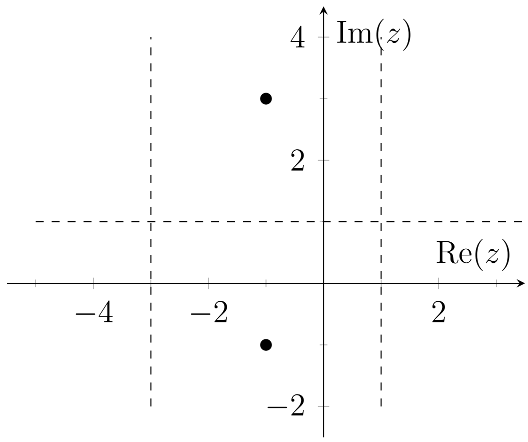
(c) Express $(1 - i)^5$ in the form $x + iy$. **2**

Solution:

$$\begin{aligned} (1-i)^5 &= \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^5 && \textbf{convert to euler form} \\ &= 4\sqrt{2}e^{-i\frac{5\pi}{4}} && \textbf{apply power} \\ &= 4\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) && \textbf{return to } a + ib \textbf{ form} \\ &= -4 + 4i \end{aligned}$$

(d) Find all complex numbers z such that $|z-1| = |z+3| = |z-i|$. **2**

Solution:



The fastest approach to this problem is to sketch out the lines and observe solutions must lie on the line $\mathrm{Re}(z) = -1$**, with distance 2 from the two vertical lines. Therefore the two solutions will be 2 above and 2 below the line** $\mathrm{Im}(z) = 1$**, or** $z = -1 - i$ **and** $z = -1 + 3i$**.**

(e) Given $(1+2i)$ is a root of the equation $x^3 - 4x^2 + 9x - 10 = 0$, find the other two roots. **2**

Solution:

As the polynomial has real coefficients we know $1 - 2i$ **is also a root by the conjugate root theorem.**

Taking the sum of the roots, we get:

$$\begin{aligned} 1 + 2i + 1 - 2i + \alpha &= 4 \\ \alpha &= 2 \end{aligned}$$

$\therefore$ **the polynomial has roots** $1 + 2i$, $1 - 2i$ **and 2.**

(f) Prove that every square number has a remainder 0 or 1 when divided by 4. **2**

Solution:

For an even number $n = 2k$ **where** $n, k \in \mathbb{N}$ **we find:**

$$n^2 = (2k)^2$$
$$= 4k^2$$

which has remainder 0 when divided by 4.

In the case where n **is odd, that is** $n = 2k + 1$ **where** $n, k \in \mathbb{N}$ **we have:**

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 4\left(k^2 + k\right) + 1$$

Which has remainder 1 when divided by 4.

Hence every square number has either remainder 0 or 1.

QED

End of Question 4

Question 5 (14 Marks) Use a SEPARATE writing booklet.

(a) (i) Factorise $z^5 + 1$ over the real field. **1**

(ii) List the roots of $z^5 + 1 = 0$ in $r(\cos\theta + i\sin\theta)$ form. **1**

(iii) Deduce that $2\cos\dfrac{\pi}{5} + 2\cos\dfrac{3\pi}{5} - 1 = 0$. **2**

Solution:

(i) $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

(ii)

let $z = re^{i\theta}$ **so we get** $z^5 = -1$

$$r^5 e^{i5\theta} = e^{i\pi} \qquad \textbf{and so } r = 1$$

$$5\theta = \pi + 2k\pi = (2k+1)\pi$$

$$\theta = \frac{\pi}{5} + \frac{2\pi}{5}k \qquad [-\pi, \pi]$$

$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, -\frac{3\pi}{5}, -\frac{\pi}{5}$$

Giving roots $e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, e^{i\pi}, e^{-i\frac{\pi}{5}}, e^{-i\frac{3\pi}{5}}$

(iii) **Given** $z^5 + 1 = 0$ **we find the sum of the roots to be 0. Equally spaced as they are around the unit circle, the imaginary component will cancel, leaving the real components only, or:**

$$\cos\frac{\pi}{5} + \cos\left(-\frac{\pi}{5}\right) + \cos\frac{3\pi}{5} + \cos\left(-\frac{3\pi}{5}\right) + (-1) = 0$$

$$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$$

as required.

(b) Find the two complex numbers z that satisfy: $z\bar{z} = 37$ and $\dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$. **3**

Solution:

$$\frac{z}{\bar{z}} = \frac{z^2}{z\bar{z}} = \frac{z^2}{37} = \frac{35}{37} + \frac{12}{37}i$$
$$z^2 = 35 + 12i$$

let $z^2 = (a + ib)^2$ **where** $a, b \in \mathbb{R}$**, so we get:**

$$a^2 - b^2 = 35$$
$$ab = 6$$

so $a = 6$ **and** $b = 1$**, giving** $a + ib = \pm(6 + i)$

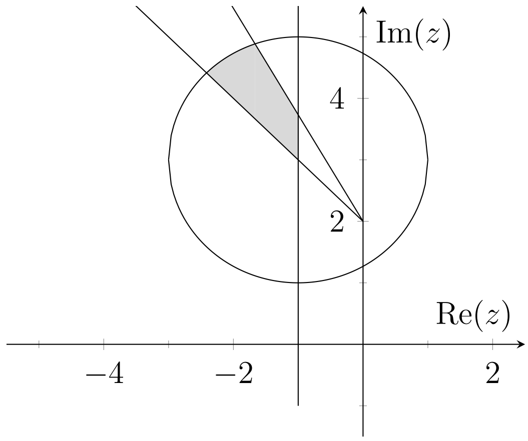
(c) (i) On **one** Argand diagram, shade the region satisfying the following inequalities, labelling each clearly. **5**

$$|z+1-3i| \leqslant 2$$

$$\frac{2\pi}{3} \leqslant \arg(z-2i) \leqslant \frac{3\pi}{4}$$

$$\text{and } |z| \geqslant |z+2|$$

Solution:



(ii) Express z, satisfying the above inequalities, in the form $a+ib$ when $\mathrm{Re}(z)$ takes its lowest value. **2**

Solution:

$$(x+1)^2 + (y-3)^2 = 2^2 \qquad \textbf{and } x = -y$$

$$(-y+1)^2 + y^2 - 6y + 9 = 2^2 \qquad \textbf{substituting}$$

$$2y^2 - 8y + 5 = 0$$

$$y = \frac{4 \pm \sqrt{6}}{2} \qquad \textbf{by completing the square}$$

And so we get $z = -\frac{4+\sqrt{6}}{2} + \frac{4+\sqrt{6}}{2}i$.

End of Question 5

Question 6 (11 Marks) Use a SEPARATE writing booklet.

(a) (i) Find the modulus and argument of each of the complex numbers z and w, where $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$. **2**

(ii) Plot the points representing z, w and $z+w$ on an Argand diagram. Deduce from the diagram that $\tan \dfrac{3\pi}{8} = \sqrt{2}+1$. **3**

Solution:

(i)

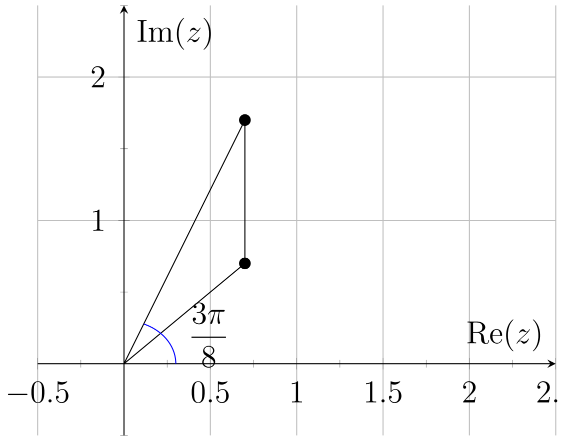
$$\begin{aligned} z &= \frac{1+i}{1-i} \\ &= \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\sqrt{2}e^{-i\frac{\pi}{4}}} \\ &= e^{i\frac{\pi}{2}} \end{aligned}$$

so $|z| = 1$ **and** $\operatorname{Arg}(z) = \dfrac{\pi}{2}$

$$\begin{aligned} w &= \frac{\sqrt{2}}{\sqrt{2}e^{-i\frac{\pi}{4}}} \\ &= e^{i\frac{\pi}{4}} \end{aligned}$$

so $|w| = 1$ **and** $\operatorname{Arg}(w) = \dfrac{\pi}{4}$

(ii)

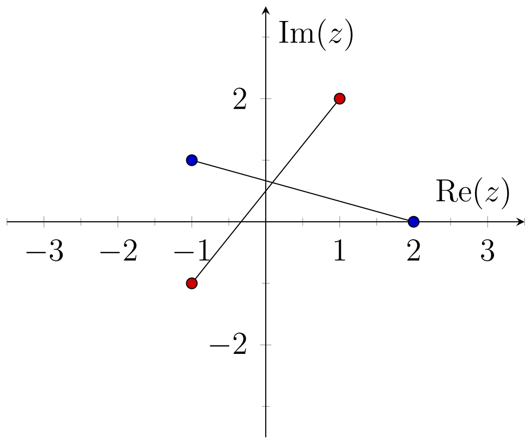


We find $\operatorname{Arg}(z+w) = \dfrac{3\pi}{8}$ **and so** $\tan \dfrac{3\pi}{8} = \dfrac{\frac{\sqrt{2}}{2}+1}{\frac{\sqrt{2}}{2}} = 1+\sqrt{2}$ **as required.**

(b) Let z be a complex number. Find the minimum value of **3**

$$|z-2|+|z-(1+2i)\,|+|z-(-1+i)\,|+|z-(-1-i)\,|$$

Solution:



The shortest distance between any two points is a straight line. A point minimising the distance from two other points will therefore lie on the line between them. Four points and two lines between them provide the diagonals of a quadrilateral and the position of the complex number z that minimises these lengths will be the intersection of these diagonals.

Therefore the minimum value is given by the sum of the diagonals of the matching quadrilateral.

$$\begin{aligned} |1+2i-(-1-i)|+|-1+i-2| &= |2+3i|+|-3+i| \\ &= \sqrt{13}+\sqrt{10} \end{aligned}$$

This can also be found with coordinate geometry or pythagoras.

(c) For how many positive integers n less than or equal to 1000 is **3**

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

Solution:

$$\begin{aligned}
LHS &= (\sin t + i\cos t)^n \\
&= \left(\cos\left(\frac{\pi}{2} - t\right) + i\sin\left(\frac{\pi}{2} - t\right)\right)^n \\
&= e^{i\left(\frac{\pi}{2}-t\right)n} \\
&= \left(\frac{e^{i\frac{\pi}{2}}}{e^{it}}\right)^n \\
&= \frac{i^n}{e^{int}}
\end{aligned}$$

and

$$\begin{aligned}
RHS &= \sin nt + i\cos nt \\
&= \frac{i}{e^{int}}
\end{aligned}$$

$RHS = LHS$ **when** $i = i^n$**, which happens when** n **is divisible by 4, or a quarter of the time; 250 times out of the integers up to 1000.**

END OF PAPER