NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 Year 12 Course Assessment Task 2

Tuesday March 23, 2021

General instructions

- Working time – 1 hour.
  (plus 5 minutes reading time)
- Commence each new question on a new Writing Booklet. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle any additional writing booklets used in the correct order within this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

Class (please ✔)

◯ 12MXX.1 – Mr Sekaran

◯ 12MXX.2 – Ms Ham

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	$\overline{11}$	$\overline{11}$	$\overline{13}$	$\overline{35}$	

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – set of all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers
- $\mathbb{C}$ – set of all complex numbers

Question 1 (11 Marks) **Marks**

(a) Expand and fully simplify: **2**

$$\left(z - 2e^{i\frac{2\pi}{3}}\right)\left(z - 2e^{-i\frac{2\pi}{3}}\right)$$

(b) i. Fully simplify **2**

$$\frac{2(1 - i\tan 2\theta)}{1 + i\tan 2\theta}$$

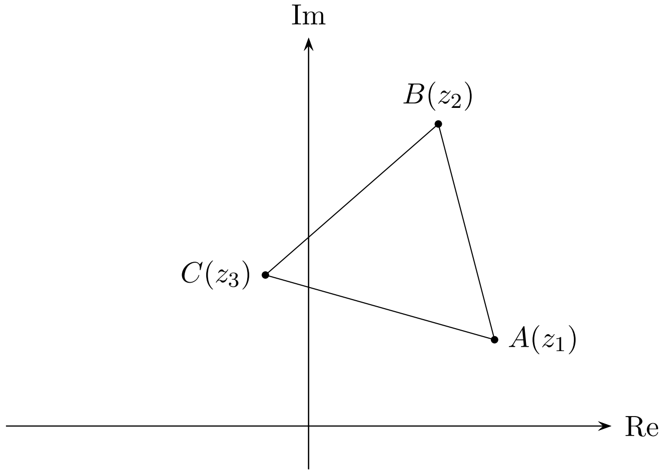
Express your answer in the complex exponential form $re^{i\alpha}$.

ii. Hence, or otherwise, rewrite $\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}$ in the complex exponential form. **1**

(c) Use $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$ to prove **2**

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

(d) In the Argand diagram below, ABC is an equilateral triangle. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.



i. Explain why $z_3 - z_1 = (z_2 - z_1)e^{i\frac{\pi}{3}}$. **1**

ii. By writing a similar expression for $z_1 - z_2$, or otherwise, deduce that **3**

$$(z_1)^2 + (z_2)^2 + (z_3)^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Examination continues overleaf...

Question 2 (11 Marks) **Marks**

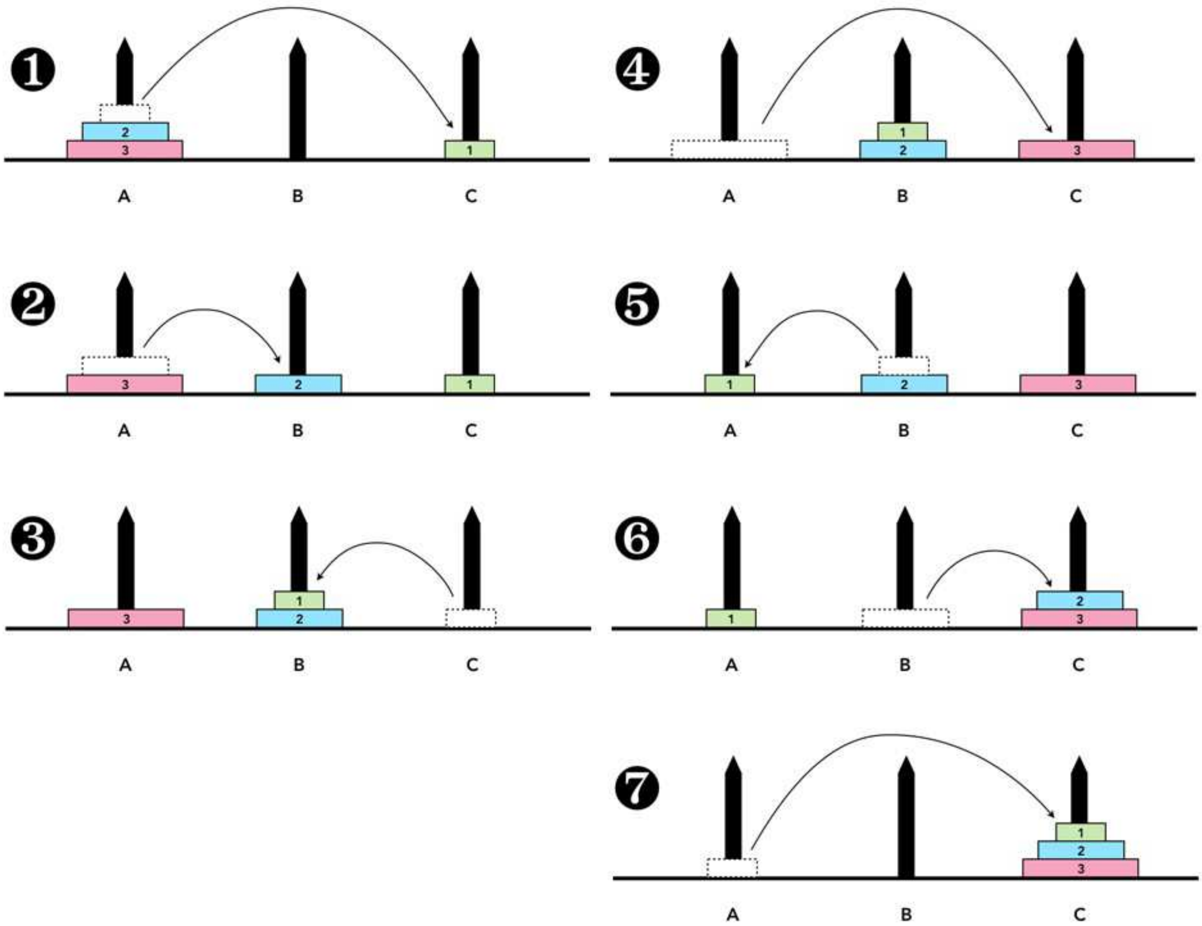
(a) Use mathematical induction to prove for $n \in \mathbb{Z}^+$, **3**

$$\frac{3^n + 5^n}{2} \geq 4^n$$

(b) *The towers of Hanoi* is a classical puzzle. You have three pillars, with n gold plates on the first pillar, all in different sizes with the smaller plates above the larger plates. Your task is to move one plate at a time, such that you end up with all n plates at the last pillar. During this procedure, a larger plate is *never* allowed to be on top of a smaller plate. **4**

Use mathematical induction to provethat this can be done using $2^n - 1$ moves.

Note: The following is a visual representation of *the towers of Hanoi* puzzle when $n = 3$.



(c) Suppose that p is a prime number and a is a positive integer.

i. Use the expansion of $(1+a)^p$ to prove that $(1+a)^p - 1 - a^p$ is divisible by p. **1**

(You may use the fact that $\binom{p}{k}$ is divisible by p for $k = 1, 2, ..., p-1$)

ii. Hence, or otherwise, prove by mathematical induction that $a^p - a$ is divisible by p for all positive integers a. **3**

Question 3 (13 Marks) **Marks**

(a) i. Write down the converse of the following statement: **1**

If an integer is divisible by 2 then it is an even number

ii. Write down the negation of the following statement: **2**

If x is positive or 0, then if y is negative, then $\frac{x}{y}$ is negative or 0.

(b) i. By proving the contrapositive, prove for an integer n that if $n^2 - 1$ is not divisible by 3 then n is divisible by 3. **3**

ii. Give a counterexample of the converse. **1**

(c) Prove that if a and b are integers, then $a^2 - 4b - 3 \neq 0$. **3**
(You may wish to consider the case when a is even and the case when a is odd)

(d) Prove that $\cos^2(55°)$ is irrational. **3**
(You may use $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$)

End of paper.

Sample Band E4 Responses

Question 1 ()

(a) (2 marks)

✓ [1] for expanding

✓ [1] for simplifying

$$z^2 - 4\left(\cos\frac{2\pi}{3}\right)z + 4 = z^2 + 2z + 4$$

(b) i. (2 marks)

✓ [1] for simplifying

✓ [1] for correct final answer

$$2 \times \frac{1 - i\dfrac{\sin 2\theta}{\cos 2\theta}}{1 + i\dfrac{\sin 2\theta}{\cos 2\theta}} = 2 \times \frac{\cos 2\theta - i\sin 2\theta}{\cos 2\theta + i\sin 2\theta} = 2 \times \frac{e^{-i2\theta}}{e^{i2\theta}} = 2e^{-i4\theta}$$

ii. (1 mark)

From (i),

$$\frac{1 - i\tan 2\theta}{1 + i\tan 2\theta} = e^{-i4\theta}.$$

Let $\theta = \frac{\pi}{6}$.

$$\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} = e^{-i\frac{2\pi}{3}}$$

(c) (2 marks)

✓ [1] for getting up to line 3

✓ [1] for correctly expanding and comparing imaginary parts

$\because e^{i(\theta_1+\theta_2)} = e^{i\theta_1} \times e^{i\theta_2}$,

$$\begin{aligned}\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\end{aligned}$$

Comparing imaginary parts,

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

(d) i. (1 mark)

$\overrightarrow{AC}$ is the rotation of $\overrightarrow{AB}$ by $\frac{\pi}{3}$ anticlockwise.

ii. (3 marks)

✓ [1] for expression for $z_1 - z_2$ or something similar

✓ [1] for expression with $z_1 - z_2$, $z_2 - z_3$ and $z_3 - z_1$
✓ [1] for showing required result

$$\begin{aligned}
\because z_1 - z_2 &= (z_3 - z_2)e^{i\frac{\pi}{3}} \\
\frac{z_3 - z_1}{z_1 - z_2} &= \frac{(z_2 - z_1)e^{i\frac{\pi}{3}}}{(z_3 - z_2)e^{i\frac{\pi}{3}}} = \frac{z_1 - z_2}{z_2 - z_3} \\
\therefore (z_1 - z_2)^2 &= (z_3 - z_1)(z_2 - z_3) \\
(z_1)^2 - 2z_1z_2 + (z_2)^2 &= z_2z_3 - (z_3)^2 - z_1z_2 + z_1z_3 \\
(z_1)^2 + (z_2)^2 + (z_3)^2 &= z_1z_2 + z_2z_3 + z_3z_1
\end{aligned}$$

Question 2 ()

(a) (3 marks)

✓ [1] for proving the base case.
✓ [1] for the inductive hypothesis, and for progress in getting the required result.
✓ [1] for final result.

Let $P(n)$ be the proposition

$$\frac{3^n + 5^n}{2} \ge 4^n \quad \forall n \in \mathbb{Z}^+$$

- Base case: $P(1)$:

$$\begin{aligned}
\text{LHS} &= 3^1 + 5^1 = 8 \\
\text{RHS} &= 4^1 = 4 \\
\therefore \text{LHS} &\ge \text{RHS}
\end{aligned}$$

  Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true, $k \in \mathbb{Z}^+$:

$$\frac{3^k + 5^k}{2} \ge 4^k$$

$$\text{i.e.} \quad 3^k + 5^k \ge 2 \times 4^k$$

  Test $P(k+1)$:

$$\text{RTP}: \frac{3^{k+1} + 5^{k+1}}{2} \ge 4^{k+1}$$

$$\text{i.e.} \quad 3^{k+1} + 5^{k+1} \ge 2 \times 4^{k+1}$$

$$\begin{aligned}
\text{LHS} &= 3 \times 3^k + 5 \times 5^k \\
&\ge 3 \times (2 \times 4^k - 5^k) + 5 \times 5^k \quad \text{...from the assumption} \\
&= 6 \times 4^k + 2 \times 5^k \\
&= 2(3 \times 4^k + 5^k) \\
&> 2(3 \times 4^k + 4^k) \\
&= 2(4 \times 4^k) \\
&= 2 \times 4^{k+1} \\
&= \text{RHS}
\end{aligned}$$

  Hence $P(k+1)$ is true.
  By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

(b) (4 marks)

✓ [1] for proving the base case.
✓ [1] for moving the first k plates to the second pillar.
✓ [1] for moving the $(k+1)^{th}$ plate to the last pillar
✓ [1] for moving the k plates from the second pillar to the last pillar and getting the required result.

Let $P(n)$ be the proposition

n gold plates on the first pillar can be moved to the last pillar using $2^n - 1$ moves.

- Base case: $P(1)$:
  $P(1) = 2^1 - 1 = 1$
  A single plate can be moved to the last pillar with one move.
  Hence $P(1)$ is true.
- Inductive step: assume $P(k)$ is true, $k \in \mathbb{Z}^+$:
  k plates on the first pillar can be moved to the last pillar using $2^k - 1$ moves.
  Test $P(k+1)$:

  RTP : $k+1$ gold plates on the first pillar can be moved to the last pillar using $2^{k+1} - 1$moves.

  Using the assumption, the first k plates on the first pillar can be moved to the second pillar using $2^k - 1$ moves.
  Then, move the $(k+1)^{th}$ plate on the first pillar to the last pillar.
  Finally, move the k plates on the second pillar to the last pillar on top of the $(k+1)^{th}$ plate using $2^k - 1$ moves.
  Thus the total number of moves is $2^k - 1 + 1 + 2^k - 1 = 2 \times 2^k - 1 = 2^{k+1} - 1$.
  Hence $P(k+1)$ is true.
  By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

(c) i. (1 mark)

$$\begin{aligned}(1+a)^p - 1 - a^p &= \binom{p}{0} + \binom{p}{1}a + \binom{p}{2}a^2 + \ldots + \binom{p}{p-1}a^{p-1} + \binom{p}{p}a^p - 1 - a^p \\ &= \binom{p}{1}a + \binom{p}{2}a^2 + \ldots + \binom{p}{p-1}a^{p-1}\end{aligned}$$

$\therefore (1+a)^p - 1 - a^p$ is divisible by p.

ii. (3 marks)

✓ [1] for proving the base case.
✓ [1] for the inductive hypothesis, and for progress in getting the required result.
✓ [1] for final result.

Let $P(a)$ be the proposition

$$a^p - a \text{ is divisible by } p \quad \forall a \in \mathbb{Z}^+$$

- Base case: $P(1)$:

$$1^p - 1 = 0, \quad \text{which is divisible by } p$$

Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true, $k \in \mathbb{Z}^+$:

  $$k^p - k = pM, \quad \text{where } M \text{ is a polynomial in terms of } k$$

  Test $P(k+1)$:

  $$\text{RTP} : (k+1)^p - (k+1) \text{ is divisible by } p$$

  Using (i),

  $$\begin{aligned}(1+k)^p - 1 - k^p &= pN, \quad N \in \mathbb{Z}^+\\(k+1)^p - (k+1) &= pN + 1 + k^p. - k - 1\\&= pN + pM \quad \text{...using the assumption}\\&= p(N+M), \quad \text{where } N + Misapolynomial\end{aligned}$$

  Hence $P(k+1)$ is true.
  By mathematical induction, $P(a)$ is true $\forall n \in \mathbb{Z}^+$.

Question 3 ()

(a) i. (1 mark)

If an integer is an even number then it is divisible by 2.

ii. (2 marks)

✓ [1] for correct negation

✓ [1] for the second half of the statement

x is positive or 0 and y is negative but $\frac{x}{y}$ is not negative *and* 0.

(b) i. (3 marks)

✓ [1] for correct contrapositive

✓ [1] for the case $n = 3k+1$

✓ [1] for the case $n = 3k+2$

Contrapositive: If n is not divisible by 3 then $n^2 - 1$ is divisible by 3.

Let $n = 3k+1, \quad k \in \mathbb{Z}$.

$$\begin{aligned}n^2 - 1 &= (3k+1)^2 - 1\\&= 9k^2 + 6k + 1 - 1\\&= 9k^2 + 6k\\&= 3(3k^2 + 2k), \quad \text{where } 3k^2 + 2k \in \mathbb{Z}\end{aligned}$$

Let $n = 3k+2, \quad k \in \mathbb{Z}$.

$$\begin{aligned}n^2 - 1 &= (3k+2)^2 - 1\\&= 9k^2 + 12k + 4 - 1\\&= 9k^2 + 12k + 3\\&= 3(3k^2 + 4k + 1), \quad \text{where } 3k^2 + 4k + 1 \in \mathbb{Z}\end{aligned}$$

$\therefore n^2 - 1$ is divisible by 3.

Hence if $n^2 - 1$ is not divisible by 3 then n is divisible by 3.

ii. (1 mark)
Converse: If n is divisible by 3 then $n^2 - 1$ is not divisible by 3.

Counterexample: No counterexample

(c) (3 marks)

✓ [1] for the case when a is even
✓ [1] for the case when a is odd
✓ [1] for the required result

Assume $a^2 - 4b - 3 = 0$ for $a, b \in \mathbb{Z}^+$
Then, $a^2 - 4b = 3$

- Let $a = 2n, \quad n \in \mathbb{Z}$
  Then,
  $$\begin{aligned}(2n)^2 - 4b &= 3\\ 4n^2 - 4b &= 3\\ 2(n^2 - 2b) &= 3,\end{aligned}$$
  which is a contradiction since the LHS is even and the RHS is odd.

- Let $a = 2m + 1 \quad n \in \mathbb{Z}^+$
  Then,
  $$\begin{aligned}(2m+1)^2 - 4b &= 3\\ 4m^2 + 4m + 1 - 4b &= 3\\ m^2 + m - b &= \frac{1}{2},\end{aligned}$$
  which is a contradiction since the LHS is an integer.

$$\therefore a^2 - 4b - 3 \neq 0$$

(d) (3 marks)

✓ [1] for $\cos 110^\circ$ has to be rational
✓ [1] for $\cos 330^\circ$ has to be rational
✓ [1] for the required result

Assume $\cos^2(55^\circ)$ is rational.

$$\cos 110^\circ = 2\cos^2(55^\circ) - 1$$

If $\cos^2 55^\circ$ is rational, then $\cos 110^\circ$ is rational.

$$\cos 330^\circ = 4\cos^3 110^\circ - 3\cos 110^\circ$$

If $\cos 110^\circ$ is rational, then $\cos 330^\circ$ is rational.
However there is a contradiction since $\cos 330^\circ = \frac{\sqrt{3}}{2}$, which is irrational.
Thus $\cos^2(55^\circ)$ is irrational.