12MATX2A
Mr B Bridges

12MATX2B
Mr C Huynh

Name ....

THE KING'S SCHOOL

2020

Year 12
Higher School Certificate Course
Assessment Task 2 (12AT2)
March 2020

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time – 1.5 hours
- Write using **black pen**
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 50

Section I 5 marks

- Answer Questions 1-5 on the multiple choice answer sheet provided
- Allow about 10 minutes for this section

Section II 45 marks

- Answer Questions 6 -8 in the spaces provided
- Allow about 80 minutes for this section

Examiners' Use Only			**Setter/Checker: GJD/BDB**
Question	**Complex Numbers**	**Proof**	**Totals**
Section I			
1-5	3,4, 5 / 3	1, 2 / 2	**/ 5**
Section II			
6	/15		**/15**
7	a,b,c / 8	d,e / 7	**/15**
8		/15	**/15**
Total	**/26**	**/24**	**/50**

Name ....

Section I

5 marks
Attempt Questions 1 - 5
Allow about 10 minutes for this section

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1. Consider the statements:

   P: n is a positive integer.
   Q: n is an even number greater than 0.

   Which of the following is false?

   (A) $Q \Rightarrow P$

   (B) $P \Leftarrow Q$

   (C) P is implied by Q

   (D) P implies Q

2. Suppose that both x and y are odd. Which of the following statements is false?

   (A) $x + y$ is even

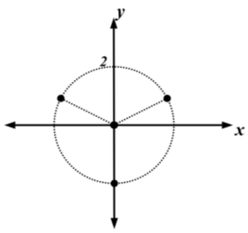
   (B) $x - y$ is even

   (C) $3x + 5y$ is even

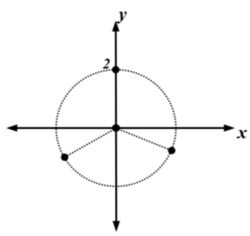
   (D) xy is odd

3. Which complex plane best represents the cube roots of $8i$?

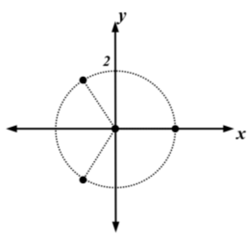
(A)



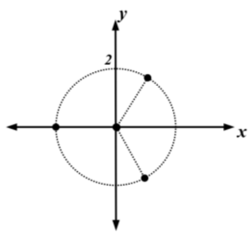
(B)



(C)



(D)



4. The complex number z satisfies $|z-2|=2$. What is the greatest distance that z can be from the point i in the complex plane?

(A) 1

(B) 2

(C) $\sqrt{5}+2$

(D) $2\sqrt{2}$

5. For a certain complex number z where $\operatorname{Arg}(z)=\frac{\pi}{5}$
$\operatorname{Arg}(z^7)$ is:

(A) $\frac{-7\pi}{5}$

(B) $\frac{-3\pi}{5}$

(C) $\frac{2\pi}{5}$

(D) $\frac{3\pi}{5}$

Name ....

Section II
45 marks
Attempt Questions 6-8
Allow about 80 minutes for this section
All questions are of equal value

Question 6 (15 marks) START A NEW WRITING BOOKLET **Marks**

(a) Prove that $2 + i$ is a root of $z^3 - 3z^2 + z + 5 = 0$, and hence find the other roots. **3**

(b) (i) Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$, in the form $a + ib$ where a and b are real. **1**

(ii) By expressing $\sqrt{3} + i$ and $\sqrt{3} - i$ in polar form, write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in polar form. **2**

(iii) Hence, find $sin\frac{\pi}{3}$ in surd form. **1**

(c) (i) Find the square roots of $-6 + 8i$. **2**

(ii) Hence solve the equation $z^2 - \sqrt{2}iz + 1 - 2i = 0$. **2**

(d) Find the eighth roots of 1 and show these roots in the complex plane. **4**

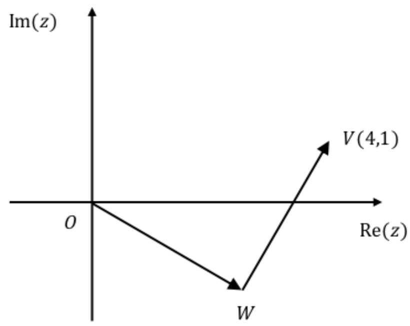
End of Question 6

Question 7 (15 marks) START A NEW WRITING BOOKLET **Marks**

(a) The point V represents the complex number $4 + i$. **3**

$\angle OWV = \frac{\pi}{2}$ and $|VW| = |OW|$.

Find the complex number represented by W.



(b) Prove that if ω is a non-real cube root of unity,

then $(1 + \omega - \omega^2)^7 = -128\omega^2$. **2**

(c) Identify subsets of the complex plane determined by the relation: **3**

$$|z - 2i| \geq 4|z + 3|$$

(d) Prove by Mathematical Induction that $4^{n+1} + 6^n$ is divisible by 10 when n is even. **3**

(e) Given $a_1 = 2$ and $a_n = \frac{a_{n-1}}{n}$, for $n \geq 2$,
prove, using Mathematical Induction, that $a_n = \frac{2}{n!}$ for $n \geq 1$. **4**

End of Question 7

Name ....

Question 8 (15 marks) START A NEW WRITING BOOKLET **Marks**

(a) Find the negation of the following:

$\forall$ real numbers x, $x < 2m+1$ for integral values of m . **1**

(b) Prove by contrapositive that if $\frac{mn}{2}$ is integral then m or n must be even. **3**

(c) Prove $\sqrt{2} + \sqrt{3} < 10$ by contradiction. **3**

(d) Prove that a number is divisible by 4 if and only if the last two digits are a multiple of 4. Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $b, c \leq 9$.

Do NOT use Proof by Induction. **4**

(e) Prove

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2 + z^4} + \frac{1}{xz}\sqrt{z^2y^2 + x^4} \text{ for } x, y, z > 0.$$ **4**

End of assessment

Name ____________________

THE KING'S SCHOOL

2020	Year 12 Higher School Certificate Course Assessment Task 2 (12AT2) March 2020

Mathematics Extension 2 Solutions

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Examiners' Use Only			Setter/Checker: GJD/BJB
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1-5	3,4, 5 / 3	1, 2 / 2	/5
Section II			
6	/15		**/15**
7	a,b,c /8	d,e /7	**/15**
8		/15	**/15**
Total	**/26**	**/24**	**/50**

Name ....

Section I

5 marks
Attempt Questions 1 - 5
Allow about 10 minutes for this section

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1. Consider the statements:

P: n is a positive integer.
Q: n is an even number greater than 0.

Which of the following is false?

(A) $Q \Rightarrow P$ True Since a number must be an integer if it is even

(B) $P \Leftarrow Q$ True Since all even numbers are integers

(C) P is implied by Q True Since all even numbers are integers

(D) P implies Q False Since if a number is an integer it might not be even

2. Suppose that both x and y are odd. Which of the following statements is false?

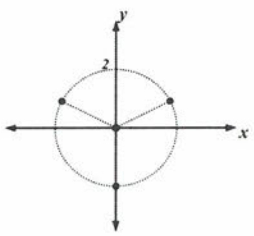
(A) $x + y$ is even

(B) $x - y$ is even

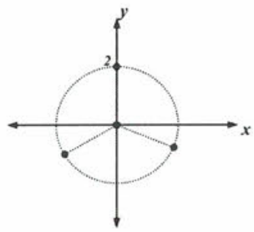
(C) $3x + 5y$ is even

(D) xy is odd

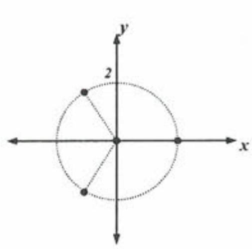
3. Which complex plane best represents the cube roots of $8i$?



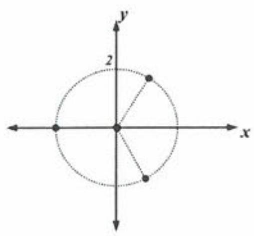
(B)



(C)



(D)



4. The complex number z satisfies $|z-2|=2$. What is the greatest distance that z can be from the point i in the complex plane?

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5. For a certain complex number z where $\text{Arg}(z)=\frac{\pi}{5}$
$\text{Arg}(z^7)$ is:

(A) $\frac{-7\pi}{5}$

(B) $\frac{-3\pi}{5}$

(C) $\frac{2\pi}{5}$

(D) $\frac{3\pi}{5}$

Name ....

Section 2
45 marks
Attempt Questions 6-8
Allow about 80 minutes for this section
All questions are of equal value

Question 6 (15 marks) START A NEW WRITING BOOKLET **Marks**

(a) Prove that $2 + i$ is a root of $z^3 - 3z^2 + z + 5 = 0$, and hence find the other roots.

$(2+i)^3 - 3(2+i)^2 + (2+i) + 5$

$= 2^3 + 3(2)^2 i + 3(2)(i)^2 + i^3 - 3(2^2 + 4i + i^2) + (2+i) + 5$

$= 8 + 12i - 6 - i - 12 - 12i + 3 + 2 + i + 5$

$= 0$

$\therefore 2 + i$ is a root.

$\therefore 2 - i$ is a root.

$$\therefore 2 + i + 2 - i + \gamma = -\frac{b}{a}$$

$$= -\frac{-3}{1}$$

$$= 3$$

$\therefore 4 + \gamma = 3$

$\therefore \gamma = -1$

$\therefore$ The roots of $z^3 - 3z^2 + z + 5 = 0$ are $-1, 2 \pm i$.

6(b) (i) Write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$, in the form $a + ib$ where a and b are real.

$$\frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$$
$$= \frac{3 + 2\sqrt{3}i + 1}{3 + 1}$$
$$= \frac{4 + 2\sqrt{3}i}{4}$$
$$= 1 + \frac{\sqrt{3}}{2}i$$

(ii) By expressing $\sqrt{3} + i$ and $\sqrt{3} - i$ in polar form, write $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in polar form.

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}$$
$$= \frac{2}{2}\left(\cos\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6} - -\frac{\pi}{6}\right)\right)$$
$$= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

(iii) Hence, find $sin\frac{\pi}{3}$ in surd form.

Equating imaginary parts of **a** and **b**

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

6(c) (i) Find the square roots of $-6+8i$.

$|-6+8i| = \sqrt{(-6)^2+8} = 10, \mathrm{Re}(-6+8i)$

$= -6$

The square roots of $-6+8i$ are

$$\pm\left(\sqrt{\frac{10-6}{2}} + \sqrt{\frac{10+6}{2}}i\right) = \pm(2+2\sqrt{2}i)$$

(ii) Hence solve the equation $z^2 - \sqrt{2}iz + 1 - 2i = 0$.

$$z = \frac{\sqrt{2}i \pm \sqrt{\left(-\sqrt{2}i\right)^2 - 4(1)(1-2i)}}{2(1)}$$

$$= \frac{\sqrt{2}i \pm \sqrt{-2-4+8i}}{2}$$

$$= \frac{\sqrt{2}i \pm \sqrt{-6+8i}}{2}$$

$$= \frac{\sqrt{2}i \pm (2+2\sqrt{2}i)}{2}$$

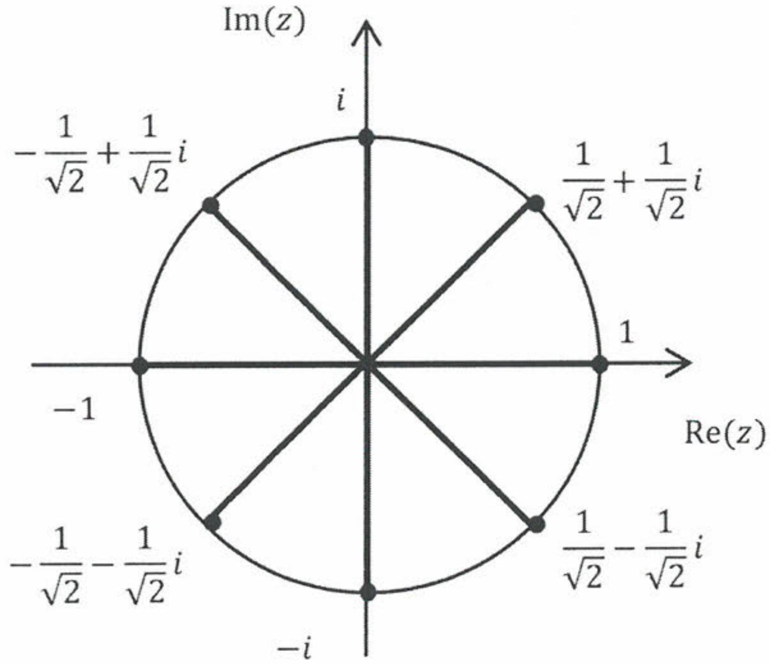
$$= -2-\sqrt{2}i, 2+3\sqrt{2}i$$

6(d) Find the eighth roots of $1(z^8 = 1)$ in exponential form and show these roots in the complex plane. **4**

1 is an eighth root of 1, and the other roots must be spaced by $\frac{2\pi}{8} = \frac{\pi}{4}$ as shown. The eighth roots of 1 are $\pm 1, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, \pm i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$.

Alternatively in exponential form the roots are

$1, e^{\pm\frac{\pi}{4}i}, \pm i, e^{\pm\frac{3\pi}{4}i}$



End of Question 6

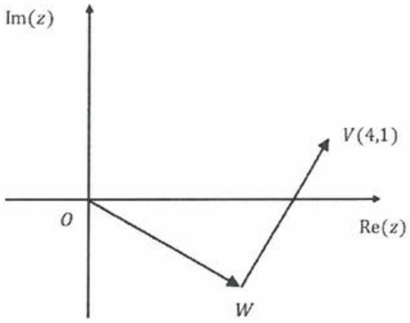
Name

Question 7 (15 marks) START A NEW WRITING BOOKLET **Marks**

(a) The point V represents the complex number $4 + i$. **3**

$\angle OWV = \frac{\pi}{2}$ and $|VW| = |OW|$.

Find the complex number represented by W.



let W represent $a + ib$

$$-i\overrightarrow{WV} = \overrightarrow{OW}$$

$$-i(4 + i - a - ib) = a + ib$$

$$-4i + 1 + ai - b = a + bi$$

$$-b + 1 + i(a - 4) = a + bi$$

$$\therefore -b + 1 = a \qquad b = 1 - a$$

$$a - 4 = b$$

$$a - 4 = 1 - a$$

$$2a = 5$$

$$a = \frac{5}{2}$$

$$b = 1 - \frac{5}{2}$$

$$= -\frac{3}{2}$$

W represents $\frac{5-3i}{2}$

Name

7(b) Prove that if ω is a non-real cube root of unity,

then $(1+\omega-\omega^2)^7 = -128\omega^2$. **2**

$$1+\omega+\omega^2 = 0$$

$$\therefore (1+\omega-w^2)^7$$

$$= (1+\omega+\omega^2-2\omega^2)^7$$

$$= (-2\omega^2)^7$$

$$= -128\omega^{14}$$

$$= -128(\omega^3)^4\omega^2$$

$$= -128\omega^2$$

(c) Identify subsets of the complex plane determined by relation: **3**

$$|z-2i| \geq 4|z+3|$$

$Let\ z = x+iy.$

$\therefore |x+iy-2i| \geq 4|x+iy+3|$

$|x+(y-2)i| \geq 4|(x+3)+iy|$

$\sqrt{x^2+(y-2)^2} \geq 4\sqrt{(x+3)^2+y^2}$

$x^2+(y-2)^2 \geq 16((x+3)^2+y^2)$

$x^2+y^2-4y+4 \geq 16(x^2+6x+9+y^2)$

$x^2+y^2-4y+4 \geq 16x^2+96x+144+16y^2$

$15x^2+96x+15y^2++4y+140 \leq 0$

$x^2+\frac{96}{15}x+y^2+\frac{4}{15}y+\frac{140}{15} \leq 0$

$x^2+\frac{96}{15}x+\left(\frac{48}{15}\right)^2+y^2+\frac{4}{15}y+\left(\frac{2}{15}\right)^2+\frac{140}{15} \leq \left(\frac{48}{15}\right)^2+\left(\frac{2}{15}\right)^2$

$\left(x+\frac{48}{15}\right)^2+\left(x+\frac{2}{15}\right)^2 \leq \frac{208}{225}$

Name

7(d) Prove by Mathematical Induction that $4^{n+1} + 6^n$ is divisible by 10 when n is even. **3**

Let $P(n)$ represent the proposition.

$P(2)$ is true since $4^{2+1} + 6^2 = 100 = 10(10)$

If $P(k)$ is true for some arbitrary $k \geq 2$ then $4^{k+1} + 6^k = 10m$ for integral m

RTP $P(k+2)$ $\quad 4^{k+3} + 6^{k+2} = 10p$ for integral p

$$\begin{aligned}
\text{LHS} &= 16(4^{k+1}) + 36(6^k) \\
&= 16(4^{k+1} + 6^k) + 20(6^k) \\
&= 16(10m) + 20(6^k) \quad \text{from } P(k) \\
&= 10(16m + 2 \times 6^k) \\
&= 10p \qquad \text{for integral } p \text{ since } m \text{ and } k \text{ are integral} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+2)$

$\therefore P(n)$ is true for even $n \geq 2$ by induction $\square$

Name

7(e) Given $a_1 = 2$ and $a_n = \frac{a_{n-1}}{n}$, for $n \geq 2$,
prove, using Mathematical Induction, that $a_n = \frac{2}{n!}$ for $n \geq 1$. **4**

Let $P(n)$ represent the proposition.

$P(1)$ is true since $a_1 = 2$ and $a_1 = \frac{2}{1!} = 2$

If $P(k)$ is true for some arbitrary $k \geq 1$ then $a_k = \frac{2}{k!}$

RTP $P(k+1)$ $\quad a_{k+1} = \frac{2}{(k+1)!}$

$$\begin{aligned} \text{LHS} &= a_{k+1} \\ &= \frac{a_k}{k+1} && \text{from the recursive formula} \\ &= \frac{2}{k!} \div (k+1) && \text{from } P(k) \\ &= \frac{2}{(k+1)!} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(n)$ is true for $n \geq 1$ by induction $\square$

Question 8 *(15 marks)* START A NEW WRITING BOOKLET **Marks**

(a) Find the negation of the following:

$\forall$ real numbers x, $x < 2m+1$ for integral m.

$\exists$ real numbers $x, x \geq 2m + 1$ for integral m

(b) Prove by contrapositive that if $\frac{mn}{2}$ is integral then m or n must be even.

Suppose m is odd and n is odd.

Let $m = 2k + 1, n = 2j + 1$ for integral k, j

$$\frac{mn}{2} = \frac{(2k+1)(2j+1)}{2}$$

$$= \frac{4kj + 2k + 2j + 1}{2}$$

$$= 2kj + k + j + \frac{1}{2}$$

Which is not integral since j and k are integral

$\therefore$ if m and n are both odd then $\frac{mn}{2}$ is not integral.

$\therefore$ if $\frac{mn}{2}$ is integral then m or n must be even.

(c) Prove $\sqrt{2} + \sqrt{3} < 10$ by contradiction.

Suppose $\sqrt{2} + \sqrt{3} \geq \sqrt{10}$ *

$\left(\sqrt{2} + \sqrt{3}\right)^2 \geq 10$ since $\sqrt{2}, \sqrt{3}, \sqrt{10} > 0$

$2 + 2\sqrt{6} + 3 \geq 10$

$2\sqrt{6} \geq 5$

$\sqrt{24} \geq \sqrt{25}$

$24 \geq 25$ #

Which is a contradiction, so $\sqrt{2} + \sqrt{3} < \sqrt{10}$

Name

8(d) Prove that a number is divisible by 4 if and only if the last two digits are a multiple of 4. Let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $b, c \leq 9$. Do not use Proof by Induction.

let the number be $x = 100a + 10b + c$ where a, b, c are positive integers and $b, c \leq 9$
If the last two digits are a multiple of 4 then $10b + c = 4m$ for integral m

$\therefore x = 4(25a) + 4m$
$= 4(25a + m)$
$= 4p$ for integral p since a, m are integral

$\therefore$ if the last two digits are a multiple of 4 then the number is divisible by 4.

Conversely, we will show by contrapositive that if a number is divisible by 4 then the last two digits are a multiple of 4.

If the last two digits are not a multiple of 4 then $10b + c = 4m + k$ for integral m, k with k not a multiple of 4.

$\therefore x = 4(25a) + 4m + k$
$= 4(25a + m) + k$
$\neq 4p$ for integral p since a, m are integral and k is not a multiple of 4

$\therefore$ if the last two digits are not a multiple of 4 then the number is not divisible by 4.
$\therefore$ if the number is divisible by 4 then the last two digits are a multiple of 4
$\therefore$ a number is divisible by 4 if and only if the last two digits are a multiple of 4.

Name

8(e) Prove $\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4}+\frac{1}{xz}\sqrt{z^2y^2+x^4}$ for $x, y, z > 0$. **4**

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \sqrt{\frac{x^2}{y^2}\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)} \qquad (\text{AM}-\text{GM})$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \sqrt{\frac{x^2}{y^2}\left(\frac{x^2y^2+z^4}{x^2z^2}\right)}$$

$$\frac{\frac{x^2}{y^2}+\left(\frac{y^2}{z^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4} \qquad (1)$$

Similarly

$$\frac{\frac{y^2}{z^2}+\left(\frac{x^2}{y^2}+\frac{z^2}{x^2}\right)}{2} \geq \frac{1}{xz}\sqrt{y^2z^2+x^4} \qquad (2)$$

$(1)+(2)$:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2+z^4}+\frac{1}{xz}\sqrt{y^2z^2+x^4}$$

End of Assessment