NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 Year 12 Course Mock Assessment Task 1

Wednesday, 15 December 2021

General instructions

- Working time – 1 hour.
  (plus 5 minutes reading time)
- Commence each new question on a new Writing Booklet. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

- ○ 12MXX.1 – Ms Ham
- ○ 12MXX.2 – Mr Sekaran
- ○ 12MXX.3 – Mr Sun

Marker's use only

QUESTION	1-8	9	10	11	Total	%
MARKS	$\overline{\ \ }$ 8	$\overline{\ \ }$ 10	$\overline{\ \ }$ 9	$\overline{\ \ }$ 11	$\overline{\ \ }$ 38	

Section I

8 marks Attempt Questions 1– 8
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1– 8.

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1. Let $z = a + ib$, where $a \neq 0$ and $b \neq 0$. Which statement is INCORRECT?

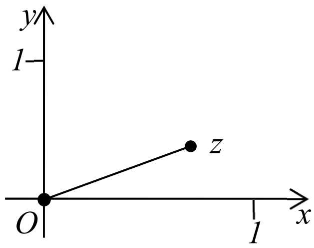
(A) $z - \bar{z} = 2bi$ (B) $|z|^2 = |z||\bar{z}|$

(C) $|z| + \overline{|z|} = |z + \bar{z}|$ (D) $\arg(z) + \arg(\bar{z}) = 0$

2. z satisfies the equation $\left|z - \sqrt{2} - i\sqrt{2}\right| = 1$. Find the minimum value of $\arg(z)$?

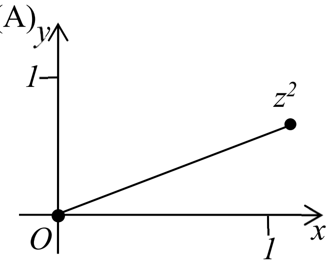
(A) -75^o (B) -15^o (C) 15^o (D) 75^o

3. The Argand diagram below shows the complex number z.

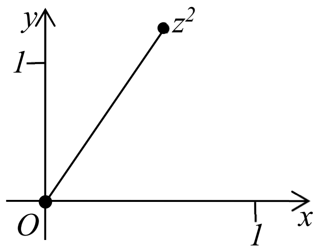


Which diagram best represents z^2?

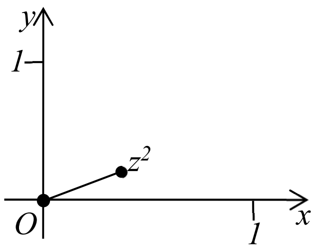
(A)



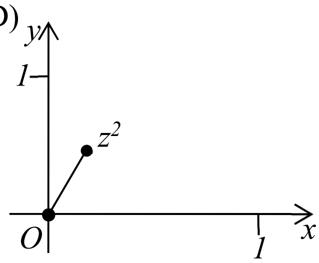
(B)



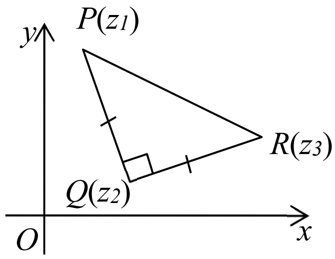
(C)



(D)



4. The vertices of the triangle PQR are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle PQR is isosceles and right-angled at Q, as shown in the diagram.



Which of the following statement is true?

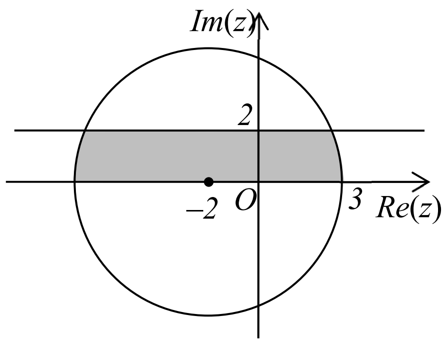
(A) $z_2 - z_1 = i(z_3 - z_2)$ (B) $z_1 - z_2 = i(z_3 - z_2)$

(C) $z_2 - z_1 = i(z_1 - z_3)$ (D) $z_1 - z_2 = i(z_1 - z_3)$

5. Let the cube roots of unity be 1, ω, ω^2. The value of $(1 - \omega - \omega^2)(1 + 2\omega + 2\omega^2)$ is

(A) 0 (B) ω (C) 2 (D) –2

6. A circle with centre (–2, 0) and radius 5 units is shown on an argand diagram below.

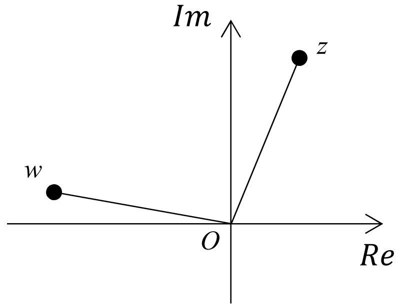


Which set of the following inequalities represents the shaded region?

(A) $Re(z) \leq 0$, $Im(z) \leq 2$ and $|z - 2| \leq 3$ (B) $Im(z) \geq 0$, $Im(z) \leq 2$ and $|z - 2| \geq 3$

(C) $Im(z) \geq 0$, $Im(z) \leq 2$ and $|z - 2| \leq 5$ (D) $Re(z) \leq 2$, $Im(z) \geq 0$ and $|z - 2| \leq 5$

7. The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant.



Which complex number could lie in the 3rd quadrant?

(A) $-w$ (B) $2iz$ (C) $\bar{z}$ (D) $w - z$

8. It is given that $z = 2 + i$ is a root of $z^3 + az^2 - 7z + 15 = 0$, where a is a real number.

What is the value of a?

(A) -1 (B) 1 (C) 7 (D) -7

Section II   31 marks

Attempt Questions 9 - 11
Allow about 50 minutes for this section
Answer each question in the appropriate writing booklet.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 9 **(10 marks)** **Marks**

(a) Find the complex square roots of $7+6\sqrt{2}i$ giving your answers in the form $x + iy$, where x and y are real. **3**

(b) If $z = \frac{1}{\sqrt{2}}(1-i)$, find z^{2022} in the form $a + ib$, where a and b are real. **2**

(c) i. On the same diagram, sketch the region of the complex plane defined by each of the following:

α. $|z-(3+2i)| = 2$ **1**

β. $|z+3| = |z-5|$. **1**

ii. Hence write down all values of z which satisfy simultaneously $|z-(3+2i)| = 2$ and $|z+3| = |z-5|$. **1**

iii. Use your diagram in (i) to determine the values of k for which the simultaneous equations $|z-(3+2i)| = 2$and $|z-2i| = k$ have exactly one solution for z. **2**

Question 10 (9 marks)

(a) It is known that $2 + 5i$ is a zero of the polynomial $P(z) = z^3 - 8z^2 + 45z - 116$.

i. Find the third zero of P(z). **2**

ii. Hence write $P(z)$ as a product of two factors with real coefficients. **2**

(b) Let $w = cos\frac{2\pi}{9} + i\,sin\frac{2\pi}{9}$.

i. Show that w^k is a solution of $z^9 - 1 = 0$, where k is an integer. **1**

ii. Prove that $w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$. **1**

iii. Write $w^9 - 1$ as a product of one linear and four quadratic factors, all with real coefficients. **1**

vi. Hence or otherwise, show that $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}$. **2**

Question 11 (11 marks)

(a) Suppose that $z = 4\sqrt{3}e^{\frac{i\pi}{3}} - 4e^{\frac{i5\pi}{6}}$.

i. Simplify z, writing your answer in exponential form. **2**

ii. Show that $\left(\frac{z}{8}\right)^3 + i\left(\frac{z}{8}\right)^2 + \frac{z}{8} = 2i$, clearly show all working. **2**

iii. Find the three cube roots of z in exponential form. **2**

(b) Let the points $A_1, A_2, \ldots, A_n$ represent the nth roots of unity, $w_1, w_2 \ldots, w_n$, and suppose P represents any complex number z such that $|z| = 1$.

i. Prove that $w_1 + w_2 + \ldots + w_n = 0$. **1**

ii. Show that $PA_i^2 = (z - w_i)(\overline{z} - \overline{w_i})$ for $i = 1, 2, \ldots, n$. **1**

iii. Prove that $\sum_{z=1}^{n} PA_i^{\ 2} = 2n$. **3**

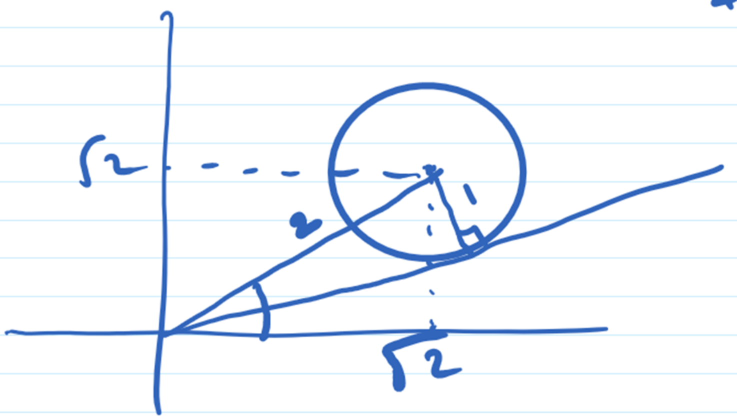
End of paper

Sample solutions mock assessment task 1-2022

Q1 C

Q2 C $|z-\sqrt{2}-\sqrt{2}i|=1$

$4+2=\sqrt{6}$



$\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{2}\right)$

$=45^\circ-30^\circ$

$=15^\circ$

Q3 D

Q4 B

Q5 D $(1-\omega-\omega^2)(1+2\omega+2\omega^2)$

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$1+\omega+\omega^2=0$

$=(1+1)(1+2(-1))$

$=-2$

Q6 C

Q7 D

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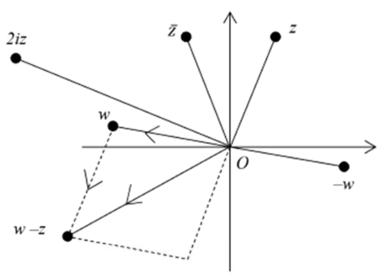

Q8 A

$z=2+i, z^2=4+4i+i^2=3+4i,$

$z^2(z+a)-7z+15=0,$

$(3+4i)(2+i+a)-7(2+i)+15$

$\Rightarrow 6+3i+3a+8i-4+4ai-14-7i+15=0$

$\Rightarrow 3a+4ai=-3-4i \Rightarrow a=-1 \therefore \text{A}$

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Q9

a) $(x+iy)^2 = 7+6\sqrt{2}i,\ x^2-y^2 = 7$ and $2xy = 6\sqrt{2}$

Screer i) $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 7^2 + (6\sqrt{2})^2 = 121$

$x^2+y^2 = 11 (x^2+y^2 \geq 0),\ 2x^2 = 18,\ x = \pm 3,\ y = \pm\sqrt{2}$

So the two square roots are $\pm(3+i\sqrt{2})$

b) $z = \frac{1}{\sqrt{2}}(1-i) = e^{-i\frac{\pi}{4}}$

$504 + \frac{3}{2}$

$z^{2022} = e^{-i\frac{1011}{2}\pi}$

$= 1 \cdot e^{-i\frac{3\pi}{2}} = -i(-1)$

$= i = 0+i$

c)

α) The locus is a circle centre $3+2i$ and radius 2.

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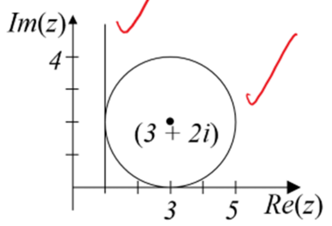
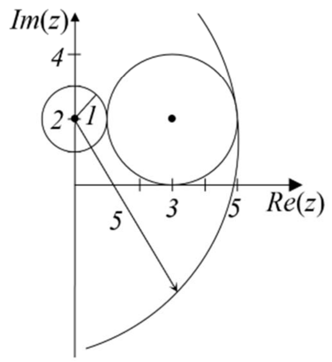


β) the locus is the points equidistant from –3 and 5. ie $x = 1$

ii) On one point: $z = 1+2i$

iii) $|z-2i| = k$ represents a circle, centre $2i$ and radius k. This circle touches the first circle if $k = 1$ or $k = 5$



Question 10 **(8 marks)**

(a) It is known that $2 + 5i$ is a zero of the polynomial $P(z) = z^3 - 8z^2 + 45z - 116$.

i. Find the third zero of P(z). **2**

ii. Hence write $P(z)$ as a product of two factors with real coefficients. 2

i) zeros are $2+5i,\ 2-5i,\ \alpha$

$\therefore\ 2+5i+2-5i+\alpha = 8$

$\alpha = 4$

$\therefore$ third zero is 4

ii) $P(z) = (z-(2+5i))(z-(2-5i))(z-4)$

$= (z^2-(2+5i+2-5i)z+(2+5i)(2-5i))(z-4)$

$= (z^2-4z+29)(z-4)$

b)

i) $w^k = \cos\frac{2\pi k}{9} + i\sin\frac{2\pi k}{9}$

$\therefore z^9 - 1 = (w^k)^9 - 1$

$= (\cos 2\pi k + i\sin 2\pi k) - 1 = (1+0) - 1 = 0$

If k is an integer, w^k is a solution to $z^9 - 1 = 0$

ii) Using the sum of a G.P $w + w^2 + w^3 + \dots + w^8$

$= \frac{w(1-w^8)}{1-w} = \frac{w-w^9}{1-w} = \frac{w-1}{1-w} = -1$

iii) roots are $w_1 = 1$

$w_2 = \text{Cis}\frac{2\pi}{9}$, $w_3 = \overline{w_2}$

$w_4 = \text{Cis}\frac{4\pi}{9}$, $w_5 = \overline{w_4}$

$w_6 = \text{Cis}\frac{6\pi}{9}$, $w_7 = \overline{w_6}$

$w_8 = \text{Cis}\frac{8\pi}{9}$, $w_9 = \overline{w_8}$

$w^9 - 1 = (w-1)(w-\text{Cis}\frac{2\pi}{9})(w-\overline{\text{Cis}\frac{2\pi}{9}})(w-\text{Cis}\frac{4\pi}{9})(w-\overline{\text{Cis}\frac{4\pi}{9}})$
$(w-\text{Cis}\frac{6\pi}{9})(w-\overline{\text{Cis}\frac{6\pi}{9}})(w-\text{Cis}\frac{8\pi}{9})(w-\overline{\text{Cis}\frac{8\pi}{9}})$

$= (w-1)(w^2-2\cos\frac{2\pi}{9}w+1)(w^2-2\cos\frac{4\pi}{9}w+1)(w^2-2\cos\frac{6\pi}{9}w+1)(w^2-2\cos\frac{8\pi}{9}w+1)$

iv) Sub $w = i$

$i - 1 = (i-1)(-2\cos\frac{2\pi}{9}i)(-2\cos\frac{4\pi}{9}i)(-2\cos\frac{6\pi}{9}i)(-2\cos\frac{8\pi}{9}i)$

$\therefore\ 1 = (-2)^4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\frac{1}{2}\right)\left(-\cos\frac{\pi}{9}\right)$

$\therefore \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{\pi}{9} = \frac{1}{8}$

Alternate method

iv) ~~iii)~~ $z^9 - 1 = 0$ has 9 distinct roots: $w^k = \cos\frac{2\pi k}{9} + i\sin\frac{2\pi k}{9}$, where $k = 0,1,2\ldots9$

From (ii): $(w+w^8)+(w^2+w^7)+(w^3+w^6)+(w^4+w^5) = -1$

Note that in each bracket there is a pair of conjugate complex numbers.

For example: $w + w^8 = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} + \cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}$

$= \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} + \cos(\frac{-2\pi}{9}) + i\sin(\frac{-2\pi}{9}) = 2\cos\frac{2\pi}{9}.$

The other brackets yeild similar results.

$$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{6\pi}{9} + 2\cos\frac{8\pi}{9} = -1$$

$$2(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}) + 2(\cos\frac{6\pi}{9} + \cos\frac{8\pi}{9}) = -1$$

$$4\cos\frac{\pi}{9}\cos\frac{3\pi}{9} + 4\cos\frac{\pi}{9}\cos\frac{7\pi}{9} = -1$$

$$4\cos\frac{\pi}{9}(\cos\frac{3\pi}{9} + \cos\frac{7\pi}{9}) = -1$$

$$8\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\cancel{4}\,5\pi}{9} = -1$$

$$-\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = -\frac{1}{8} \Rightarrow \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$$

$\therefore \cos\frac{2\pi}{9}$ co

Question 11

z

$= 4\sqrt{3}e^{\frac{i\pi}{3}} - 4e^{\frac{5i\pi}{6}}$

$= 4\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 4\sqrt{3}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= 4\sqrt{3} + 4i$

$= 8\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= 8e^{\frac{i\pi}{6}}$

Using part a we have

$\frac{z}{8} + i\left(\frac{z}{8}\right)^2 + \left(\frac{z}{8}\right)^3$

$= \frac{8e^{\frac{i\pi}{6}}}{8} + i\left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^2 + \left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^3$

$= e^{\frac{i\pi}{6}} + i\left(e^{\frac{i\pi}{6}}\right)^2 + \left(e^{\frac{i\pi}{6}}\right)^3$

$= e^{\frac{i\pi}{6}} + ie^{\frac{i\pi}{3}} + e^{\frac{i\pi}{2}}$

$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + i\cos\frac{\pi}{3} + i^2\sin\frac{\pi}{3} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{i}{2} - \frac{\sqrt{3}}{2}i + 0 + i$

$= 2i$

Let $\lambda = re^{i\theta}$ be a cube root of z. It follows that

$\lambda^3 = r^3e^{3i\theta} = 8e^{\frac{i\pi}{6}}$

Comparing modulus, we see that $r^3 = 8$ and so $r = 2$. Then comparing argument, we see that $3\theta = 2n\pi + \frac{\pi}{6} = \frac{(12n+1)\pi}{6}$, where n is an integer. Hence,

$\theta = \frac{(12n+1)\pi}{18}$

Thus, the roots are of the form

$\lambda = 2e^{\frac{(12n+1)\pi}{18}}$ where n is an integer

Taking $n = -1, 0, 1$, we see that the three cube roots of z are,

$\lambda = 2e^{-\frac{11i\pi}{18}}, 2e^{\frac{i\pi}{18}}, 2e^{\frac{13i\pi}{18}}$

i) $w_1, w_2, \dots w_n$ are solutions of the polynomial $z^n - 1 = 0$. Sum of roots: $w_1 + w_2 + \dots w_n = \frac{0}{1} = 0$

ii) $PA_i = |z - w_i|, PA_i^2 = |z - w_i|^2$

$= (z - w_i)\overline{(z - w_i)} = (z - w_i)(\bar{z} - \bar{w}_i)$

$$PA_i^2 = (w_i - z)(\bar{w}_i - \bar{z})$$

$$= w_i\bar{w}_i - w_i\bar{z} - \bar{w}_i z + z\bar{z}$$

$$= \underbrace{|w_i|^2}_{=1} - w_i\bar{z} - \bar{w}_i z + \underbrace{|z|^2}_{=1}$$

$$= 2 - w_i\bar{z} - \bar{w}_i z$$

$$\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n}(2 - w_i\bar{z} - \bar{w}_i z)$$

$$= \sum_{i=1}^{n} 2 - \sum_{i=1}^{n} w_i\bar{z} - \sum_{i=1}^{n} \bar{w}_i z$$

$$= 2n - \bar{z}\sum_{i=1}^{n} w_i - z\sum_{i=1}^{n} \bar{w}_i$$

$$= 2n - \bar{z}\cdot 0 - z\,\underbrace{\overline{(w_1 + w_2 \dots + w_n)}}_{=0}$$

$$= 2n$$

$$\sum_{i=1}^{n} 2 = 2 + 2 + \dots + 2$$

$$= 2n$$