Northern Beaches Secondary College
Manly Campus

2024

HSC Assessment 2

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 90 minutes • Write using black pen • NESA approved calculators may be used • A reference sheet is provided with this paper • For questions in Section II, show relevant mathematical reasoning and calculations
Total Marks: 52	**Section I – 10 marks (pages 1-5)** • Attempt Questions 1–10 • Allow about 15 minutes for this section • Answer on the Multiple Choice Answer Sheet provided **Section II – 42 marks (pages 6-8)** • Attempt Questions 11-13 • Allow about 75 minutes for this section • Answer each question in a separate booklet

Section I (10 marks)

Attempt Questions 1 – 10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1 – 10

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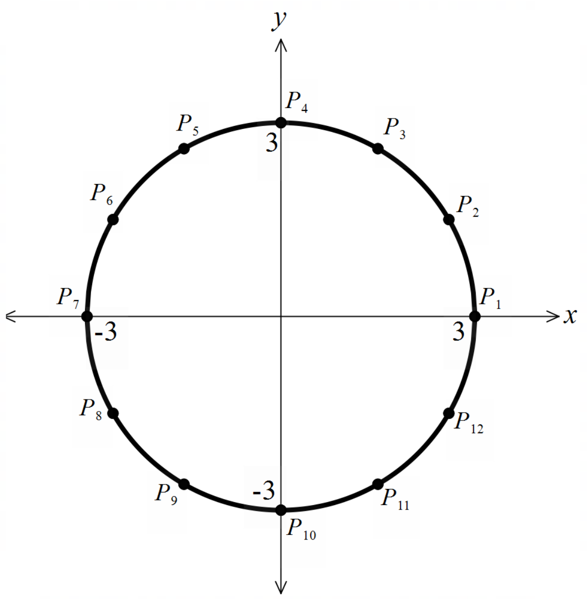
1) For the statement $x^2 = 16 \;\Rightarrow\; x = \pm 4$, the statement $x \neq \pm 4 \Rightarrow x^2 \neq 16$ is the:

(A) inverse (B) converse (C) negation (D) contrapositive

2) On the Argand diagram below, the points P_1 , P_2 , P_3 , ... , P_{12} are evenly spaced around the circle of radius 3 centred at (0, 0).



Which of the points P_1 , P_2 , P_3 , ... , P_{12} represent the complex numbers z for which $z^3 = -27i$?

(A) P_4 , P_8 , P_{12}

(B) P_1 , P_5 , P_9

(C) P_3 , P_7 , P_{11}

(D) P_2 , P_6 , P_{10}

3) If $z = -3 + 2i$ is a root of the polynomial equation $5x^3 + mx^2 + 59x - 13 = 0$ where $m \in \mathbb{R}$, find the value of m.

(A) 31 (B) -31 (C) 29 (D) -29

4) Which of the following is equivalent to the value of $\dfrac{2i}{e^{\frac{5\pi i}{6}}}$?

(A) $2\left[\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right]$

(B) $2\left[\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right]$

(C) $-2\left[\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right]$

(D) $2\left[\cos\left(-\dfrac{2\pi}{3}\right) - i\sin\left(-\dfrac{2\pi}{3}\right)\right]$

5) Which of the following is true?

(A) $\forall\, x, y \in \mathbb{Z}^+,\ \exists\, z \in \mathbb{Q} :\ z\sqrt{y} = \dfrac{x}{\sqrt{y}}$

(B) $\forall\, x, y \in \mathbb{R},\ \exists\, z \in \mathbb{N} :\ z\sqrt{y} = \dfrac{x}{\sqrt{y}}$

(C) $\forall\, x, y \in \mathbb{Q},\ \exists\, z \in \mathbb{Z} :\ z\sqrt{y} = \dfrac{x}{\sqrt{y}}$

(D) $\forall\, x, y \in \mathbb{Q},\ \exists\, z \in \mathbb{R} :\ z\sqrt{y} = \dfrac{x}{\sqrt{y}}$

6) The polynomial equation $P(z) = 0$ has real coefficients and has roots which include $z = 2$ and $z = -2 + i$. If $z = 2$ is a root of multiplicity 2, what is the minimum degree of $P(z)$?

(A) 3 (B) 4 (C) 5 (D) 6

7) Consider the statement: $\exists\, x \in \mathbb{R} :\ \log_e x = 1$ and $x > 2$

Which of the following is the negation of this statement?

(A) $\exists x \in \mathbb{R},\ \log_e x \neq 1$ and $x \leq 2$

(B) $\exists x \in \mathbb{R},\ \log_e x \neq 1$ or $x \leq 2$

(C) $\forall x \in \mathbb{R},\ \log_e x \neq 1$ and $x \leq 2$

(D) $\forall x \in \mathbb{R},\ \log_e x \neq 1$ or $x \leq 2$

8) Let $z = x + iy$ where $x, y \in \mathbb{R}$ and $x, y \neq 0$.

If $z + \frac{1}{z} \in \mathbb{R}$, which one of the following must be true?

(A) $Arg\ z = \frac{\pi}{4}$

(B) $a = -b$

(C) $|z| = 1$

(D) $z^2 = 1$

9) The complex numbers z, iz and $z + iz$ where $z \in \mathbb{C}$ and $z \neq 0$ are plotted in the Argand plane, forming the vertices of a triangle.

The area of this triangle is given by:

(A) $|z| + |z|^2$

(B) $\frac{1}{2}|z|^2$

(C) $|z|$

(D) $\frac{\sqrt{3}}{2}|z|^2$

10) It is given that a, b are purely real and c, d are purely imaginary.

Which pair of inequalities must always be true?

(A) $a^2c^2 + b^2d^2 \geq 2abcd$, $a^2b^2 + c^2d^2 \leq 2abcd$

(B) $a^2c^2 + b^2d^2 \geq 2abcd$, $a^2b^2 + c^2d^2 \geq 2abcd$

(C) $a^2c^2 + b^2d^2 \leq 2abcd$, $a^2b^2 + c^2d^2 \leq 2abcd$

(D) $a^2c^2 + b^2d^2 \leq 2abcd$, $a^2b^2 + c^2d^2 \geq 2abcd$

Section II (42 marks)

Attempt Questions 11-13

Allow about 75 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks)

a)

(i) Find the two square roots of $7 + 24i$. Express the results in the form $x + iy$. **2**

(ii) Hence, solve the equation $2z^2 - iz - (1 + 3i) = 0$ **2**

b) The sequence $\{u_n\}$ is defined by

$$u_1 = 2, \quad u_{n+1} = \frac{u_n}{2(n+1)} \qquad \forall\, n \in \mathbb{Z}^+$$

Use mathematical induction to prove that

$$u_n = \frac{2^{2-n}}{n!} \qquad \forall\, n \in \mathbb{Z}^+$$ **3**

c) Simplify

$$i^{1!} + i^{2!} + i^{3!} + i^{4!} + i^{5!} + \ldots + i^{100!}$$ **3**

d) Let ω be a non-real solution to the equation $z^3 = 1$.

Evaluate $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)(1 - \omega^7)(1 - \omega^8)$. **3**

End of Question 11

Question 12 (15 marks)

a)

(i) Find the fourth roots of -4 . Express the results in the form $x + iy$. **3**

(ii) Hence, or otherwise, write $z^4 + 4$ as a product of quadratic factors with real coefficients. **2**

b) Prove by contraposition that if $m, n \in \mathbb{Z}^+$ such that $m + n$ and $m - n$ have no common factors, then m and n have no common factors. **3**

c) Let z be a complex number with $\operatorname{Im} z > 0$. Find the value of $Arg\left(\frac{z\overline{z}}{z - \overline{z}}\right)$. **2**

d)

(i) Let $z = \cos\theta + i\sin\theta$

Prove that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ **2**

(ii) Find real constants A, B and C such that **3**

$\cos^5\theta = A\cos 5\theta + B\cos 3\theta + C\cos\theta$

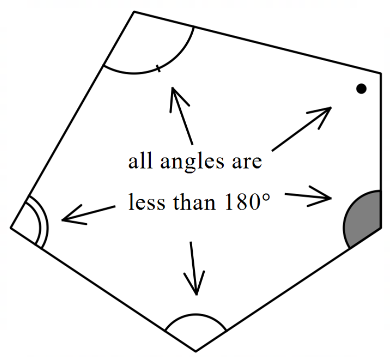
End of Question 12

Question 13 (14 marks)

a) Let $z = \text{cis}\theta$ for $0 < \theta < \dfrac{\pi}{2}$.

Find the modulus and argument of $z + 1$ in terms of θ. **3**

b) A convex polygon is any closed figure with non-intersecting straight sides and with all interior angles less than $180°$. For example, the figure below is a convex pentagon.



Prove by induction that a convex polygon with n sides has $S(n) = \dfrac{1}{2}n(n-3)$ diagonals for all integers $n \geq 3$ **4**

c)

(i) Show that if a, b are positive real numbers then $\dfrac{a^2 + b^2}{2} \geq ab$ **1**

(ii) Show that if a, b, c, d are positive real numbers then **3**

$$\sqrt{(a^2 + c^2)(b^2 + d^2)} \geq ab + cd$$

d) Prove by contradiction that there are no positive integers p and q such that $4p^2 - q^2 = 25$ **3**

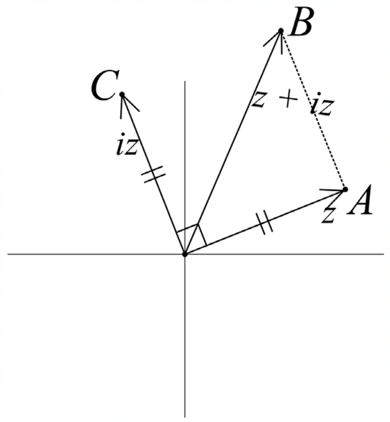
End of Question 13

End of paper

2024 Maths Extension 2 Half Yearly marking guidelines

Q	Solution	Marks
1	$x^2 = 16 \Rightarrow x = \pm 4$ is of the form $A \Rightarrow B$ $x \neq \pm 4 \Rightarrow x^2 \neq 16$ is $\sim B \Rightarrow \sim A$, i.e. the contrapositive	D **39 (out of 41) students answered correctly**
2	$(3i)^3 = 3^3 \times i^3 = -27i$ Therefore one solution is represented by the point P_4 The cube roots of $-27i$ are evenly spaced around the circle, and so they will have angles of 120 degrees between them Therefore the points P_4, P_8 and P_{12} represent the solutions	A **28 (out of 41) students answered correctly**
3	Since the coefficients are real, $\overline{z} = -3 - 2i$ is a zero of $P(x)$. Let the zeroes be $\alpha = -3 + 2i$, $\beta = -3 - 2i$ and γ $\alpha + \beta + \gamma = -\frac{b}{a}$ $-3 + 2i + (-3 - 2i) + \gamma = -\frac{m}{5}$ $-6 + \gamma = -\frac{m}{5}$ $\frac{m}{5} = 6 - \gamma$ $\alpha\beta\gamma = -\frac{d}{a}$ $(-3 + 2i)(-3 - 2i)\gamma = \frac{13}{5}$ $13\gamma = \frac{13}{5}$ $\gamma = \frac{1}{5}$ $\therefore \quad \frac{m}{5} = 6 - \frac{1}{5} = \frac{29}{5}$ $m = 29$ Alternative solution: $5(-3 + 2i)^3 + m(-3 + 2i)^2 + 59(-3 + 2i) - 13 = 0$ After expanding: $(-145 + 5m) + i(348 - 12m) = 0 + 0i$ Equating real and im. parts: $-145 + 5m = 0 \Rightarrow m = 29$ $348 - 12m = 0 \Rightarrow m = 29$	C **25 (out of 41) students answered correctly**

4	$\frac{2i}{e^{\frac{5i\pi}{6}}} = \frac{2e^{\frac{i\pi}{2}}}{e^{\frac{5i\pi}{6}}} = 2e^{i\left(\frac{\pi}{2}-\frac{5\pi}{6}\right)}$ $= 2e^{-\frac{\pi}{3}i}$ $= 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$ $= 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]$ $= 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$ $= 2\left[-\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right]$ $= -2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$	C **26 (out of 41) students answered correctly**
5	The expression $\frac{x}{\sqrt{y}}$ where x and y are positive integers can be rewritten as: $\frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y} = \left(\frac{x}{y}\right)\sqrt{y}$ Hence, for all positive integers x and y, there exists a rational number $z = \frac{x}{y}$ such that $z\sqrt{y} = \frac{x}{\sqrt{y}}$ ($y = 0$ is a counterexample to the other three options)	A **21 (out of 41) students answered correctly**
6	Since coefficients are real, $\bar{z} = -2 - i$ is also a root of $P(z) = 0$ and $z = 2$ is a root from the factor $(z-2)^2$. Therefore the minimum degree is 4	B **34 (out of 41) students answered correctly**
7	The statement says "There is a real number whose natural log is 1 and which is greater than 2." The negation is: "There is **no** real number whose natural log is 1 and which is greater than 2" i.e. "For all real numbers, either the natural log is not 1 or the number is not greater than 2" i.e. 'For all real numbers, either the natural log is not 1 or the number is less than or equal to 2"	D **37 (out of 41) students answered correctly**

8	$z + \frac{1}{z} \in \mathbb{R}$ $z + \frac{\overline{z}}{z\overline{z}} \in \mathbb{R}$ $z + \frac{\overline{z}}{\lvert z\rvert^2} \in \mathbb{R}$ $\therefore Im\left(z + \frac{\overline{z}}{\lvert z\rvert^2}\right) = 0$ But, we also have $z + \overline{z} \in \mathbb{R}$ $\therefore Im(z + \overline{z}) = 0$ Therefore $\lvert z\rvert^2 = 1$ $\lvert z\rvert = 1 \quad (\lvert z\rvert \geq 0)$	C **32 (out of 41) students answered correctly**
9	 $z + iz$ is the diagonal of the square $OABC$ where $OA = \lvert z\rvert$ and $AB = \lvert iz\rvert = \lvert z\rvert$. Area of triangle OAB: $Area = \frac{1}{2} \times OA \times OB$ $= \frac{1}{2} \lvert z\rvert\ \lvert z\rvert$ $= \frac{1}{2} \lvert z\rvert^2$	B **40 (out of 41) students answered correctly**
10	a, b are real and c, d are purely imaginary Therefore, ab and cd are real and ac and cd are purely imaginary Therefore, $ab - cd$ is real and $ac - bd$ are purely imaginary $\therefore \quad (ab - cd)^2 \geq 0$ $a^2b^2 - 2abcd + c^2d^2 \geq 0$ $a^2b^2 + c^2d^2 \geq 2abcd$ and $\therefore \quad (ac - bd)^2 \leq 0$ $a^2c^2 - 2abcd + b^2d^2 \leq 0$ $a^2c^2 + b^2d^2 \leq 2abcd$	D **23 (out of 41) students answered correctly**

11ai	$$\begin{aligned} (x+iy)^2 &= 7+24i \\ x^2-y^2+2xyi &= 7+24i \\ \therefore \quad x^2-y^2 &= 7 \quad \text{and } 2xy = 24 \\ y &= \frac{12}{x} \\ \therefore \quad x^2-\left(\frac{12}{x}\right)^2 &= 7 \\ x^4-7x^2-144 &= 0 \\ (x^2-16)(x^2+9) &= 0 \\ x^2 &= 16 \quad (x\in\mathbb{R}) \\ x &= \pm 4 \\ \therefore \quad y &= \pm 3 \\ \therefore \quad x+iy &= \pm(4+3i) \end{aligned}$$	2 marks for correct solution 1 mark for obtaining $x^2-y^2=7$ and $2xy=24$
11aii	$$z=\frac{i\pm\sqrt{(-i)^2+4(2)(1+3i)}}{2(2)}$$ $$z=\frac{i\pm\sqrt{-1+8+24i}}{4}$$ $$z=\frac{i\pm\sqrt{7+24i}}{4}$$ $$z=\frac{i\pm(4+3i)}{4}$$ $$z=\frac{4+4i}{4} \text{ or } \frac{-4-2i}{4}$$ $$z=1+i \text{ or } \frac{-2-i}{2}$$	2 marks 1 mark for writing a correct expression for z **25 (out of 41) students got full marks for 11a**
11b	For $n=1$: $LHS = u_1 = 2$ $RHS = \frac{2^{2-1}}{1!} = 2$ Therefore true for $n=1$ Assume true for $n=k$: $u_k = \frac{2^{2-k}}{k!}$ RTP for $n=k+1$: $u_{k+1} = \frac{2^{2-(k+1)}}{(k+1)!}$ $u_{k+1} = \frac{2^{1-k}}{(k+1)!}$	3 marks for correct solution 2 marks for proving S(1) true and correctly substituting the assumption S(k) 1 mark for proving S(1) true **34 (out of 41) students got full marks**

	$LHS = u_{k+1}$ $= \dfrac{u_k}{2(k+1)}$ $= \dfrac{1}{2(k+1)} \times \dfrac{2^{2-k}}{k!}$ (from assumption) $= \dfrac{2^{2-k-1}}{(k+1)!}$ $= \dfrac{2^{1-k}}{(k+1)!}$ $\therefore S(1)$ true and $S(k) \Rightarrow S(k+1)$ true $\therefore S(n)$ true $\forall\ n \in \mathbb{Z}^+$	
11c	$i^{1!} = i$ $i^{2!} = i^2 = -1$ $i^{3!} = i^6 = i^4 i^2 = -1$ $i^{4!} = i^{24} = (i^4)^6 = 1$ $i^{5!} = i^{120} = (i^4)^{30} = 1$ For $n \geq 4$, n! is a multiple of 4, so $i^{n!} = 1$ $\therefore\ S = i - 1 - 1 + (1 + 1 + 1 + \ldots + 1)$ $S = i - 2 + (97)$ $S = 95 + i$	3 marks for correct solution 2 marks for correct solution with one error (e.g. 96 instead of 97) 1 mark for evaluating the first four terms or for identifying a correct pattern **20 (out of 41) students got full marks**
11d	$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)(1-\omega^7)(1-\omega^8)$ $= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$ $= (1-\omega)^3(1-\omega^2)^3$ $= \left[(1-\omega)(1-\omega^2)\right]^3$ $= [1-\omega^2-\omega+\omega^3]^3$ $= (1-\omega^2-\omega+1]^3$ $= \left[2-(\omega^2+\omega)\right]^3$ $= (2-(-1))^3$ $= 27$	3 marks for correct solution 2 marks for correct solution with one error or insufficient working 1 mark for using $\omega^3 = 1$ **26 (out of 41) students got full marks**

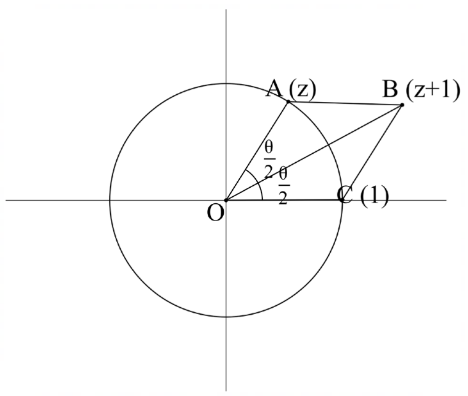
12ai	$z^4 = -4$ $z^4 = 4\text{cis}(\pi + 2k\pi),\ k \in \mathbb{Z}$ $z = 4^{\frac{1}{4}}(\text{cis}(\pi + 2k\pi))^{\frac{1}{4}}$ $z = \sqrt{2}\ \text{cis}\left(\frac{\pi + 2k\pi}{4}\right)$ by dMT Letting $k = 0,1,2,3$: $z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}, \sqrt{2}\ \text{cis}\,\frac{3\pi}{4},\ \sqrt{2}\ \text{cis}\,\frac{5\pi}{4}, \sqrt{2}\ \text{cis}\,\frac{7\pi}{4}$ $z = \sqrt{2}\left(\pm\frac{1}{\sqrt{2}} \pm\frac{1}{\sqrt{2}}\,i\right)$ $z = \pm1\pm i$ i.e. $z = 1 + i,\ 1 - i,\ -1 + i,\ -1 - i$ Alternative method: $z^4 = -4$ $(x + iy)^4 = -4$ Expand the brackets and equate real and imaginary parts: $x^4 - 6x^2y^2 + y^4 = -4$ $4x^3y - 4xy^3 = 0$ $xy(x^2 - y^2) = 0$ $x = \pm y \quad (x, y \neq 0)$ $\therefore\ x^4 - 6x^4 + x^4 = -4$ $-4x^4 = -4$ $x^4 = 1$ $x = 1$ or $-1 \quad (x \in \mathbb{R})$ $x = 1 \Rightarrow y = \pm1$ $x = -1 \Rightarrow y = \pm1$ $\therefore \quad z = \pm(1 + i), \pm(1 - i)$ A third option is: $z^4 = -4$ $z^2 = 2i$ or $-2i$ $(x + iy)^2 = 2i$ or $-2i$ etc	3 marks 2 marks for correct solution with one error 2 marks for correct roots but not in Cartesian form 1 mark for some progress, e.g. correct modulus and use of k =-1,0,1,2; or forming two equations involving the real and imaginary parts of z
12aii	$z^4 + 4$ $= [z - (1 - i)][z - (1 + i)][z - (-1 + i)][z - (-1 - i)]$ $= (z - 1 + i)(z - 1 - i)(z + 1 + i)(z + 1 - i)$ $= \left[(z - 1)^2 + 1\right]\left[(z + 1)^2 + 1\right]$ $= (z^2 - 2z + 2)(z^2 + 2z + 2)$	2 marks 1 mark for obtaining any correct factored form (using the roots obtained in (i)) **12 (out of 41) students got full marks for Q12a**
12b	Contrapositive statement:	3 marks

	If m and n have at least one common factor, then $m+n$ and $m-n$ have at least one common factor. Suppose that m and n have least one common factor. $\therefore \exists\, p, q, k \in \mathbb{Z}$ s.t. $m = kp$ and $n = kq$ $m + n = kp + kq = k(p+q)$, $\therefore k \mid (m+n)$ $m - n = kp - kq = k(p-q)$ $\therefore k \mid (m-n)$ Therefore, if m and n share a common factor, then $m+n$ and $m-n$ share a common factor (indeed, the same common factor). As the contrapositive is true, the original statement is true	2 marks for correct proof but with incomplete reasoning/ one error. e.g. m, n have a common factor. Therefore, $m = kn$. 1 mark for correctly writing the contrapositive statement **28 (out of 41) students got full marks**
12c	$Arg\left(\frac{z\bar{z}}{z-\bar{z}}\right)$ $= Arg\left(\frac{\lvert z\rvert^2}{2i \times Imz}\right)$ $= Arg\left(\frac{i\lvert z\rvert^2}{-2Imz}\right)$ $= Arg\left[\left(\frac{\lvert z\rvert^2}{-2Imz}\right)i\right]$ $= -\frac{\pi}{2}$ since $\frac{\lvert z\rvert^2}{-2Imz} < 0$	2 marks 1 mark for simplifying $\frac{z\bar{z}}{z-\bar{z}}$ to $\frac{\lvert z\rvert^2}{2iImz}$ Note that $Im(x+iy) = y$ and not iy. This caused problems for some students. 8 (out of 41) students got full marks
12di	$z^n + \frac{1}{z^n}$ $= (\text{cis}\theta)^n + (\text{cis}\theta)^{-n}$ $= \text{cis}(n\theta) + \text{cis}(-n\theta)$ by dMT $= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ $= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$ $= 2\cos(n\theta)$	2 marks 1 mark for using de Moivre's theorem to obtain $\text{cis}(n\theta) + \text{cis}(-n\theta)$
12dii	$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$ (binomial expansion) $\left(z+\frac{1}{z}\right)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$ $(2\cos\theta)^5 = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$ from (i) $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$ $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$ $\therefore$ $A = \frac{1}{16}, B = \frac{5}{16}, C = \frac{5}{8}$ An alternative worse method which doesn't use part (i) or complex numbers:	3 marks 2 marks for correct solution with one error 1 mark for expanding $\left(z+\frac{1}{z}\right)^5$ **15 (out of 41) students got full marks for Q12d**

$$
\begin{aligned}
\cos^5\theta &= \cos\theta\cos^4\theta \\
&= \cos\theta(\cos^2\theta)^2 \\
&= \cos\theta\left[\frac{1}{2}(1+\cos 2\theta)\right]^2 \\
&= \frac{1}{4}\cos\theta(1+2\cos 2\theta+\cos^2 2\theta) \\
&= \frac{1}{4}\cos\theta\left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right] \\
&= \frac{1}{8}\cos\theta(2+4\cos 2\theta+1+\cos 4\theta) \\
&= \frac{1}{8}\cos\theta(3+4\cos 2\theta+\cos 4\theta) \\
&= \frac{1}{8}(3\cos\theta+4\cos 2\theta\cos\theta+\cos 4\theta\cos\theta) \\
&= \frac{1}{8}\left[3\cos\theta+4\times\frac{1}{2}(\cos\theta+\cos 3\theta)+\frac{1}{2}(\cos 3\theta+\cos 5\theta)\right] \\
&= \frac{1}{8}\left[3\cos\theta+2\cos\theta+2\cos 3\theta+\frac{1}{2}\cos 3\theta+\frac{1}{2}\cos 5\theta\right] \\
&= \frac{1}{8}\left[5\cos\theta+\frac{5}{2}\cos 3\theta+\frac{1}{2}\cos 5\theta\right] \\
&= \frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta
\end{aligned}
$$

13a

$|z| = 1$ and $0 < \arg z < \dfrac{\pi}{2}$, so z lies on the unit circle in the first quadrant.



Let z be the vector $\overrightarrow{OA}$ and $z+1$ be the vector $\overrightarrow{OB}$

$\therefore OABC$ is a rhombus

Since $\overrightarrow{OB}$ bisects $\angle AOC$, $\angle BOC = \dfrac{\theta}{2}$

$\therefore Arg(z+1) = \dfrac{\theta}{2}$

In ΔOMC, $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{OM}{OC} = \dfrac{OM}{1} = OM$

$\therefore \quad OB = 2OM \Rightarrow |z+1| = 2\cos\dfrac{\theta}{2}$

3 Marks:
Correct solution

2 Marks:
Correct derivation of either |z+1| or Arg (z+1).

1 Mark:
Correct expression for |z+1| or Arg(z+1) without justification.

14 (out of 41) students got full marks

	Alternative solution: $\angle OAB = \pi - \theta$ (co-int. angles, AB//OC) $OA = AB = 1$ Therefore triangle OAB is isosceles. $\therefore 2\angle AOB + \pi - \theta = \pi$ (angle sum of Δ) $2\angle AOB = \theta$ $\angle AOB = \frac{\theta}{2}$ $\therefore \quad \arg(z+1) = \angle BOC = \frac{\theta}{2}$ Using cosine rule in triangle OAB, $\|z+1\|^2 = 1^2 + 1^2 - 2(1)(1)\cos(\pi - \theta)$ $\|z+1\|^2 = 2 + 2\cos\theta$ $\|z+1\| = \sqrt{2 + 2\cos\theta}$ or $\sqrt{2 + 2\left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right)}$ $= \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}$ $= 2\cos\frac{\theta}{2}$	
13b	For $n = 3$: $LHS = S(3) = 0$ as a triangle has no diagonals $RHS = \frac{1}{2}(3)(3-3) = 0$ $\therefore S(3)$ *true* Assume true for $n = k$: $S(k) = \frac{1}{2}k(k-3)$ RTP true for $n = k + 1$: $S(k+1) = \frac{1}{2}(k+1)(k-2)$ Consider the diagonals of a k-gon with vertices $P_1, P_2, P_3, \ldots, P_k$ When a new vertex P_{k+1} is added to form a $(k + 1)$-gon, new diagonals can be drawn from P_{k+1} to those vertices which are not adjacent to P_{k+1} (i.e. $P_2, P_3, \ldots, P_{k-1}$), which makes $(k - 2)$ new diagonals. Additionally, the interval $P_1 P_k$ which was a side length of the k-gon is now a diagonal of the $(k + 1)$-gon. Therefore,	**4 Marks:** Correct solution. Must state a correct form of S(k) and S(k+1) **3 Marks:** Consistent result obtained from incomplete reasoning. **2 Marks:** Correct initial case AND correct statement of S(k) and S(k+1) **1 Mark:** Correct initial case. Must state that a triangle has zero diagonals. **28 (out of 41) students got full marks**

	$S(k+1) = S(k) + (k-2) + 1$ $= S(k) + k - 1$ $= \frac{1}{2}k(k-3) + k - 1$ $= \frac{1}{2}[k(k-3) + 2k - 2]$ $= \frac{1}{2}[k^2 - 3k + 2k - 2]$ $= \frac{1}{2}(k^2 - k - 2)$ $= \frac{1}{2}(k+1)(k-2)$ $\therefore S(1)$ true and $S(k) \Rightarrow S(k+1)$ true $\therefore S(n)$ true $\forall$ integral $n \geq 3$	
13ci	$(a-b)^2 \geq 0$ $a^2 - 2ab + b^2 \geq 0$ $a^2 + b^2 \geq 2ab$ $\frac{a^2+b^2}{2} \geq ab$	**1 Mark:** Correct answer
13cii	$(a^2+c^2)(b^2+d^2) - (ab+cd)^2$ $= a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 - (a^2b^2 + 2abcd + c^2d^2)$ $= a^2d^2 + b^2c^2 - 2abcd$ $= (ad-bc)^2$ ≥ 0 $\therefore (a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$ $\sqrt{(a^2+c^2)(b^2+d^2)} \geq ab+cd$ Alternative solution: $\frac{a^2+b^2}{2} \geq ab$ Replacing $a \to ad$ and $b \to bc$ $(ad)^2 + (bc)^2 \geq (ad)(bc)$ $a^2d^2 + b^2c^2 \geq 2abcd$ $a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 \geq (ab)^2 + 2abcd + (cd)^2$ $(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$ $\sqrt{(a^2+c^2)(b^2+d^2)} \geq ab+cd$ Another alternative solution:	**3 Marks:** Correct answer **2 Marks:** Expands $(ac\text{-}bd)^2$ and make some relevant and sufficient progress towards the final result. **1 Mark:** Expands $(ac\text{-}bd)^2$ or equivalent correct starting point. **11 (out of 41) students got full marks for Q13c**

	$RTP: \sqrt{(a^2+c^2)(b^2+d^2)} \geq ab+cd$ i.e. $(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$ i.e. $a^2b^2+a^2d^2+b^2c^2+c^2d^2 \geq a^2b^2+c^2d^2+2abcd$ i.e. $a^2d^2+b^2c^2 \geq 2abcd$ From (i), $\dfrac{a^2d^2+b^2c^2}{2} \geq (ab)(cd)$ $a^2d^2+b^2c^2 \geq 2abcd$	
13d	Suppose that there exist positive integers p and q such that $4p^2-q^2=25$ Then, $(2p+q)(2p-q)=25$ Since p and q are positive integers, $2p+q$ and $2p-q$ are integers, and $2p+q>2p-q$. The only way to express 25 as the product of two distinct positive integers is $25=25\times 1$. Hence we must have: $2p+q=25$ and $2p-q=1$. $\therefore\ 2p+q+2p-q=26$ $4p-2q=13$ $2p-q=6.5$, a contradiction since $2p-q\in\mathbb{Z}$ Therefore, there are no positive integers p and q such that $4p^2-q^2=25$	**3 Marks:** Correct solution **2 Marks:** Case by case answer with three correct cases OR Factorises $4p^2-q^2$ and equates factors to 1 or 25. **1 Mark:** Factorises $4p^2-q^2$ AND attempts to equate one, or both, factors to 1,5, 25 OR Recognises q^2 must be odd but no further relevant progress. **14 (out of 41) students got full marks**