Killara High School

MATHEMATICS DEPARTMENT

Student Number

2024/2025 **Year 12 Examination**

Mathematics Extension 2

Task 1

Week 10 Wednesday 17/12/2024

Q	Marks
MC	/4
5	/13
6	/15
7	/8
Total	/40

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 40

Section I - 4 marks

- Allow about 5 minutes for this section

Section II - 36 marks

- Allow about 55 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

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Section I

3 marks
Attempt Questions 1–3.
Allow about 5 minutes for this section.

Use the multiple-choice sheet for Question 1–3.

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1 If $z = 4 + 2i$ then $\bar{z}$ is equal to

A. $4 - 2i$

B. $-4 - 2i$

C. $-4 + 2i$

D. $2 + 4i$

2 If $z = 7 + 5i$ then $\dfrac{1}{z-2}$ is equal to:

A. $\dfrac{1-i}{10}$

B. $\dfrac{1+i}{10}$

C. $\dfrac{1-i}{25}$

D. $\dfrac{1+i}{25}$

3 If $z = \sqrt{3} + 3i$ then z^{63} is:

A. Real and negative

B. Equal to a negative real multiple of i

C. Real and positive

D. Equal to a positive real multiple of i

4 Which of the following, where $z \in \mathbb{C}$ does not represent a circle

A. $z\,\overline{z} = 4$

B. $|z+1| + |z-1| = 3$

C. $(z - 3 + i)(\overline{z} - 3 - i) = 9$

D. $|z-3| = 2$

End of Section I

Section II

33 marks
Attempt Questions 4-7.
Allow about 50 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (13 marks) Answer each question in a NEW booklet.

(a) Factorise the following over the complex numbers **1**

$$x^2 + 2x + 5$$

(b) Given $z = -3 + \sqrt{3}i$ and $w = -1 + i$.

Find the following in the form $a + bi$

(i) $2\bar{z} + w$ **1**

(ii) $\dfrac{z}{w}$ **2**

Using z and w from above express the following in mod arg form

(iii) $\dfrac{z}{w}$ **1**

(iv) $z\overline{w}$ **2**

(v) w^3 **2**

(c) Let $g(x)$ be a polynomial with real coefficients such that $g(2 + i) = g(3 + i) = 0$ and the degree of $g(x)$ is less than or equal to 4. Find all roots of $g(x)$. **1**

(d) Express z in cartesian form given that $z^2 = -20 + 21i$ **3**

End of Question 5

Question 6 (15 marks) Answer each question in a NEW booklet.

(a) If the six solutions of $z^6 = -64$ are written in the form $a + bi$, where a and b are real, find the product of the roots whose real part, a, is positive. **3**

(b) (i) Sketch all possible values of z such that **3**

$$\arg\left(\frac{z-5+i}{z+1+i}\right) = \frac{\pi}{4}$$

(ii) With reference to the diagram find the maximum value of $|z|$ **3**

(c) If $z = 1 + i$, what is the lowest value of n, $n \in \mathbb{Z}^+$, such that z^n is a real number. **2**

(d) (i) Show that $z^n + z^{-n} = 2\cos n\theta$ **1**

(ii) Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ **3**

End of Question 6

Question 7 (8 marks) Answer each question in a NEW booklet.

(a) Let u and v be complex numbers such that $|u| = |v| = 1$, and $u \neq v$. Show that the following is real **2**

$$\frac{4uv}{(u+v)^2}$$

(b) Suppose that $2 + i$ and $8 - 3i$ are opposite vertices of a regular hexagon in the complex plane. What are the complex numbers that correspond to the other four vertices? **3**

(c) The angles α, β and γ satisfy $e^{i\alpha} + e^{i\beta} + e^{i\gamma} = 0$. **3**
Show that $e^{3i\alpha} = e^{3i\beta} = e^{3i\gamma}$.

End of Question 7

End of paper

2024 Year 12 Mathematics Extension 1 Task 1Marking Scheme

Section I: Multiple Choice Questions (4 marks)

Question	Solution	Marking Criteria	Feedback
1	Answer: A	1 mark– correct answer	
2	Answer: A $$\frac{1}{z-2}=\frac{1}{5+5i}$$ $$=\frac{5-5i}{2\times 25}$$ $$=\frac{1-i}{10}$$	1 mark– correct answer	
3	Answer: A	1 mark– correct answer	
4.	Answer:B A and C are both complex numbers multiplied by their conjugates, giving modulus squared is equal to RHS, and modulus equals positive real number is a circle		

Section II: Question 5 (8 marks)

4a	$x^2+2x+5=$ $x^2+2x+1-(-4)=(x+1+2i)(x+1-2i)$	1 -mark correct answer from correct working	We need to factorise this expression and we do not need find the zeros.
B(i)	$z\,\bar{z}+w=2\left(-3-\sqrt{3}i\right)-1+i$ $=2\left(-3-\sqrt{3}i\right)-1+i$ $=-7+\left(2\sqrt{3}-1\right)i$	1- Mark correct mark from correct working	
(ii)	$\frac{z}{w}=\frac{-3+\sqrt{3}i}{-1+i}\times\frac{1+i}{1+i}$ $=\frac{-3+\sqrt{3}i-3i-\sqrt{3}}{-(1+1)}$ $=\frac{\left(3+\sqrt{3}\right)}{2}+\frac{3-\sqrt{3}}{2}i$	2 marks – correct answer from correct working 1 mark – small error	
Iii	$\frac{z}{w}=\sqrt{6}(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right))$	1 mark correct answer (must not be expressed in cis form)	
(iv)	$z=\left(2\sqrt{3}\right)\left(\cos\frac{5\pi}{6}+\sin\frac{5\pi}{6}\right)$ $\bar{w}=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$ $z\bar{w}=2\sqrt{6}\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]$		
(v)	$w^3=\left(\sqrt{2}\left(\cos\frac{9\pi}{4}+\sin\frac{9\pi}{4}\right)\right)^3$		
c)	$g(2+i)=g(3+i)=0$ Given the coefficients are real the roots must be conjugates. Hence the 4 roots are $2+i, 2-i, 3+i, 3-i$	1 mark must mention the conjugate root theorem	
d)	$z^2=-20+21i$ Let $z=x+iy$ $x^2+y^2=\sqrt{20^2+21^2}=29$ $x^2-y^2=-20$ Note $2xy=21$ Therefore $x=\pm\frac{3\sqrt{2}}{2}, y=\pm\frac{7\sqrt{2}}{2}$ $z=\frac{3\sqrt{2}}{2}+\frac{7\sqrt{2}}{2}i$ or $z=-\frac{3\sqrt{2}}{2}-\frac{7\sqrt{2}}{2}i$	1 mark for setting up the simultaneous equations 2 marks for finding at least one pair of x and y values and above. 3 marks for correct answer from correct working	

Section II: Question 6 (8 marks)

6(a)	$$z = re^{i\theta}$$ $$r^6e^{6i\theta} = 64e^{i(\pi+2k\pi)}, k \in z$$ $r = 2$ and $\theta = \frac{\pi+2k\pi}{6}$ $\theta = \pm\frac{\pi}{6}\ or\ \pm\frac{3\pi}{6}\ or\ \pm\frac{5\pi}{6}$ The only two roots with positive a positive real part are $2e^{\frac{\pi}{6}i}$ and $2e^{-\frac{\pi}{6}i}$ Therefore $2e^{\frac{\pi}{6}i} \times 2e^{-\frac{\pi}{6}i} = 4$	1 mark for finding the 6 roots 2 marks for computing the product of 6 roots and above. 3 marks for computing the product of the 2 roots with positive real parts.	
6(b) (i)	Due to symmetry, the x coordinate of the centre is 2. The angle at the centre is $\frac{\pi}{2}$, which is $2 \times \frac{\pi}{4}$. Hence the radius of the circle is $3\sqrt{2}$. Move(3,-1) up by 3 units and we will get the coordinates of the centre $(2, 2)$.	1 mark for sketching a major arc passing through (-1, -1) and (5,-1) 2marks for above and label the angles. 3 marks for above and two open ends.	

6(b) (ii)	The centre (2, 2) and (-1, -1) are on the line y=x. Connect the two points and extend this line segment until it meets circle in the first quadrant. This will be the diameter and it passes through the origin. From the discussion above, we know that the radius is $3\sqrt{2}$. The maximum distance of z from the origin is $3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$ units The maximum value of $\lvert z\rvert=5\sqrt{2}$	1 mark for calculating the centre and the radius 2 marks for recognising (-1, -1), (0, 0) and (2, 2) are collinear or equivalent and above 3 marks for calculate the maximum value and above.	
6©	$$z=1+i=\sqrt{2}e^{\frac{\pi}{4}i}$$ $$z^n=4e^{\frac{n\pi}{4}i}$$ It is real a number when n is a multiple of 4. Therefore the smallest positive integer for n is 4.	1 mark for converting $z=1+i$ to the exponential form. 2 marks for the correct answer and above.	
6(d) (i)	Given that $\lvert z\rvert=1$ $z^n+z^{-n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta=2\cos n\theta$	1 mark for correct working	
6(d)(ii)	To prove $\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$ Use $(z+\frac{1}{z})^4=(2\cos\theta)^4=z^4+z^{-4}+4z^2+4z^{-2}+6$ $$16(\cos\theta)^4=2\cos 4\theta+8\cos 2\theta+6$$ Hence $$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$	1 mark for $(z+\frac{1}{z})^4=(2\cos\theta)^4$ and expansion 2 marks for $z^4+z^{-4}+4z^2+4z^{-2}+6$ $=2\cos 4\theta+8\cos 2\theta+6$ and above. 3 marks for correct working.	

Section II: Question 6 (8 marks)

6a			
B			

Section II: Question 7 (9 marks)

6a	To show $z = \frac{4uv}{(u+v)^2}$ is real, show that $z = \bar{z}$ $\bar{z} = \overline{\left(\frac{4uv}{(v+u)^2}\right)}$ $= \frac{4\overline{uv}}{\overline{(u+v)^2}}$ $= \frac{4\bar{u} \times \bar{v}}{(\bar{u}+\bar{v})^2}$ Because the magnitudes are equal to 1, $\bar{u} = \frac{1}{u}$ and $\bar{v} = \frac{1}{v}$ $= \frac{4 \times \frac{1}{u} \times \frac{1}{v}}{\left(\frac{1}{u}+\frac{1}{v}\right)^2}$ Multiplying by u^2v^2 $= \frac{4uv}{(u+v)^2}$		
B	$e^{\frac{\pi i}{3}}[(2+i)-(5-i)]+(5-i)]$ $= \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(-3+2i)+(5-i)$ $= \frac{7-2\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}i$ $e^{\frac{2\pi i}{3}}[(2+i)-(5-i)]+(5-i)]$ $= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(-3+2i)+(5-i)$ $= \frac{13-2\sqrt{3}}{2} - \frac{(4+3\sqrt{3})}{2}i$ $e^{\frac{\pi i}{3}}[(8-3i)-(5-i)]+(5-i)]$		

	$e^{\frac{2\pi i}{3}}[(8-3i)-(5-i)]+(5-i)]$ $=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(3-2i)+(5-i)$ $=\frac{7+2\sqrt{3}}{2}+\frac{2\sqrt{3}}{2}i$		
c)	Give $e^{i\alpha}$, $e^{i\beta}$ and $e^{i\gamma}$ are the roots of a polynomial, the polynomial must be a cubic, I,e, $p(z)=az^3+bz^2+cz+d$, where a, b, c and d are complex numbers. Given $e^{i\alpha}+\ e^{i\beta}+e^{i\gamma}=\frac{b}{a}=o$, we know $b=0$ Taking the conjugate of the equation, we get $e^{-i\alpha}+\ e^{-i\beta}\ +e^{-i\gamma}=0$ Multiplying by $e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$ we get $e^{i\beta}e^{i\gamma}+e^{i\alpha}\ e^{i\gamma}+e^{i\alpha}\ e^{i\beta}=0=\frac{c}{a}$, so we know c=0 Therefore $p(z)=az^3$+$e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$ $P\left(e^{i\alpha}\right)=ae^{i3\alpha}\ +e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}=0$ $ae^{i3\alpha}\ =-e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$ Similarly $P\left(e^{i\beta}\right)=ae^{i3\beta}\ +e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}=0$ $ae^{i3\beta}\ =-e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$ $P\left(e^{i\gamma}\right)=ae^{i3\gamma}\ +e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}=0$ $ae^{i3\gamma}\ =-e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$ Therefore $ae^{i3\alpha}\ =ae^{i3\beta}\ =ae^{i3\gamma}\ =-e^{i\alpha}\ \ e^{i\beta}e^{i\gamma}$		