NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2020 Year 12 Course Assessment Task 1
Monday December 2, 2019

General instructions

- Working time – 1 hour.
  (plus 5 minutes reading time)
- Write your answers in the space provided.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle any additional writing booklets used in the correct order within this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

Class (please ✔)

◯ 12MAT.1 – Ms Ham

◯ 12MAT.2 – Mr Lam

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Marker's use only.

QUESTION	1	2	3	4	5	6	Total	%
MARKS	$\overline{6}$	$\overline{5}$	$\overline{7}$	$\overline{4}$	$\overline{12}$	$\overline{5}$	$\overline{39}$	

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 1 (6 Marks) **Marks**

(a) If $z = 1 + 2i$, find in the form $a + ib$, where a, $b \in \mathbb{R}$, the values of:

i. z^2 **1**

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ii. $\dfrac{1}{z}$ **1**

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(b) i. Express $z = -1 + \sqrt{3}i$ in the form $z = re^{i\theta}$. **2**

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ii. Hence show that $z^8 - 16z^4$ is purely imaginary. **2**

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Question 2 (5 Marks) **Marks**

(a) i. Find the values of the real numbers a and b such that $(a+ib)^2 = 5-12i$. **2**

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ii. Hence or otherwise, solve the equation $ix^2 + 3x + (3-i) = 0$. **2**

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(b) The complex number z has $|z| = 1$ and $\text{Arg}(z) = \dfrac{\pi}{6}$. The points represented by the complex numbers z, 0 and ω form an equilateral triangle on the Argand diagram. **1**

Find a possible complex number ω.

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Question 3 (7 Marks) **Marks**

A circle C and ray ℓ have equations $\left|z-2\sqrt{3}-i\right|=4$ and $\operatorname{Arg}(z+i)=\dfrac{\pi}{6}$ respectively.

(a) Show that the circle C passes through the point where $z=-i$. **1**

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(b) Show that the ray ℓ passes through the centre of C. **2**

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(c) Sketch C and ℓ on the same Argand diagram. **2**

(d) Shade on your sketch in part (c), the region satisfying both **2**

$$\left|z-2\sqrt{3}-i\right|\leq 4 \qquad \text{and} \qquad 0\leq \operatorname{Arg}(z+i)\leq \frac{\pi}{6}$$

Examination continues overleaf...

Question 4 (4 Marks) **Marks**

In the Argand diagram, $OABC$ is a square, where O, A, B and C are in anti-clockwise cyclic order. The complex number z is represented by the vector $\overrightarrow{OA}$.

(a) Find in terms of z, the complex numbers represented by the vectors $\overrightarrow{OC}$ and $\overrightarrow{OB}$. **2**

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(b) If the vector $\overrightarrow{OB}$ represents the complex number $4+2i$, find z in the form $a+ib$, where a and $b \in \mathbb{R}$. **2**

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Examination continues overleaf...

Question 5 (12 Marks) **Marks**

(a) The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p, $q \in \mathbb{R}$, has roots α, β and γ.

It is given that $\alpha = 2 + 2\sqrt{3}i$.

i. Write down another root, β, of the equation. **1**

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ii. Find the third root γ. **2**

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iii. Find the values of p and q. **2**

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iv. By expressing α in modulus-argument form, show that **1**

$$\left(2 + 2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$$

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Question 5 continues overleaf...

Question 5 continued from the previous page...

v. Hence, show that **2**

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$$

where $n \in \mathbb{Z}$.

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(b) i. If $z = e^{i\theta}$, show that **2**

$$z + z^{-1} = 2\cos\theta$$

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ii. Given further that $z + z^{-1} = \sqrt{2}$, find the value of **2**

$$z^{10} + \frac{1}{z^{10}}$$

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Examination continues overleaf...

Question 6 (5 Marks) **Marks**

(a) i. Solve $z^4 + 1 = 0$, giving your answers in modulus-argument form. **3**

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ii. Plot these solutions on the Argand diagram. **1**

iii. Find the exact area of the quadrilateral that they form. **1**

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End of paper.

2020 Mathematics Extension 2 Year 12 Course Assessment Task 1 STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

2. Which topics did I need more help with, and what parts specifically?

- Q1 - Complex Numbers (Arithmetic)
- Q2 - Complex Numbers (Equations)
- Q3 - Complex Numbers (Graphs in the Argand Diagram)
- Q4 - Complex Numbers (Vectors)
- Q4 - Complex Numbers (De Moivre's Theorem/Polynomials)
- Q5 - Complex Numbers (Roots of unit/polynomials)

3. What other parts from the feedback session can I take away to refine my solutions for future reference?

Sample Band E4 Responses

Question 1 (Ham)

(a) i. (1 mark)

$$\begin{aligned}(1+2i)^2 &= \left(1^2-2^2\right)+4i\\ &= -3+4i\end{aligned}$$

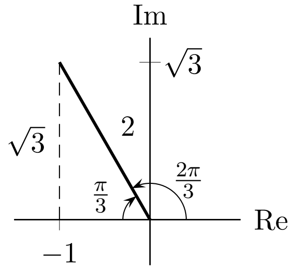
ii. (1 mark)

$$\begin{aligned}\frac{1}{1+2i}\times\tfrac{1-2i}{1-2i} &= \frac{1-2i}{1^2+2^2}\\ &= \frac{1}{5}-\frac{2}{5}i\end{aligned}$$

(b) i. (2 marks)

✓ [1] for modulus

✓ [1] for argument



$$z=-1+\sqrt{3}i=2e^{i\left(\frac{2\pi}{3}\right)}$$

ii. (2 marks)

✓ [1] for finding the correct value of z^4.

✓ [1] for showing $z^8-16z^4\notin\mathbb{R}$.

$$\begin{aligned}z^4 &= \left(2e^{i\left(\frac{2\pi}{3}\right)}\right)^4\\ &= 2^4e^{i\left(\frac{8\pi}{3}\right)}\\ &= 16e^{i\left(\frac{2\pi}{3}\right)}\\ z^8-16z^4 &= z^4\left(z^4-16\right)\\ &= 16e^{i\left(\frac{2\pi}{3}\right)}\left(16e^{i\left(\frac{2\pi}{3}\right)}-16\right)\\ &= 256\left(e^{i\left(\frac{4\pi}{3}\right)}-e^{i\left(\frac{2\pi}{3}\right)}\right)\\ &= 256\left(e^{i\left(-\frac{2\pi}{3}\right)}-e^{i\left(\frac{2\pi}{3}\right)}\right)\\ &= -256\left(e^{i\left(\frac{2\pi}{3}\right)}-e^{i\left(\frac{-2\pi}{3}\right)}\right)\\ &= -256\left(2i\sin\frac{2\pi}{3}\right)\\ &= -256\sqrt{3}i\notin\mathbb{R}\end{aligned}$$

Question 2 (Ham)

(a) i. (2 marks)

✓ [1] for obtaining both equations (1) and (2).

✓ [1] for each correct value of a and b.

$$\begin{aligned}(a+ib)^2 &\equiv 5-12i\\ (a^2-b^2)+i(2ab) &\equiv 5-12i\end{aligned}$$

Equating real and imaginary parts,

$$\begin{cases}a^2-b^2=5 & (1)\\ 2ab=-12 & (2)\end{cases}$$

From (2),

$$\begin{aligned}ab &= -6\\ \therefore b &= -\frac{6}{a}\end{aligned}$$

Substitute into (1):

$$\begin{aligned}a^2-\left(-\frac{6}{a}\right)^2 &= 5\\ a^2-\frac{36}{a^2} &= 5\\ a^4-36 &= 5a^2\\ a^4-5a^2-36 &= 0\\ \left(a^2-9\right)\left(a^2+4\right) &= 0\end{aligned}$$

As $a\in\mathbb{R}$,

$$\begin{aligned}\therefore a &= \pm 3\\ b &= -\frac{6}{a}=\mp 2\end{aligned}$$

ii. (2 marks)

✓ [−1] for unsimplified answers.

$$ix^2+3x+(3-i)=0$$

Applying the quadratic formula,

$$\begin{aligned}x &= \frac{-3\pm\sqrt{3^2-4(i)(3-i)}}{2i}\\ &= \frac{-3\pm\sqrt{9-12i-4}}{2i}\\ &= \frac{-3\pm\sqrt{5-12i}}{2i}\end{aligned}$$

From part (a),

$$\begin{aligned} x &= \frac{-3 \pm (3-2i)}{2i} \\ &= \frac{-3+3-2i}{2i} \text{ or } \frac{-3-(3-2i)}{2i} \\ &= -1 \text{ or } 1+3i \end{aligned}$$

(b) (1 mark)

- Simply add or subtract $\frac{\pi}{3}$ to the argument of $z = e^{i\frac{\pi}{6}}$ to rotate z by $\frac{\pi}{3}$ to obtain an isosceles triangle with an apex angle about O of $\frac{\pi}{3}$.
- Hence, $\omega = e^{-i\frac{\pi}{6}}$ or $\omega = e^{i\frac{\pi}{2}} = i$.

Question 3 (Ham)

(a) (1 mark)

- Substitute $z = -i$ into the modulus equation to see whether the value equals to 4:

$$\begin{aligned} &\left|z - 2\sqrt{3} - i\right|\Big|_{z=-i} \\ &= \left|-i - 2\sqrt{3} - i\right| \\ &= \left|-2\sqrt{3} - 2i\right| \\ &= |-2| \times \underbrace{\left|\sqrt{3} - i\right|}_{(1,\sqrt{3},2)\text{ triangle}} \\ &= 2 \times 2 = 4 \end{aligned}$$

(b) (2 marks)

✓ [1] for correct coordinates of the centre

✓ [1] for obtaining argument of $2\sqrt{3} + 2i$

- As the circle's centre is $(2\sqrt{3}, 1)$, substitute $2\sqrt{3}+i$ into the argument expression to see if it evaluates to $\frac{\pi}{6}$:

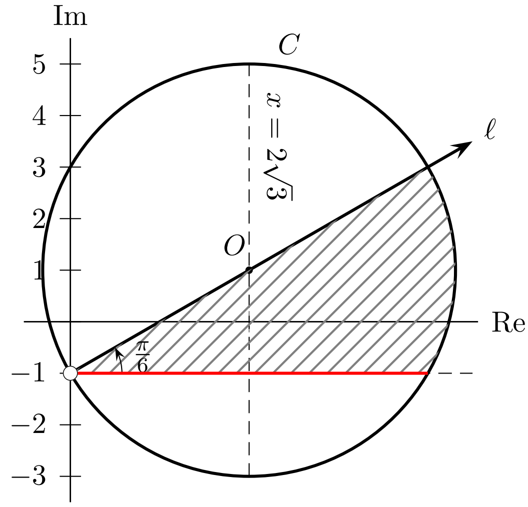
$$\begin{aligned} \text{Arg}\left(2\sqrt{3} + i + i\right) &= \text{Arg}\left(2\sqrt{3} + 2i\right) \\ &= \text{Arg}\left(2\left(\sqrt{3} + i\right)\right) \\ &= \frac{\pi}{6} \end{aligned}$$

(c) (2 marks)

✓ [1] for circle at centre $\left(2\sqrt{3}, 1\right)$ and radius of 4.

✓ [1] for correct ray emanating from $z = -i$ angled at $\frac{\pi}{6}$

✓ [−1] for not having an open circle at −1.



(d) (2 marks)

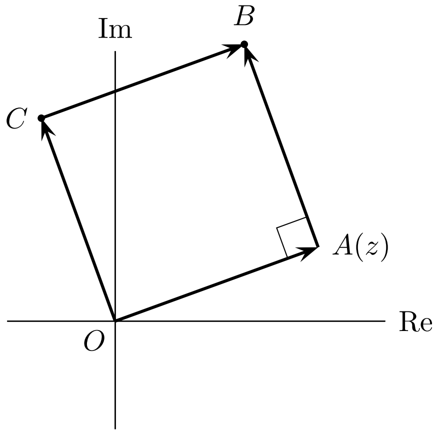
✓ [2] for fully correct shading, including the solid line at $y = -1$ within the circle.

✓ [−1] for each newly introduced error.

Question 4 (Lam)

(a) (2 marks)

✓ [1] for each of $\overrightarrow{OC}$ and $\overrightarrow{OB}$ in terms of z.



- $\overrightarrow{AB}$ is $\overrightarrow{OA}$ rotated anticlockwise by $90°$, i.e.
$$\overrightarrow{AB} = iz$$
- Also $\overrightarrow{AB} = \overrightarrow{OC} = iz$ (opposite sides of a square).
- Hence,

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= z + iz\\ &= z(1+i)\end{aligned}$$

(b) (2 marks)

✓ [1] for the correct real part based on the previous answer.

✓ [1] for the correct imaginary part based on the previous answer.

$$\underset{\div(1+i)}{4+2i} = \underset{\div(1+i)}{z(1+i)}$$

$$\begin{aligned}z &= \frac{4+2i}{1+i}\\ &= \frac{2(2+i)}{1+i} \qquad \times \tfrac{1-i}{1-i}\\ &= \frac{\cancel{2}(2+i)(1-i)}{\cancel{1+1}}\\ &= 2-2i+i+1\\ &= 3-i\end{aligned}$$

Question 5 (Lam)

(a) i. (1 mark)

$$\beta = 2 - 2\sqrt{3}i$$

ii. (2 marks)

✓ [1] for obtaining the value of $\alpha\beta$.

✓ [1] for correct final answer.

$$2z^3 + pz^2 + qz + 16 = 0$$

$$\begin{aligned}\alpha\beta &= \left(2+2\sqrt{3}i\right)\left(2-2\sqrt{3}i\right)\\ &= 4 + 4(3)\\ &= 16\end{aligned}$$

Apply the sum of roots:

$$\begin{aligned}\alpha\beta\gamma &= -\frac{d}{a}\\ \therefore 16\gamma &= -\frac{16}{2} = -8\\ \gamma &= -\frac{1}{2}\end{aligned}$$

iii. (2 marks)

✓ [1] for each correct value of p and q.

Apply the sum of roots:

$$\alpha + \beta + \gamma = -\frac{p}{2}$$

$$\left(2+2\sqrt{3}i\right) + \left(2-2\sqrt{3}i\right) - \frac{1}{2} = -\frac{p}{2}$$

$$\frac{7}{2} = -\frac{p}{2}$$

$$\therefore p = -7$$

Apply the pairs of roots:

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{q}{2}$$

$$\alpha\beta + \gamma(\alpha+\beta) = 16 + \left(-\frac{1}{2}\right)(4) = \frac{q}{2}$$

$$\therefore q = 28$$

iv. (1 mark)

$$\begin{aligned}\left(2+2\sqrt{3}i\right)^n &= \left(2\left(1+\sqrt{3}\right)\right)^n\\ &= 2^n\left(2e^{i\frac{\pi}{3}}\right)^n\\ &= 2^n\left(2^n e^{i\frac{n\pi}{3}}\right)\\ &= 2^{2n}\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)\\ &= 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)\end{aligned}$$

v. (2 marks)

✓ [1] for obtaining the value of $\alpha\beta$.

✓ [1] for correct final answer.

$$\alpha^n = \left(2+2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)$$
$$\beta^n = \left(2-2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$$
$$\gamma^n = \left(-\frac{1}{2}\right)^n$$
$$\begin{aligned}&\therefore \alpha^n+\beta^n+\gamma^n\\ &= 4^n\left(2\cos\frac{n\pi}{3}\right)+\left(-\frac{1}{2}\right)^n\\ &= 2^{2n}\times 2^1\cos\frac{n\pi}{3}+\left(-\frac{1}{2}\right)^n\\ &= 2^{2n+1}\cos\frac{n\pi}{3}+\left(-\frac{1}{2}\right)^n\end{aligned}$$

(b) i. (2 marks)

✓ [1] for rewriting $e^{i\theta}$ into modulus-argument form.

✓ [1] for proper communication of the result.

$$\begin{aligned}&e^{i\theta}+e^{-i\theta}\\ &=\cos\theta+\cancel{i\sin\theta}+\cos\theta-\cancel{i\sin\theta}\\ &=2\cos\theta\end{aligned}$$

ii. (2 marks)

✓ [1] for finding $\theta=\frac{\pi}{4}$

✓ [1] for proper communication of the result, including showing why $z^{10}+z^{-10}=2\cos(10\theta)$

$$2\cos\theta=\sqrt{2}$$
$$\therefore\cos\theta=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$
$$\therefore z=e^{i\frac{\pi}{4}}$$
$$\therefore z^{10}=e^{i\frac{10\pi}{4}}=e^{i\frac{\pi}{2}}=i$$
$$z^{-10}=\left(z^{10}\right)^{-1}=\frac{1}{i}=-i$$
$$\therefore z^{10}+z^{-10}=i+(-i)=0$$

Note: terminating error if student claims

$$z^{10}+z^{-10}=\left(z+z^{-1}\right)^{10}=2\cos(10\theta)$$

Question 6 (Lam)

(a) (3 marks)

✓ [1] for obtaining $e^{i\frac{\pi+2k\pi}{4}}$, $k=0,1,2,3$.

✓ [1] for enumerating through values of k

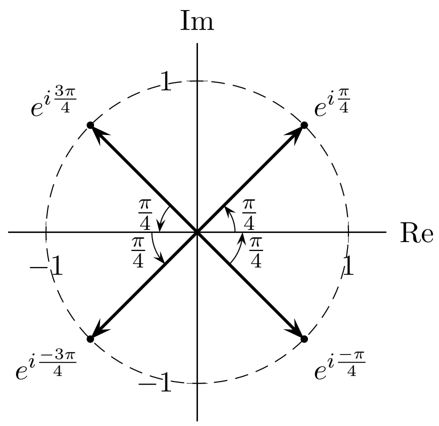
✓ [1] for reverting to principal argument and rewriting in modulus-argument form.

$$z^4+1=0$$
$$\therefore z^4=-1=e^{i(\pi+2k\pi)}$$
$$z=e^{i\frac{\pi+2k\pi}{4}}\quad k=0,1,2,3$$

Enumerate through values of k and reverting to principal argument:

$$\begin{aligned}k=0&\quad z=e^{i\frac{\pi}{4}}\\ k=1&\quad z=e^{i\frac{3\pi}{4}}\\ k=2&\quad z=e^{i\frac{5\pi}{4}}=e^{-i\frac{3\pi}{4}}\\ k=3&\quad z=e^{i\frac{7\pi}{4}}=e^{-i\frac{\pi}{4}}\end{aligned}$$
$$\therefore z=\cos\left(\pm\frac{\pi}{4}\right)+i\sin\left(\pm\frac{\pi}{4}\right),\quad \cos\left(\pm\frac{3\pi}{4}\right)+i\sin\left(\pm\frac{3\pi}{4}\right)$$

(b) (1 mark)



(c) (1 mark)

- Shape formed is a rhombus.
- Diagonal length is 2 units.
- $A=\frac{1}{2}xy=\frac{1}{2}\times2\times2=2$ units^2.