Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

Q	Marks
1	/14
2	/16
3	/12
Total	/42

2023

Year 12

Mathematics Extension 2

Task 3
Validation Task
26/03/2023

General Instructions:

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks:
42

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

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Question 1 (14 marks)

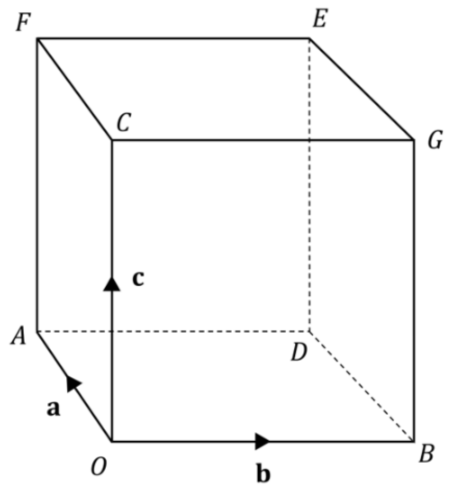
(a) In the triangle ABC, $\overrightarrow{AB} = -2i + 3j - 4k$ and $\overrightarrow{AC} = -5i - 4j - 7k$. **3**

Show that $\angle BAC = 81.9°$ to one decimal place.

(b) Can the numbers $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ be direction cosines of a line? Give relevant working. **2**

(c) The coordinates of points A and B are $(-2, 5, -7)$ and $(-7, k, 3)$ respectively. Given that the distance from A to B is $5\sqrt{14}$, find possible values of k. **2**

(d) The diagram below shows a cube whose vertices are O, A, B, C, D, E, F and G.



a, b and **c** are vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively. A has coordinates $(5, 0, 0)$.

(i) Express vector $\overrightarrow{OF}$ in component form. **2**

(ii) Explain why $\overrightarrow{FE} = \mathbf{b}$. **1**

(iii) Hence, find the exact distance OE. **2**

(e) Find the vector perpendicular to both lines given below: **2**

$$x+1=\frac{y+1}{-3}=\frac{z+1}{2}$$
$$\frac{x-3}{2}=\frac{y-5}{3}=z-7$$

End of Question 1

Question 2 (16 marks)

(a) Relative to the fixed origin O, the following vector equations are given.

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 1 \\ 13 \\ 1 \end{pmatrix}$$

(i) Find the vector equation for the straight line l_1 which passes through A and B. **1**

(ii) The straight line l_2 has vector equation: **2**

$$\underset{\sim}{r} = \begin{pmatrix} 7 \\ 0 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Where μ is a scalar parameter.

Show that l_1 and l_2 do not intersect.

(iii) Find the position vector of C, given that it lies on l_2 and $\angle ABC = 90°$. **2**

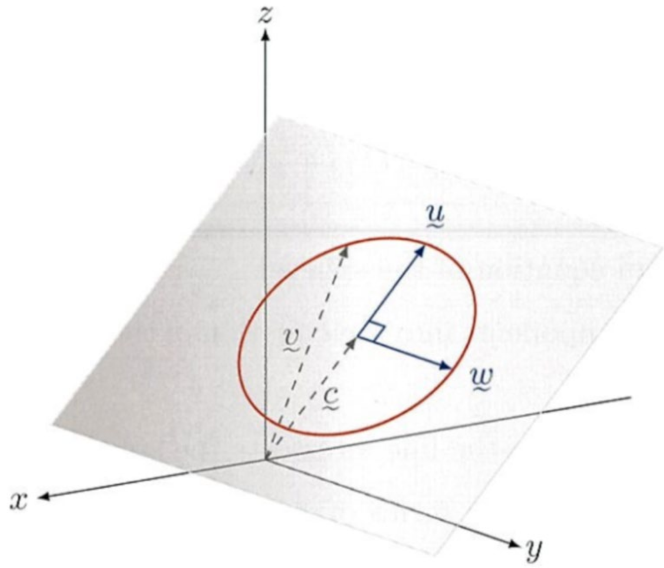
(b) The straight line L has vector equation, **4**

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in [-2, 5].$$

The point A has position vector $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

By finding the reflection of A on the line L, calculate the area of the kite thus formed.

(c) Consider a circle with radius r on some plane P containing two perpendicular vectors $\vec{u}$ and $\vec{w}$ both with length r.



Let the centre of the circle have position vector $\vec{c}$ and let $\vec{v}$ represent the position vector of any point on the circle.

(i) Write the standard parametrisation for the circle centred at the origin with radius r in the xy plane. **1**

(ii) Use your previous answer to explain how **2**

$$\vec{v} = \vec{c} + \vec{u}\cos\theta + \vec{w}\sin\theta$$

represents a circle in 3D space.

(iii) Find the radius of the circle and verify that it is r. **1**

(d) Find the shortest distance between the lines: **3**

$$l_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad l_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

End of Question 2

Question 3 (12 marks)

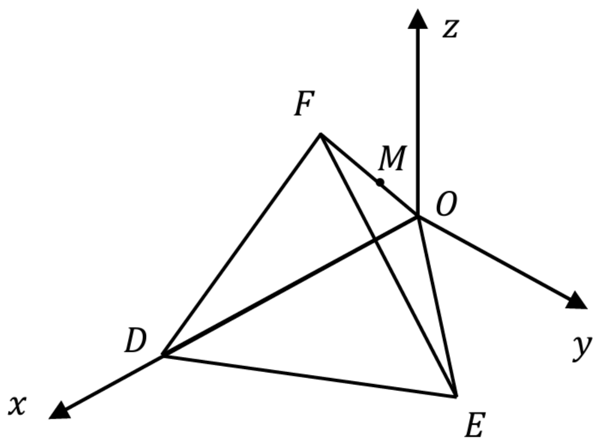
(a) Astronomers have traced the path of two asteroids travelling through space. At a particular time t, the position of asteroids Ceres and Pallas can be represented as shown below:

$$Ceres\text{:}\ \ (-1+t, 4-t, -1+2t)$$
$$Pallas\text{:}\ \ (-7+2t, -6+2t, -1+t)$$

(i) Do the paths of the two asteroids cross? If so, where? **4**

(ii) Following these paths, will the asteroids collide? If so, where? **3**
If not, find the closest distance between them.

(b) The faces of tetrahedron $ODEF$ are comprised of equilateral triangles of side length 1 unit. Its base lies flat on the xy plane with vertices at O, $D(1,0,0)$ and $E\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ as shown. Prove the coordinates of M, the midpoint of FO, is $\left(\frac{1}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{6}}{6}\right)$. **3**



(c) Sketch the curve given by the equations: **2**

$$x = t^2, \qquad y = \frac{1}{t^2}$$

End of Examination

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Section I – Multiple Choice Answer Sheet

Use this multiple-choice answer sheet for questions 1 – 2. Detach this sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9
A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start Here → **1.** A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

2023 Task 3

Question 3(c) solution

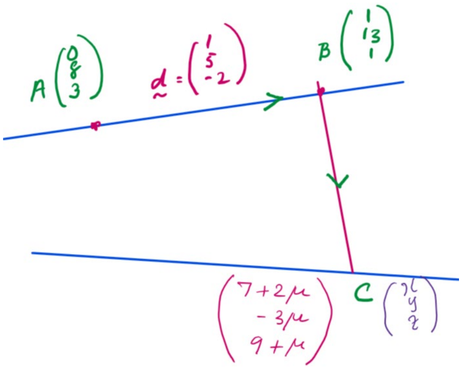
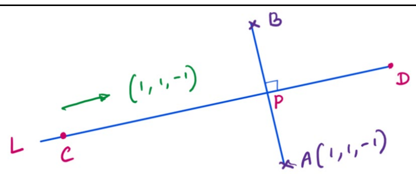
(d) (i)

3a	$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2} \neq 1$ Hence, these numbers do not represent direction cosines of a line.		
3b	$AB = -2i + 3j - 4k$ and $\overrightarrow{AC} = -5i - 4j - 7k$ $\overrightarrow{BC} = \begin{pmatrix}-5\\-4\\-7\end{pmatrix} - \begin{pmatrix}-2\\3\\-1\end{pmatrix} = \begin{pmatrix}-3\\-7\\-6\end{pmatrix}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{2^2+3^2+4^2} = \sqrt{14}$ $\lvert\overrightarrow{AC}\rvert = \sqrt{5^2+4^2+7^2} = \sqrt{90}$ $\lvert\overrightarrow{BC}\rvert = \sqrt{3^2+7^2+6^2} = \sqrt{94}$ $\cos A = \frac{90+14-94}{2\sqrt{90}\sqrt{94}}$ $\angle BAC = \cos^{-1}\left(\frac{90+14-94}{2\sqrt{90}\sqrt{94}}\right)$ $= 81.9°\ \ 1\text{ d.p.}$	1 mark: determines $\overrightarrow{BC}$ 1 mark: calculates the lengths of AB, BC, CA 1 mark: calculates the angle	
3 (c)	DISTANCE = MAGNITUDE (PYTHAGORAS) $(-7--2)^2 + (k-5)^2 + (3--7)^2 = (5\sqrt{14})^2$ $(-5)^2 + (k-5)^2 + 10^2 = 350$ $(k-5)^2 = 225$ $k-5 = \sqrt{225} = \pm 15$ $k = \pm 15 + 5$ $\boxed{k=20 \quad k=-10}$		
3d (i)	$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \mathbf{a} + \mathbf{c}$ as $\overrightarrow{AF} = \overrightarrow{OC} = \mathbf{c}$ Thus $\overrightarrow{OF} = \begin{pmatrix}5\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\5\end{pmatrix} = 5\boldsymbol{i} + 5\boldsymbol{k}$	1 mark: correct calculations with **a** and **c** and correct reasoning 1 mark: correct vectors in terms of 5	

		and expresses in component form.	
3d (ii)	$\overrightarrow{FE}$ and $\overrightarrow{OB}$ are parallel and equal as the sides of the cube Hence, $\overrightarrow{FE} = \overrightarrow{OB} = \boldsymbol{b}$	. 1 mark: correct reason	
3d (iii)	$\overrightarrow{OE} = \overrightarrow{OF} + \overrightarrow{FE} = \begin{pmatrix}5\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\5\end{pmatrix} + \begin{pmatrix}0\\5\\0\end{pmatrix}$ $= \begin{pmatrix}5\\5\\5\end{pmatrix} = \sqrt{5^2+5^2+5^2} = 5\sqrt{5}$	1 mark: gives the position vector of F 1 mark: correct answer with working	
3e	Let (a, b, c) be the perpendicular vector Then $(1, -3, 2) . (a, b, c) = 0$ Also, $(2, 3, 1) . (a, b, c) = 0$ $a - 3b + 2c = 0$ $2a + 3b + c = 0$ Adding, $3a + 3c = 0$, then $c = -a$ Also, $b = -\frac{a}{3}$ $\begin{pmatrix}a\\-\frac{a}{3}\\-a\end{pmatrix} = \frac{a}{3}\begin{pmatrix}3\\-1\\-3\end{pmatrix}$ Thus, the vector perpendicular to l_1 and l_2 is $(3, -1, -3)$	1 mark: sets up dot product of direction vectors and the arbitrary vector (a, b, c) correctly 1 mark: correct evaluation and then the correct answer	

Question 4

4a (i)	$\overrightarrow{AB} = \begin{pmatrix}1\\13\\1\end{pmatrix} - \begin{pmatrix}0\\8\\3\end{pmatrix} = \begin{pmatrix}1\\5\\-2\end{pmatrix}$ $\underset{\sim}{r_1} = \begin{pmatrix}0\\8\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}\lambda\\8+5\lambda\\3-2\lambda\end{pmatrix}$	1 mark: Gives the correct equation of the straight line	
4a (ii)	$\underset{\sim}{r_2} = \begin{pmatrix}7\\0\\9\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}7+2\mu\\-3\mu\\9+\mu\end{pmatrix}$ Checking $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ for intersection $\begin{pmatrix}\lambda\\8+5\lambda\\3-2\lambda\end{pmatrix} = \begin{pmatrix}7+2\mu\\-3\mu\\9+\mu\end{pmatrix}$ $\lambda = 7 + 2\mu$ $5\lambda + 8 = -3\mu$		

	$5(7+2\mu)+8 = -3\mu$ $35+10\mu+8 = -3\mu$ $13\mu = -43$ $\mu = -\frac{43}{13}$ Then, $\lambda = \frac{5}{13}$ Check: $3-2\left(\frac{5}{13}\right) = -\frac{43}{13}+9$ $\frac{29}{13} \neq \frac{74}{13}$ ∴the lines do not intersect.	1 marks: Equates the I, j, k components and solves for λ and μ 1 mark: Tests the solution and gives the correct judgement	
4a (iii)	 $\overrightarrow{AB}.\overrightarrow{BC} = 0$ $= \begin{pmatrix}1\\5\\-2\end{pmatrix}.\begin{pmatrix}x-1\\y-13\\z-1\end{pmatrix} = 0$ $x-1+5y-65-2z+2 = 0$ $x+5y-2z+64 = 0$ Sub. x, y, z $7+2\mu-15\mu-18-2\mu+64 = 0$ $-15\mu = 75$ $\mu = -5$ Then $C(-3, 15, 4)$	2 marks: correct answer from correct working 1 mark: Sets up $\overrightarrow{AB}.\overrightarrow{BC} = 0$ and some minor error in working (must use the direction vector)	
4b	 $\boldsymbol{\lambda} = -2, \quad C\begin{pmatrix}1-2\\-2-2\\5+2\end{pmatrix} = \begin{pmatrix}-1\\-4\\7\end{pmatrix}$ $\boldsymbol{\lambda} = 5, \quad D\begin{pmatrix}1+5\\-2+5\\5-5\end{pmatrix} = \begin{pmatrix}6\\3\\0\end{pmatrix}$	1 mark: determines the end points C and D and calculates length. 2 marks: finds the projection of CA onto L, or otherwise,	

	CD $= \sqrt{(6+1)^2 + (3+4)^2 + (7)^2} = 7\sqrt{3}$ Let $P(x, y, z)$ Project $\overrightarrow{CA}$ onto line L $\overrightarrow{CA} = \begin{pmatrix} 1 - -1 \\ 1 - -4 \\ -1 - 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -8 \end{pmatrix}$ $Proj_L\, \overrightarrow{CA} = \dfrac{\begin{pmatrix} 2 \\ 5 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $= \dfrac{2+5+8}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (5, 5, -5)$ Thus, the position vector of P is $(-1, -4, 7) + (5, 5, -5) = (4, 1, 2)$ $\underset{A}{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \begin{matrix} \xrightarrow{+3} \\ \xrightarrow{+0} \\ \xrightarrow{+3} \end{matrix} \underset{P}{\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}} \begin{matrix} \xrightarrow{+3} \\ \xrightarrow{+0} \\ \xrightarrow{+3} \end{matrix} \underset{B}{\begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}}$ $AB = \sqrt{(7-1)^2 + 0 + (5+1)^2}$ $= 6\sqrt{2}$ $\text{Area of the kite } = \dfrac{1}{2} \times 7\sqrt{3} \times 6\sqrt{2}$ $= 21\sqrt{6}\ \text{sq. units}$	determines the foot of the perpendicular P. 1 mark: Uses a valid method to find P with some error 1 mark: finds the reflected point B 1 mark: calculates the area.	

Question 5

5a (i)	$Ceres\text{:}\ \ (-1+t, 4-t, -1+2t)$ $Pallas\text{:}\ (-7+2t, -6+2t, -1+t)$ Consider the lines: $(-1+\lambda, 4-\lambda, -1+2\lambda)$ $(-7+2\mu, -6+2\mu, -1+\mu)$ For some values it $t = \lambda$ and μ Then, the equations of lines can be written as $(-1, 4, -1) + \lambda(1, -2, 2)$ and $(-7, -6, -1) = \mu(2, 2, 1)$ The direction vectors are not parallel as $(1, -2, 2) \neq k(2, 2, 1),\ \ k \in \mathbb{R}$	1 mark: tests for parallel	

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Also, $\underset{\sim}{d_1}.\underset{\sim}{d_2} =$
$1 \times 2 + \ -1 \times 2 + 2 \times 1 = 2 \neq 0$
Hence the lines are not perpendicular to each other.

Checking for point of intersection:
$$\begin{pmatrix} -1+\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix} = \begin{pmatrix} -7+2\mu \\ -6+2\mu \\ -1+\mu \end{pmatrix}$$
$$2\mu - \lambda = 6$$
$$2\mu + \lambda = 10$$
$$4\mu = 14 \rightarrow \quad \mu = 4$$
Then, $\lambda = 2\mu - 6 = 2$
Test: $-1 + 2\lambda = \ -1 + \mu$
$$-1 + 2(2) = \ -1 + 4$$
$$3 = 3 \ \ True$$
Hence, the paths do cross each other.

Point of intersection
$$\begin{pmatrix} -1+2 \\ 4-2 \\ -1+2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
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1 mark: tests for perpendicular

1 mark: proves the lines do intersect

1 mark: finds the point of intersection
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<td>5a (ii)</td>
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Using the result from (i), the asteroids reach the point (1,2,3) at $t = 2$ and $t = 3$ respectively. That is, they do not collide with each other.

At time t seconds, let C be the position vector of Ceras and P be that of Pallas.
Then the vector
$$\overrightarrow{CP} = \begin{pmatrix} -1+t \\ 4-t \\ -1+2t \end{pmatrix} - \begin{pmatrix} -7+2t \\ -6+2t \\ -1+t \end{pmatrix}$$
$$= \begin{pmatrix} -6+t \\ -10+3t \\ -t \end{pmatrix}$$
$$\left|\overrightarrow{CP}\right|^2 = \ (t-6)^2 + (3t-10)^2 + t^2$$
$$= t^2 - 12t + 36 + 9t^2 - 60t + 100 + t^2$$
Let $f(t) = 11t^2 - 72t + 136$

Minimum value of $f(t)$ when
$$t = \frac{72}{2 \times 11} = \frac{36}{11}$$
Substituting, the minimum distance is 4.26 units.
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