NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2020 Year 12 Course Assessment Task 3

Tuesday, 30 June 2020

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Commence each new question on a new Writing Booklet. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, arrange the booklets used in the correct order after this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

Class (please ✔)

◯ 12MXX.1 – Ms Ham

◯ 12MXX.2 – Mr Lam

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	$\overline{13}$	$\overline{17}$	$\overline{10}$	$\overline{40}$	

Question 1 (13 Marks) **Marks**

(a) Prove that **3**

A three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Hint: Let the digits be a, b and c.

(b) Prove the following statement by contrapositive: **2**

For $n \in \mathbb{Z}$, if $n^2 - 6n + 5$ is even, then n is odd.

(c) If a, b, c and d are all positive real numbers,

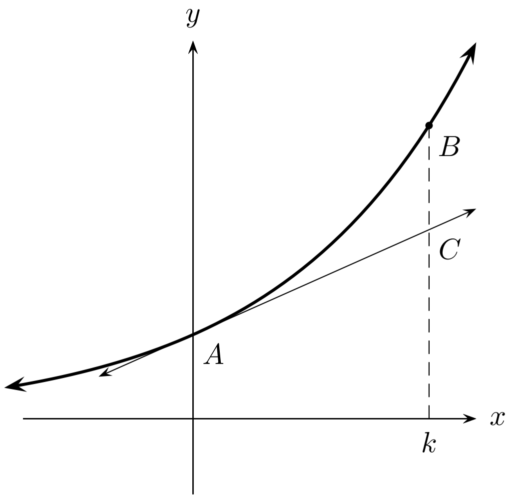
i. Show that **2**

$$a^2 + b^2 + c^2 + d^2 \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

ii. Hence or otherwise, show that **2**

$$\left(\frac{a+b+c+d}{4}\right)^2 \geq \frac{1}{6}(ab + ac + ad + bc + bd + cd)$$

(d) The curve $y = e^x$ cuts the y axis at A. B is a second point on the curve such that $x = k$, where $k > 0$. The tangent to the curve $y = e^x$ at A cuts the vertical line $x = k$ at the point C.



i. By considering areas, show that **3**

$$\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k\left(1 + e^k\right)$$

ii. Hence deduce that $2.5 < e < 3$. **1**

Question 2 (17 Marks) **Marks**

(a) i. Find the partial fraction decomposition of $\dfrac{1}{x^2-1}$. **2**

ii. Hence or otherwise, evaluate $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$. **2**

(b) Use the substitution $x = \tan^2\theta$ to evaluate **4**

$$\int_0^3 \frac{\sqrt{x}}{(1+x)^{\frac{5}{2}}}\,dx$$

Expressing your answer in simplest exact form.

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x+2\sin x}\,dx$$

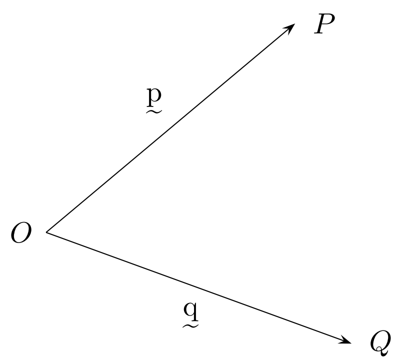
(d) i. Given $I_n = \displaystyle\int_0^1 x^n 2^x\,dx$ where $n \in \mathbb{Z}^+$, show that **2**

$$I_n = \frac{2}{\ln 2} - \frac{n}{\ln 2} I_{n-1}$$

ii. Hence evaluate $\displaystyle\int_0^1 x^3 2^x\,dx$. **3**

Question 3 (10 Marks) **Marks**

(a) In the following diagram, let $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$.



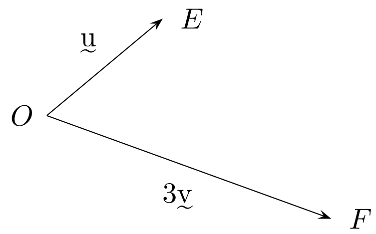
i. Reproduce the diagram on to your writing booklet. In addition, draw the vector **1**

$$\overrightarrow{OR} = \underset{\sim}{p} + \underset{\sim}{q}$$

ii. State the conditions on $\left|\underset{\sim}{p}\right|$ and $\left|\underset{\sim}{q}\right|$ such that $\overrightarrow{OR}$ bisects $\angle POQ$, giving brief reason(s). **2**

Examination continues overleaf...

(b) The following figure shows $\overrightarrow{OE} = \underset{\sim}{u} = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}$ and $\overrightarrow{OF} = 3\underset{\sim}{v} = \begin{pmatrix} 15 \\ 21 \\ 6 \end{pmatrix}$.



i. Find a vector $\overrightarrow{OG}$ in column vector notation that bisects $\angle EOF$. **2**

ii. State the name of the shape formed by $OEGF$, and a brief reason for why $OEGF$ forms this shape. **2**

iii. Show that the area A of, $OEGF$ is **3**

$$A = 2\sqrt{5123}$$

Hint: Consider $\text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$.

End of paper.

Suggested Band E4 Responses

Question 1

(a) (3 marks)

✓ [1] to define the sufficient and necessary conditions.

✓ [1] for proving the forward statement.

✓ [1] for proving the converse statement.

- Let the number n be

$$100a + 10b + c$$

such that a, b, c are single digit numbers from 0 to 9 with $a \neq 0$.

  – Let p be the statement

    The number $100a + 10b + c$ is divisible by 9

  – Let q be the statement

    The sum of its digits $a + b + c$ is divisible by 9

- **Forward statement:** $p \to q$
If

$$n = 100a + 10b + c$$

is divisible by 9, then

$$100a + 10b + c = 9A$$

where $A \in \mathbb{Z}^+$, then

$$\begin{aligned} &100a + 10b + c - (99a + 9b) \\ &= 9A - 9(11a + b) \\ &= 9(A - 11a + b) \\ &= 9B \end{aligned}$$

where $B \in \mathbb{Z}$ is also divisible by 9. Therefore,

$$100a + 10b + c - (99a + 9b) = a + b + c$$

is divisible by 9, i.e. the sum of its digits $a + b + c$ is divisible by 9.

- **Converse statement:** $q \to p$
If $a + b + c$ is divisible by 9, then

$$a + b + c = 9A \quad A \in \mathbb{Z}^+$$

$$\begin{aligned} &a + b + c + (99a + 9b) \\ &= 9A + 99a + 9b \\ &= 9(A + 11a + b) \end{aligned}$$

is also divisible by 9, as $99a + 9b = 9(11a + b)$ is divisible by 9. Therefore,

$$a + b + c + 99a + 9b = 100a + 10b + c$$

is divisible by 9, i.e. n is divisible by 9.

Hence $p \leftrightarrow q$, and a number is divisible by 9 iff the sum of its digits is divisible by 9.

(b) (2 marks)

- Let p be the statement

  $n^2 - 6n + 5$ is even

- Let q be the statement

  n is an odd number

Required to prove:

$$\neg q \to \neg p$$

i.e. if n is not an odd number, then $n^2 - 6n + 5$ is not even, or

If n is an even number, then $n^2 - 6n + 5$ is odd.

Proof Let $n = 2k$ where $k =\in \mathbb{Z}$.

$$\begin{aligned} &n^2 - 6n + 5 \\ &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 2\left(2k^2 - 6k + 2\right) + 1 \\ &= 2B + 1 \end{aligned}$$

where $B \in \mathbb{Z}$. $\therefore \neg q \to \neg p$. The contrapositive of the statement is true, hence

$$p \to q$$

is a true statement.

(c) i. (2 marks)

✓ [1] Uses $(a-b)^2 \geq 0$ and its subsequent derivation for $a^2 + b^2 \geq 2ab$.

✓ [1] Uses the fact, enumerates through pairs and deduces the result.

As $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$,

$$\therefore a^2 + b^2 \geq 2ab \qquad (1.1)$$

Similarly,

$$\begin{aligned} a^2+c^2 &\geq 2ac && (1.2)\\ a^2+d^2 &\geq 2ad && (1.3)\\ b^2+c^2 &\geq 2bc && (1.4)\\ b^2+d^2 &\geq 2bd && (1.5)\\ c^2+d^2 &\geq 2cd && (1.6) \end{aligned}$$

Add (1.1) to (1.6) and factorising,

$$3\left(a^2+b^2+c^2+d^2\right)$$
$$\geq 2(ab+ac+ad+bc+bd+cd)$$

$$\therefore a^2+b^2+c^2+d^2$$
$$\geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

ii. (2 marks)

✓ [1] Correctly expands $(a+b+c+d)^2$.

✓ [1] Uses the expansion and deduces the result.

$$(a+b+c+d)^2$$
$$= a^2+b^2+c^2+d^2$$
$$+ 2(ab+ac+ad+bc+bd+cd) \qquad (1.7)$$

From part (i) and (1.7),

$$\begin{aligned} &(a+b+c+d)^2\\ &\geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)\\ &\quad + 2(ab+ac+ad+bc+bd+cd)\\ &= \frac{8}{3}(ab+ac+ad+bc+bd+cd) \end{aligned}$$

Dividing $(a+b+c+d)^2$ by 16,

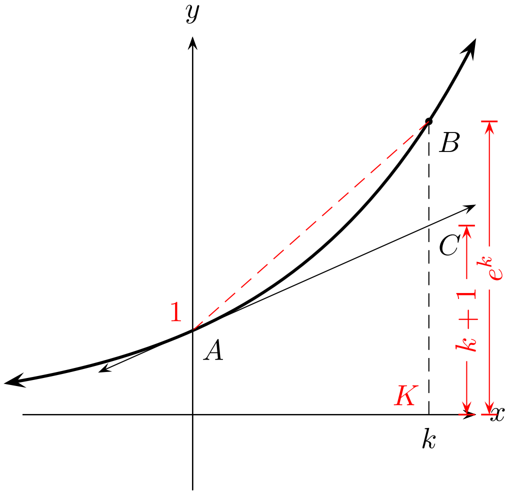
$$\begin{aligned} &\left(\frac{a+b+c+d}{4}\right)^2\\ &\geq \frac{1}{16}\times\frac{8}{3}(ab+ac+ad+bc+bd+cd)\\ &= \frac{1}{6}(ab+ac+ad+bc+bd+cd) \end{aligned}$$

(d) i. (3 marks)

✓ [1] Finds the lower bound for e^k-1 by comparing the definite integral with the underestimate.

✓ [1] Find the upper bound for e^k-1 by comparing the definite integral with the overestimate.

✓ [1] For comparing the exact area with the lower/upper bounds, and concluding the result.



- Equation of tangent at A:

$$\frac{dy}{dx} = e^x\Big|_{x=0} = 1$$
$$\therefore y = x+1$$

At $x=k$,

$$y = x+1\Big|_{x=k} = k+1$$
$$\therefore C(k, k+1)$$

- Underestimate of area: trapezium $AOKC$, where O is the origin:

$$\begin{aligned} A_{AOKC} &= \frac{1}{2}h(a+b)\\ &= \frac{1}{2}(k)(1+k+1)\\ &= \frac{1}{2}k(k+2) \end{aligned}$$

- Overestimate of area: trapezium $AOKB$, where O is the origin:

$$\begin{aligned} A_{AOKB} &= \frac{1}{2}h(a+b)\\ &= \frac{1}{2}(k)(1+e^k) \end{aligned}$$

- Exact area:

$$\begin{aligned} A &= \int_0^k e^x\,dx \\ &= \left[e^x\right]_0^k \\ &= e^k - e^0 \\ &= e^k - 1 \end{aligned}$$

The exact area is bounded by the underestimate and overestimate:

$$A_{AOKC} < \int_0^k e^x\,dx < A_{AOKB}$$
$$\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k\left(1+e^k\right)$$

ii. (1 mark) When $k=1$,

$$\frac{1}{2}(3)+1 < e^1 < \frac{1}{2}(1+e)+1$$
$$\therefore \frac{5}{2} < \underbrace{e < \frac{1}{2}(3+e)}_{\text{examine this}}$$
$$\underset{-e}{2e} < 3 + \underset{-e}{e}$$
$$\therefore e < 3$$
$$\therefore 2.5 < e < 3$$

Question 2

(a) i. (2 marks)
✓ [1] for each of A and B.

$$\underset{\times(x-1)(x+1)}{\frac{1}{(x-1)(x+1)}} \equiv \underset{\times(x-1)(x+1)}{\frac{A}{x-1}} + \underset{\times(x-1)(x+1)}{\frac{B}{x+1}}$$
$$1 \equiv A(x+1) + B(x-1)$$

- When $x=1$,

$$1 = A(1+1) + B(1-1)$$
$$\therefore A = \frac{1}{2}$$

- When $x=-1$,

$$1 = A(-1+1) + B(-1-1)$$
$$\therefore B = -\frac{1}{2}$$

$$\therefore \frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

ii. (3 marks)
✓ [1] for applying technique to split integrand.
✓ [1] for applying result from previous part.
✓ [1] for all correct primitives, subtracting 1 mark for each new error introduced.

$$\begin{aligned} &\int \frac{x^2+1}{x^2-1}\,dx \\ &= \int \frac{x^2-1+2}{x^2-1}\,dx \\ &= \int \left(1 + \frac{2}{x^2-1}\right)dx \\ &= \int \left(1 + 2\left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right)\right)dx \\ &= \int \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right)dx \\ &= x + \ln(x-1) - \ln(x+1) + C \\ &= x + \ln\left(\frac{x-1}{x+1}\right) + C \end{aligned}$$

(b) (4 marks)
✓ [1] for correct $\frac{dx}{d\theta}$ transformation.
✓ [1] For correct limits.
✓ [1] For significant progress in reducing integrand to $\sin^2\theta\cos\theta$.
✓ [1] for final result.

$$\int_0^3 \frac{\sqrt{x}}{(1+x)^{\frac{5}{2}}}\,dx$$

Make the substitution $x = \tan^2\theta$:

$$x = \tan^2\theta$$
$$\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$$

Altering the limits,

$$\begin{aligned} x = 0 \quad \tan^2\theta &= 0 \\ \therefore \theta &= 0 \\ x = 3 \quad \tan^2\theta &= 3 \\ \tan\theta &= \sqrt{3} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

Replacing x with θ:

$$\begin{aligned}
&\int_0^3 \frac{\sqrt{x}}{(1+x)^{\frac{5}{2}}}\,dx\\
&= \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\tan\theta}{(1+\tan^2\theta)^{\frac{5}{2}}} \times 2\tan\theta\sec^2\theta\,d\theta\\
&= \int_0^{\frac{\pi}{3}} \frac{2\tan^2\theta\sec^2\theta}{(\sec^2\theta)^{\frac{5}{2}}}\,d\theta\\
&= \int_0^{\frac{\pi}{3}} \frac{2\tan^2\theta\cancel{\sec^2\theta}}{\sec^{\cancel{5}\,3}\theta}\,d\theta\\
&= 2\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \times \cos^{\cancel{3}}\theta\,d\theta\\
&= 2\int_0^{\frac{\pi}{3}} \sin^2\theta\cos\theta\,d\theta\\
&= \frac{2}{3}\left[\sin^3\theta\right]_0^{\frac{\pi}{3}}\\
&= \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{2}{3}\times\frac{3\sqrt{3}}{8}\\
&= \frac{\sqrt{3}}{4}
\end{aligned}$$

(c) (4 marks)

- ✓ [1] for correct differential and alteration of limits.
- ✓ [1] for correct manipulation to quadratic trinomial in the denominator.
- ✓ [1] for applying partial fractions.
- ✓ [1] for final answer.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x+2\sin x}\,dx$$

Making the substitution $t = \tan\frac{x}{2}$ and calculating the differential:

$$\begin{aligned}
t &= \tan\frac{x}{2}\\
\frac{x}{2} &= \tan^{-1}t\\
x &= 2\tan^{-1}t\\
\frac{dx}{dt} &= \frac{2}{1+t^2}
\end{aligned}$$

Altering the limits,

$$\begin{aligned}
x = 0 &\quad t = \tan 0\\
x = \frac{\pi}{2} &\quad t = \tan\frac{\frac{\pi}{2}}{2}\\
&\quad\;\; = \tan\frac{\pi}{4}\\
&\quad\;\; = 1
\end{aligned}$$

Replacing x with t:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x+2\sin x}\,dx\\
&= \int_{t=0}^{t=1} \frac{1}{2-\left(\frac{1-t^2}{1+t^2}\right)+2\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt\\
&= \int_0^1 \frac{2}{2(1+t^2)-(1-t^2)+4t}\,dt\\
&= \int_0^1 \frac{2}{3t^2+4t+1}\,dt
\end{aligned}$$

Decompose the integrand into partial fractions:

$$\frac{2}{3t^2+4t+1} = \frac{2}{(3t+1)(t+1)} \equiv \frac{A}{3t+1}+\frac{B}{t+1}$$
$$\therefore 2 \equiv A(t+1)+B(3t+1)$$

When $t=-1$,

$$\begin{aligned}
-2B &= 2\\
B &= -1
\end{aligned}$$

When $t=-\frac{1}{3}$,

$$\begin{aligned}
2 &= \frac{2}{3}A\\
A &= 3
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^1 \frac{2}{3t^2+4t+1}\,dt\\
&= \int_0^1 \left(\frac{3}{3t+1}-\frac{1}{t+1}\right)dt\\
&= \left[\ln(3t+1)-\ln(t+1)\right]_0^1\\
&= \left[\ln\left(\frac{3t+1}{t+1}\right)\right]_0^1\\
&= \ln\frac{4}{2}-\ln 1 = \ln 2
\end{aligned}$$

(d) i. (3 marks)

✓ [1] for applying integration by parts and correctly finding the values in the u-v matrix.

✓ [1] for correct evaluation of $[uv]_0^1$.

✓ [1] for correct evaluation of $\int_0^1 v\,du$.

$$I_n = \int_0^1 \underbrace{x^n}_{=u} \underbrace{2^x}_{=dv}\,dx$$

$$u = x^n \qquad v = \frac{1}{\ln 2}2^x$$

$$du = nx^{n-1} \qquad dv = 2^x$$

$$\begin{aligned}\therefore I_n &= \left[\frac{x^n 2^x}{\ln 2}\right]_0^1 - \int_0^1 nx^{n-1} \times \frac{1}{\ln 2}2^x\,dx \\ &= \left(\frac{2}{\ln 2} - 0\right) - \frac{n}{\ln 2}\int_0^1 x^{n-1}2^x\,dx \\ &= \frac{2}{\ln 2} - \frac{n}{\ln 2} I_{n-1}\end{aligned}$$

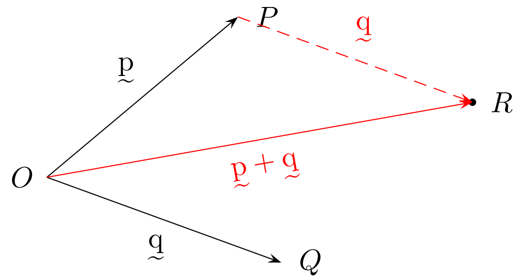
ii. (3 marks)

✓ [1] for each of I_1, I_2 and I_3.

$$\begin{aligned}I_0 &= \int_0^1 x^0 2^x\,dx = \int_0^1 2^x \\ &= \frac{1}{\ln 2}\left[2^x\right]_0^1 \\ &= \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2} \\ I_1 &= \frac{2}{\ln 2} - \frac{1}{\ln 2} I_0 \\ &= \frac{2}{\ln 2} - \frac{1}{\ln 2}\left(\frac{1}{\ln 2}\right) \\ &= \frac{2}{\ln 2} - \frac{1}{\ln^2 2} \\ I_2 &= \frac{2}{\ln 2} - \frac{2}{\ln 2} I_1 \\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln^2 2}\right) \\ &= \frac{2}{\ln 2} - \frac{4}{\ln^2 2} + \frac{2}{\ln^3 2} \\ I_3 &= \frac{2}{\ln 2} - \frac{3}{\ln 2} I_2 \\ &= \frac{2}{\ln 2} - \frac{3}{\ln 2}\left(\frac{2}{\ln 2} - \frac{4}{\ln^2 2} + \frac{2}{\ln^3 2}\right) \\ &= \frac{2}{\ln 2} - \frac{6}{\ln^2 2} + \frac{12}{\ln^3 2} - \frac{6}{\ln^4 2}\end{aligned}$$

Question 3

(a) i. (1 mark)



ii. (2 marks)

✓ [1] for reference to parallelogram and rhombus.

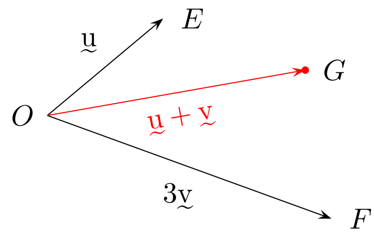
✓ [1] for correct logic and articulation of the proof.

- $OPRQ$ is a parallelogram as the addition of vectors produces a parallelogram of vectors.
- For a diagonal of a parallelogram to bisect its angles, it must be a rhombus.
- $\therefore \left|\underset{\sim}{p}\right| = \left|\underset{\sim}{q}\right|$ for $\overrightarrow{OR}$ to bisect $\angle POQ$.

(b) i. (2 marks)

✓ [1] for checking lengths of $\underset{\sim}{u}$ and $\underset{\sim}{v}$

✓ [1] for coordinates of $\overrightarrow{OG}$.



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- For $\angle EOF$ to be bisected, need $|\underset{\sim}{u}| = |\underset{\sim}{v}|$ (result from the previous part)
- Checking lengths of $\underset{\sim}{u}$ and $\underset{\sim}{v}$:

$$|\underset{\sim}{u}|^2 = \left|\begin{pmatrix}2\\5\\-7\end{pmatrix}\right|^2 = 2^2 + 5^2 + 7^2 = 78$$

$$|\underset{\sim}{v}|^2 = \left|\begin{pmatrix}5\\7\\2\end{pmatrix}\right|^2 = 5^2 + 7^2 + 2^2 = 78$$

As $|\underset{\sim}{u}| = |\underset{\sim}{v}|$ hence $\underset{\sim}{u} + \underset{\sim}{v}$ will bisect $\angle EOF$.

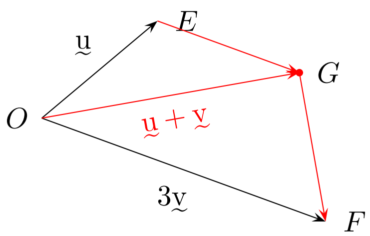
- Hence

$$\begin{aligned}\overrightarrow{OG} &= \underset{\sim}{u} + \underset{\sim}{v} \\ &= \begin{pmatrix}2\\5\\-7\end{pmatrix} + \begin{pmatrix}5\\7\\2\end{pmatrix} \\ &= \begin{pmatrix}7\\12\\-5\end{pmatrix}\end{aligned}$$

ii. (2 marks)

✓ [1] For using $\overrightarrow{EG}$ is equal to $\underset{\sim}{v}$

✓ [1] for the shape and articulation of reasoning.



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Shape is a trapezium.

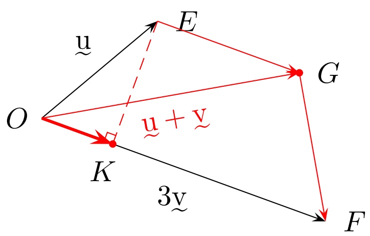
- $\overrightarrow{EG} = \underset{\sim}{v}$ (parallel to $\underset{\sim}{v}$)
- Shape has $EG \parallel OF$ - one pair of opposite sides parallel.

iii. (3 marks)

✓ [1] For finding the projection of $\underset{\sim}{u}$ on to $\underset{\sim}{v}$.

✓ [1] For applying Pythagoras' Theorem to find the length of KE.

✓ [1] for finding the area.



(NOT TO SCALE)

Vector $\overrightarrow{OK}$ is the projection of $\underset{\sim}{u}$ on to $\underset{\sim}{v}$.

$$\begin{aligned}\text{proj}_{\underset{\sim}{v}}\,\underset{\sim}{u} &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v} \\ &= \frac{1}{78}\left[\begin{pmatrix}2\\5\\-7\end{pmatrix}\cdot\begin{pmatrix}5\\7\\2\end{pmatrix}\right]\begin{pmatrix}5\\7\\2\end{pmatrix} \\ &= \frac{10+35-14}{78}\begin{pmatrix}5\\7\\2\end{pmatrix} \\ &= \frac{31}{78}\begin{pmatrix}5\\7\\2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore \left|\text{proj}_{\underset{\sim}{v}}\,\underset{\sim}{u}\right| &= \left|\frac{31}{78}\begin{pmatrix}5\\7\\2\end{pmatrix}\right| \\ &= \left|\frac{31}{78}\right|\left|\begin{pmatrix}5\\7\\2\end{pmatrix}\right| \\ &= \frac{31}{78}\sqrt{78} = \frac{31}{\sqrt{78}}\end{aligned}$$

Applying Pythagoras' Theorem to find the perpendicular height of the trapezium KE:

$$\begin{aligned}\left|\text{proj}_{\underset{\sim}{v}}\,\underset{\sim}{u}\right|^2 + KE^2 &= |\underset{\sim}{u}|^2 \\ \left(\frac{31}{\sqrt{78}}\right)^2 + KE^2 &= \left(\sqrt{78}\right)^2 \\ KE^2 = 78 - \frac{31^2}{78} &= \frac{5\,123}{78} \\ \therefore KE &= \sqrt{\frac{5123}{78}}\end{aligned}$$

Applying the formula for the area of a trapezium,

$$\begin{aligned}A &= \frac{1}{2}h(a+b) \\ &= \frac{1}{2}(KE)\left(|\underset{\sim}{v}| + |3\underset{\sim}{v}|\right) \\ &= \frac{1}{2}\left(\sqrt{\frac{5\,123}{78}}\right)\left(\sqrt{78} + 3\sqrt{78}\right) \\ &= 2\sqrt{5\,123}\end{aligned}$$