NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 Year 12 Course Assessment Task 3
Friday, 4 June 2021

General instructions

- Working time – 1 hour.
  (plus 5 minutes reading time)
- Commence each new question on a new Writing Booklet. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

○ 12MXX.1 – Mr Sekaran
○ 12MXX.2 – Ms Ham

Marker's use only

QUESTION	1	2	3	Total	%
MARKS	$\overline{12}$	$\overline{11}$	$\overline{14}$	$\overline{37}$	

Question 1 (12 marks) Commence a NEW booklet. **Marks**

(a) Evaluate $\int_0^1 xe^x\,dx$ **2**

(b) Find $\int \tan^4 x \sec^4 x\,dx$ **3**

(c) Let $I_n = \int_0^1 \frac{dx}{(x^2+1)^n}$

i. Prove that $I_n = \frac{1}{2^n} + 2n\,(I_n - I_{n+1})$. **3**

ii. Hence show that $2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n$. **1**

iii. Given that $I_1 = \frac{\pi}{4}$, show that $I_3 = \frac{8+3\pi}{32}$. **3**

Question 2 (11 marks) Commence a NEW booklet. **Marks**

(a) Let a, b, c and d be positive integers.

i. Use the result $\frac{a+b}{2} \geq \sqrt{ab}$ to prove that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$. **2**

ii. By letting $d = \frac{a+b+c}{3}$ or otherwise, show that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$. **3**

(b) i. Show that $x \geq \log_e(x+1)$ for $x > -1$. **3**

ii. Hence or otherwise, show that $e^{\frac{n(n-1)}{2}} > n!$ for all positive integers n. **3**

Question 3 (14 marks) Commence a NEW booklet. **Marks**

(a) Let $\underset{\sim}{u} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$.

i Find the projection of the vector $\underset{\sim}{u}$ along $\underset{\sim}{v}$. **1**

ii. Find the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, giving the answer to the nearest minute. **2**

(b) The lines l_1 and l_2 have vector equations

$$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ a \\ 0 \end{pmatrix} \text{ and } \underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{where } \lambda, \mu \in \mathbb{R}$$

respectively, and a is a constant.

i. Given that l_1 and l_2 intersect at the point N, find the coordinates of N and the value of a. **3**

ii. A perpendicular line from the point $P(2, 1, 1)$ to the line l_2 will intersect the line l_2. **3**

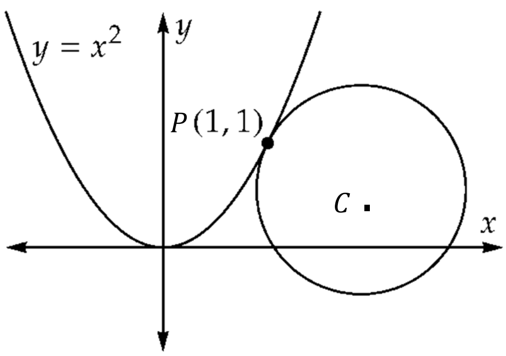
Show that the position vector of the point of intersection is $\frac{1}{3}(4\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k})$.

iii. Let P' be the point of the reflection of the point P in part (ii) about the line l_2. **2**

Find the position vector of P'.

(c) There are two possible circles with radius 1 that are tangential to the parabola $y = x^2$ at the point P(1,1). **3**

The diagram below shows one such case.



Not to scale

Using vector method, find the position vectors of the center C of both circles.

[The tangent to the circle is perpendicular to its radius at the point of contact]

End of examination

Ass task 3 ext 2 2021 Samples sol^{n}

Q1

(a) $\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx$

$= e^1 - 0 - [e^x]_0^1 = 1$

b) $\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x(\tan^2 x + 1)\sec^2 x \, dx$

$= \int \tan^6 x \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx$

$= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C$

b) $I_n = \int_0^1 \frac{1}{(x^2+1)^n} dx$

$= \left[\frac{x}{(x^2+1)^n}\right]_0^1 - \int_0^1 x \left(-2xn(x^2+1)^{-n-1}\right) dx$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(x^2+1)^{n+1}} dx$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2+1-1}{(x^2+1)^{n+1}} dx$

$$= \frac{1}{2^n} + 2n\left[I_n - I_{n+1} \right]$$

iii. $$2n I_{n+1} = \frac{1}{2^n} + 2n I_n - I_n$$

$$= \frac{1}{2^n} + (2n-1) I_n$$

iv. $n=2$ $$4 I_3 = \frac{1}{4} + 3 I_2$$

$n=1$ $$2 I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \frac{\pi}{4}$$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}$$

$$4 I_3 = \frac{1}{4} + \frac{3}{4} + \frac{3\pi}{8}$$

$$I_3 = \frac{1}{4} + \frac{3\pi}{32} = \frac{3\pi + 8}{32}$$

Q2

a) i. $a+b \geq 2\sqrt{ab}$

$c+d \geq 2\sqrt{cd}$

$a+b+c+d \geq 2\sqrt{ab} + 2\sqrt{cd}$ ✓

$= 2\left(2\sqrt{\sqrt{ab} \times \sqrt{cd}}\right)$ ✓

$= 4(abcd)^{1/4}$

$\therefore \dfrac{a+b+c+d}{4} \geq (abcd)^{1/4}$

ii. let $d = \dfrac{a+b+c}{3}$

then $a+b+c = 3d$

by (i) $\dfrac{3d+d}{4} \geq (abcd)^{1/4}$ ✓

$d \geq (abc)^{1/4} d^{1/4}$

$\div d^{1/4}$ $d^{3/4} \geq (abc)^{1/4}$ ✓

$d^3 \geq abc$

$d \geq (abc)^{1/3}$ ✓

i.e. $\dfrac{a+b+c}{3} \geq (abc)^{1/3}$

Let $f(x) = x - \ln(x+1)$

then $f'(x) = 1 - \frac{1}{x+1}$ and $f''(x) = \frac{1}{(x+1)^2}$

$f'(x) = 0 \Rightarrow x+1-1=0$ ✓ - Choosing correct $f(x)$ or equivalent

$x = 0$

$f''(0) = \frac{1}{1} = 1 > 0$ ✓. show global max

Since there is only one stationary point

$f(x)$ has global min at $x=0$

$\therefore f(0) \leq f(x) \quad \forall x > 0$ ✓

$f(0) = 0 \leq x - \ln(x+1)$

$\therefore x \geq \ln(1+x \quad \forall x > -1$ ✓ ✓

* Show $y = x$ is a tangent at $x=0$ to $y = \ln(1+x)$
* Explain $x \geq \ln(1+x)$ using graph or equivalent ✓

ii) $1 \geq \ln 2$

$2 \geq \ln 3$ ✓

$\vdots$

$n-1 \geq \ln n$

$1 + 2 + \cdots + (n-1) \geq \ln(n!)$ ✓

$\frac{(n-1)}{2}(n) \geq \ln(n!)$ ✓

$e^{\frac{n(n-1)}{2}} \geq n!$ Since $y = \ln x$ is an increasing function.

Q3

a) i. $\underline{u} = -\underline{i} + 2\underline{j} - \underline{k}$ $\quad \underline{v} = \underline{i} - \underline{j} + \underline{k}$

$$\text{proj}_{\underline{v}}\underline{u} = \left(\frac{\underline{u}\cdot\underline{v}}{|\underline{v}|^2}\right)\underline{v} = \frac{-1-2-1}{(1+1+1)}(\underline{i}-\underline{j}+\underline{k})$$

$$= -\frac{4}{3}(\underline{i}-\underline{j}+\underline{k})$$

ii.

$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \frac{-4}{\sqrt{6}\sqrt{3}} = -\frac{4}{3\sqrt{2}} = \frac{-2\sqrt{2}}{3}$$

$$\theta = 160^\circ 32' \text{ OR } 199^\circ 28'$$

b) i. $\underline{r}_1 = \underline{r}_2$

$$1 + 3\lambda = 1 - \mu$$

$$\mu = -3\lambda$$

$$2 + a\lambda = 2 + \mu$$

and $\quad 1 = \mu$

$$\therefore \lambda = -1/3$$

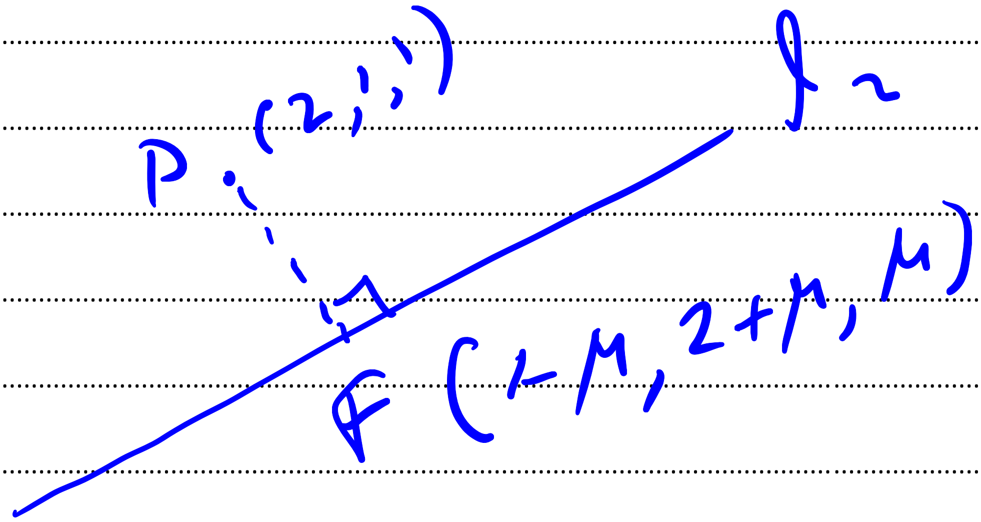
$$2 - \frac{a}{3} = 3$$

$$a = -3$$

$$\therefore \vec{ON} = \begin{pmatrix}1\\2\\0\end{pmatrix} + 1\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix}$$

$$N \equiv (0, 3, 1)$$

ii.



$$\vec{FP} = \begin{pmatrix}1+\mu\\-1-\mu\\1-\mu\end{pmatrix}$$

$$\vec{FP} \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}1+\mu\\-1-\mu\\1-\mu\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} = 0$$

$$-1-\mu-1-\mu+1-\mu = 0$$

$$-1-3\mu = 0$$

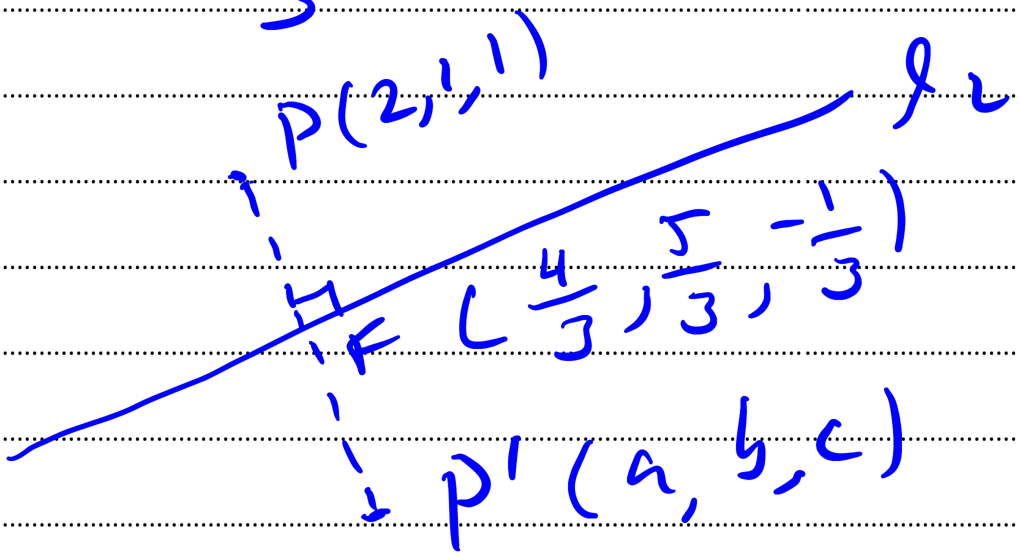
$$\mu = -1/3$$

$$F\left(1+\frac{1}{3}, 2-\frac{1}{3}, -1/3\right)$$

$$\overrightarrow{OF} = \frac{4}{3}\underset{\sim}{i} + \frac{5}{3}\underset{\sim}{j} - \frac{1}{3}\underset{\sim}{k}$$

$$= \frac{1}{3}\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}\right)$$

iii)



$$\frac{a+2}{2} = \frac{4}{3} \Rightarrow a = \frac{2}{3}$$

$$\frac{b+1}{2} = \frac{5}{3} \Rightarrow b = \frac{7}{3}$$

$$\frac{c+1}{2} = -\frac{1}{3} \Rightarrow c = \frac{-5}{3}$$

$$\overrightarrow{OP'} = \frac{1}{3}\left(2\underset{\sim}{i} + 7\underset{\sim}{j} - 5\underset{\sim}{k}\right)$$

Alternate method

$$\overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PP'}$$

$= \overrightarrow{OP} + 2\overrightarrow{PF}$ ✓

$= 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + 2\left(\frac{4}{3} - 2\right)\underset{\sim}{i}$

$+ 2\left(\frac{5}{3} - 1\right)\underset{\sim}{j}$

$+ 2\left(-\frac{1}{3} - 1\right)\underset{\sim}{k}$

$= \left(2 - \frac{4}{3}\right)\underset{\sim}{i} + \left(1 + \frac{4}{3}\right)\underset{\sim}{j} + \left(1 - \frac{8}{3}\right)\underset{\sim}{k}$

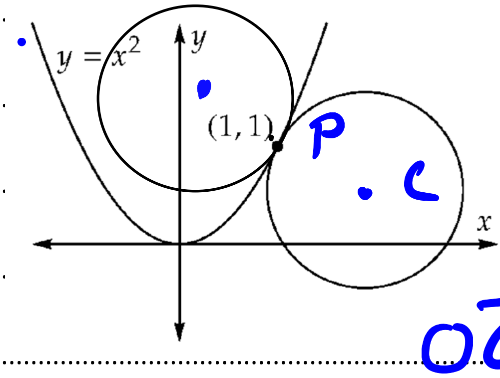
$= \frac{1}{3}\left(2\underset{\sim}{i} + 7\underset{\sim}{j} - 5\underset{\sim}{k}\right)$ ✓

c) $y' = 2x$

At $x = 1$, $y' = 2$

$\therefore \begin{pmatrix}1\\2\end{pmatrix}$ is parallel to tangent

$\therefore \begin{pmatrix}2\\-1\end{pmatrix}$ ✓ is perpendicular to tangent



$\therefore \overrightarrow{PC} = 1 \times \frac{(2\underset{\sim}{i} - \underset{\sim}{j})}{\sqrt{5}}$ ✓

$\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC}$

$= \underset{\sim}{i} + \underset{\sim}{j} + \frac{1}{\sqrt{5}}(2\underset{\sim}{i} - \underset{\sim}{j})$ ✓ $\frac{1}{2}$

$= \left(1 + \frac{2}{\sqrt{5}}\right)\underset{\sim}{i} + \left(1 - \frac{1}{\sqrt{5}}\right)\underset{\sim}{j}$

OR $\begin{pmatrix}-2\\1\end{pmatrix}$ is perpendicular to tangent

$$\overrightarrow{PC} = \frac{1}{\sqrt{5}}(-2\underset{\sim}{i} + \underset{\sim}{j})$$ $\frac{1}{2}$ ✓

$$\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC}$$
$$= \left(1 - \frac{2}{\sqrt{5}}\right)\underset{\sim}{i} + \left(1 + \frac{1}{\sqrt{5}}\right)\underset{\sim}{j}$$

Alternate method — Let $\overrightarrow{OC} = a\underset{\sim}{i} + b\underset{\sim}{j}$

Then $\overrightarrow{PC} = (a-1)\underset{\sim}{i} + (b-1)\underset{\sim}{j}$

$y' = 2x$, at $x = 1$, $y' = 2$

$\therefore$ $\underset{\sim}{i} + 2\underset{\sim}{j}$ is parallel to the tangent at $x=1$

$$\therefore (\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot ((a-1)\underset{\sim}{i} + (b-1)\underset{\sim}{j}) = 0$$
$$a - 1 + 2(b-1) = 0$$ ✓ using vector method to find one of the equation
$$a + 2b = 3 \quad \text{—— (1)}$$

and also $(a-1)^2 + (b-1)^2 = 1$ —— (2)

$$(3 - 2b - 1)^2 + (b-1)^2 = 1$$
$$(2(1-b))^2 + (b-1)^2 = 1$$
$$5(b-1)^2 = 1$$
$$b - 1 = \pm\frac{1}{\sqrt{5}}$$
$$\therefore b = 1 \pm \frac{1}{\sqrt{5}}$$

$$a = 3 - 2\left(1 \pm \frac{1}{\sqrt{5}}\right)$$
$$= 1 \mp \frac{2}{\sqrt{5}}$$

✓ making reasonable attempt to solve simultaneously

$$\therefore \overrightarrow{OC} = \left(1 \mp \frac{2}{\sqrt{5}}\right)\underset{\sim}{i} + \left(1 \pm \frac{1}{\sqrt{5}}\right)\underset{\sim}{j}$$

✓ Correct solutions