NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2022 Year 12 Course Assessment Task 2

Wednesday March 16, 2022

General instructions

- Working time – 90 minutes. (plus 5 minutes reading time)
- Write using a black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- All necessary working should be shown in every question.
- Marks may be deducted for illegible or incomplete working.
- A NESA Reference Sheet is provided.

SECTION I

- Mark your answers on the answer grid provided (on page 3)

SECTION II

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Sekaran ◯ 12MXX.3 – Mr Ho

Marker's use only.

QUESTION	1-5	6	7	8	9	10	Total	%
MARKS	$\overline{5}$	$\overline{15}$	$\overline{11}$	$\overline{7}$	$\overline{7}$	$\overline{9}$	$\overline{54}$	

Section I

5 marks
Attempt Question 1 to 5
Allow approximately 6 minutes for this section

Mark your answers on the answer grid provided (labelled as page 3).

Glossary

- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ – set of all natural numbers
- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – set of all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers
- $\mathbb{C}$ – set of all complex numbers

Questions **Marks**

1. Which of the following are the principal arguments of the solutions to the equation **1**

$$z^2 = 1 + i$$

(A) $\dfrac{\pi}{8}$ and $\dfrac{9\pi}{8}$

(B) $-\dfrac{\pi}{8}$ and $\dfrac{7\pi}{8}$

(C) $-\dfrac{7\pi}{8}$ and $\dfrac{\pi}{8}$

(D) $\dfrac{7\pi}{8}$ and $\dfrac{15\pi}{8}$

2. If $z = 3 - 4i$, which of the following is equal to $\dfrac{1}{1-z}$? **1**

(A) $\dfrac{-1-2i}{10}$ (B) $\dfrac{-1+2i}{10}$ (C) $\dfrac{-1-i}{6}$ (D) $\dfrac{-1+i}{6}$

3. Given $z_1 = \cos\alpha + i\sin\alpha$ and $z_2 = \cos\beta + i\sin\beta$, what is the value of **1**

$$\arg\left(\frac{z_1 z_2}{z_1 + z_2}\right)$$

(A) $\dfrac{1}{2}\alpha\beta$ (B) $\dfrac{1}{2}(\alpha+\beta)$ (C) $\alpha\beta$ (D) $\alpha+\beta$

4. Given $x \in \mathbb{N}$ and $y \in \mathbb{N}$, which of the following statements is **false**? **1**

(A) $\forall x\, \exists y\, (x - y = 1)$

(B) $\forall x\, \exists y\, (2x - y = 0)$

(C) $\forall x\, \exists y\, (x - 2y = 0)$

(D) $\exists x\, \exists y\, (x + y = 2)$

5. What is the negation of the statement: **1**

$$n < -2 \text{ or } n \geq 3$$

(A) $n \leq -2$ and $n > 3$

(B) $n \leq -2$ and $n \geq 3$

(C) $-2 \leq n < 3$

(D) $-2 < n \leq 3$

Answer grid for Section I

1 – (A) (B) (C) (D)
2 – (A) (B) (C) (D)
3 – (A) (B) (C) (D)
4 – (A) (B) (C) (D)
5 – (A) (B) (C) (D)

Examination continues overleaf...

Section II

49 marks
Attempt Questions 6 to 10
Allow approximately 84 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 Marks) **Marks**

(a) i. Express $-\sqrt{3}+i$ in modulus-argument form. **1**

ii. On an Argand diagram, sketch the set of points P that represents the complex number z, which satisfies **2**

$$\left|z+\sqrt{3}-i\right|=1$$

iii. Find the set of possible values of $|z|$ and $\text{Arg}(z)$, where $\text{Arg}(z)$ is the principal argument. **2**

(b) i. On an Argand diagram, draw a neat sketch of **3**

$$|z-4-4i|=2 \quad \text{and} \quad \text{Arg}(z)=\frac{\pi}{4}$$

ii. Hence write down all values of z which simultaneously satisfy **2**

$$|z-4-4i|=2 \quad \text{and} \quad \text{Arg}(z)=\frac{\pi}{4}$$

(c) The points O, A, B and C on the Argand diagram represent complex numbers 0, 1, z and $z+1$ respectively, where $z=\cos\theta+i\sin\theta$ is any complex number of modulus 1 and $0<\theta<\pi$.

i. With the aid of a diagram or otherwise, explain why $OACB$ is a rhombus. **1**

ii. Show that $\dfrac{z-1}{z+1}$ is purely imaginary. **2**

iii. Find the modulus and argument of $z+1$. Write both the modulus and argument in terms of θ. **2**

Question 7 (11 Marks) **Marks**

(a) Consider the complex polynomial $P(z) = z^4 - 3z^3 + 6z^2 - 12z + 8$. **3**

Given that $P(2i) = 0$, factorise $P(z)$ into linear and quadratic factors with real coefficients.

(b) i. Show that $\left(3 - e^{i2\theta}\right)\left(3 - e^{-i2\theta}\right)$ is purely real. **3**

ii. Now consider $S = e^{i\theta} + \dfrac{e^{i3\theta}}{3} + \dfrac{e^{i5\theta}}{3^2} + \dfrac{e^{i7\theta}}{3^3} + \cdots$. **3**

Show that

$$S = \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos 2\theta}$$

iii. Hence or otherwise, show that **2**

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{3^2} + \frac{\sin 7\theta}{3^3} + \cdots = \frac{6\sin\theta}{5 - 3\cos 2\theta}$$

Question 8 (7 Marks) **Marks**

(a) Using De Moivre's Theorem, show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(b) Hence, show that **3**

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

(c) Briefly explain why **1**

$$\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

Examination continues overleaf...

Question 9 (7 Marks) **Marks**

(a) Use mathematical induction to prove for $n \geq 2$, **3**

$$u_n = \frac{1}{2}n(n+1)$$

given $u_1 = 1$ and $u_n = u_{n-1} + n$.

(b) Suppose $x_i > 0$ for $i = 1, 2, \cdots, n$, and $n \geq 2$. **4**

Prove by mathematical induction that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq \frac{n^2}{s}$$

where $s = x_1 + x_2 + \cdots x_n$.
Hint: you may use the fact that $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{4}{c}$ where $a > 0$, $b > 0$ and $c = a + b$, without proving it.

Question 10 (9 Marks) **Marks**

(a) Prove that $\log_2 6$ is irrational. **2**

(b) Let P be the proposition

If a pair of angles are alternate and stand on parallel lines, then the pair of angles are equal.

i. Write the converse of the proposition P, and state whether this is true or not. **2**

ii. State the contrapositive of P. **1**

(c) Prove that for all $n \in \mathbb{Z}^+$ and $p \in \mathbb{Z}^+$ where p is prime, that there exists no positive integer m such that **4**

$$(3m+2)^2 = n^2 + p$$

End of paper.

2022 Mathematics Extension 2 Year 12 Course Assessment Task 2 STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

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2. Which topics did I need more help with, and what parts specifically?

- Q1-3, 6 - Complex Numbers (Arithmetic, Mod Arg form)

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- Q7 - Complex Numbers (Applications)

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- Q8 - Complex Numbers (De Moivre's Theorem)

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- Q9 - Mathematical Induction

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- Q10 - The Nature of Proof

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3. What other parts from the feedback session can I take away to refine my solutions for future reference?

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Sample Band E4 Responses

1. (C) **2.** (A) **3.** (B) **4.** (C) **5.** (C)

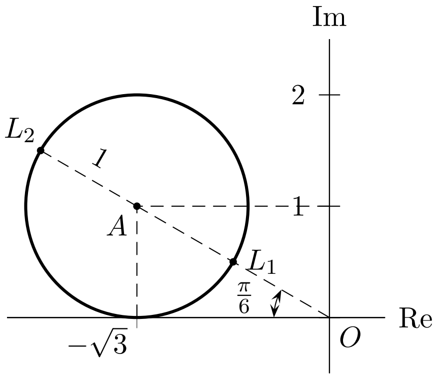
Question 6 (Ham)

(a) i. (1 mark)

$$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

(Use diagram to assist with quickly finding the modulus and argument)

ii. (2 marks)



iii. (2 marks)

✓ [1] for correct range of values for the modulus.

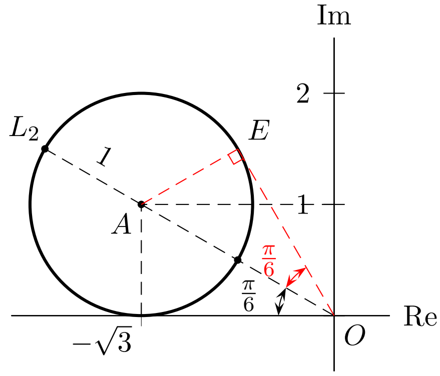
✓ [1] for correct range of values for the argument.

Modulus

- $AO = 2$ due to Pythagoras, and with AL_1 being the radius of the circle, hence $OL_1 = 1$
- The furthest point away from O on the circle is the point L_2, which will have modulus of 3.
- Therefore,

$$1 \le |z| \le 3$$

Argument adding more lines to the diagram:



- Construct a new point E on the circle such that $AE \perp OE$.
- Hence the minimum argument for z is

$$\pi - \left(\frac{\pi}{6}\right) - \left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$$

- Max argument for z is π.
- Therefore,

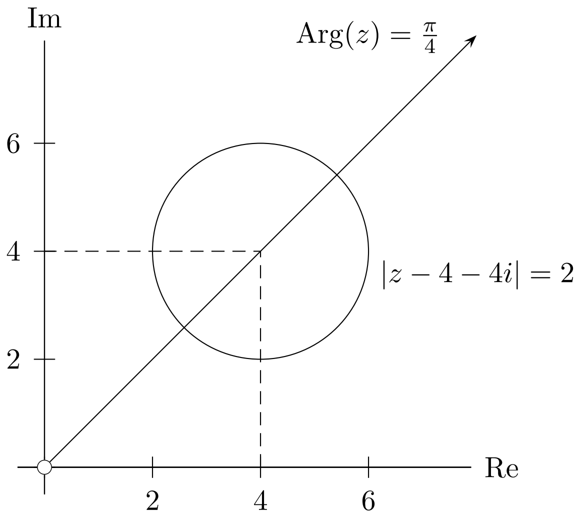
$$\frac{2\pi}{3} \le \operatorname{Arg}(z) \le \pi$$

(b) i. (3 marks)

✓ [1] for correctly sketching the circle.

✓ [1] for correctly sketching the line

✓ [1] for open circle on the line at $0 + 0i$



ii. (2 marks)

✓ [1] for obtaining a quadratic equation in terms of x only.

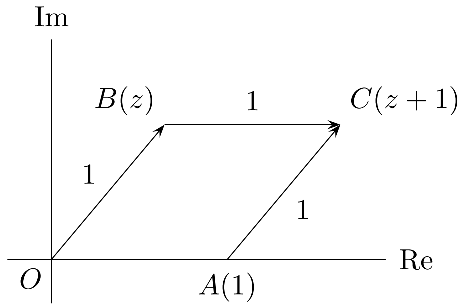
✓ [1] for both correct x and y values.

$$\begin{cases} y = x & (1) \\ (x-4)^2 + (y-4)^2 = 4 & (2) \end{cases}$$

Solving simultaneously by substituting (1) into (2):

$$\begin{aligned}(x-4)^2+(x-4)^2&=4\\ 2(x-4)^2&=4\\ (x-4)^2&=2\\ \therefore x-4&=\pm\sqrt{2}\\ x&=4\pm\sqrt{2}\\ \therefore y&=4\pm\sqrt{2}\\ \therefore z&=\left(4\pm\sqrt{2}\right)(1+i)\end{aligned}$$

(c) i. (1 mark)



- As $z=\cos\theta+i\sin\theta$, $|z|=1$.
- $\left|\overrightarrow{OA}\right|=1$ as well.
- OB and OA form adjacent sides of a parallelogram $OACB$. However, one pair of adjacent sides are equal.

  Hence $OACB$ is a rhombus.

ii. (2 marks)

$$\begin{aligned}&\frac{z-1}{z+1}\\ &=\frac{\cos\theta+i\sin\theta-1}{\cos\theta+i\sin\theta+1}\\ &=\frac{(\cos\theta-1)+i\sin\theta}{(\cos\theta+1)+i\sin\theta}\,\frac{\times[(\cos\theta+1)-i\sin\theta]}{\times[(\cos\theta+1)-i\sin\theta]}\\ &=\frac{(\cos\theta-1)(\cos\theta+1)-i\sin\theta(\cos\theta-1)+i\sin\theta(\cos\theta+1)+\sin^2\theta}{(\cos\theta+1)^2+\sin^2\theta}\\ &=\frac{\cancel{\cos^2\theta}-\cancel{1}+2i\sin\theta+\cancel{\sin^2\theta}}{(\cos\theta+1)^2+\sin^2\theta}\\ &=i\left(\frac{2\sin\theta}{(\cos\theta+1)^2+\sin^2\theta}\right)\end{aligned}$$

Alternatively,

$$z-1=\overrightarrow{AB}\qquad z+1=\overrightarrow{OC}$$

As $\overrightarrow{AB}\perp\overrightarrow{OC}$ due to the diagonals of a rhombus,

$$\therefore z-1=\lambda i(z+1)$$

where $\lambda\in\mathbb{R}$, $\lambda\neq 0$. Dividing throughout by $(z+1)$:

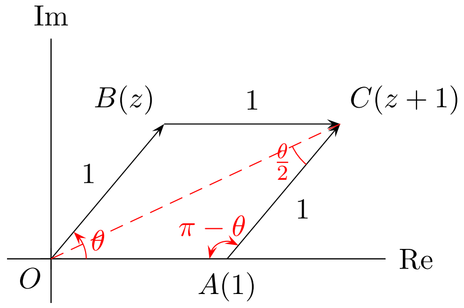
$$\therefore \frac{z-1}{z+1}=\lambda i$$

and $\dfrac{z-1}{z+1}$ is purely imaginary.

iii. (2 marks)

✓ [1] for correct expression for $\text{Arg}(z+1)$

✓ [1] for $|z+1| = \sqrt{2(1+\cos\theta)}$



- $\text{Arg}(z+1) = \dfrac{\theta}{2}$ as $\text{Arg}(z) = \theta$ (Diagonals of a rhombus bisects the angle)
- As $\triangle AOC$ is isosceles (base side lengths of 1),

$$\therefore \angle OCA = \frac{\theta}{2}$$

and hence

$$\angle OAC = \pi - 2\left(\frac{\theta}{2}\right) = \pi - \theta$$

- Applying the cosine rule in $\triangle OCA$:

$$\begin{aligned} |z+1|^2 &= 1^2 + 1^2 - 2(1)(1)\cos(\pi - \theta) \\ &= 2 + 2\cos\theta \\ |z+1| &= \sqrt{2(1+\cos\theta)} \\ &= \sqrt{2 \times 2\cos^2\frac{\theta}{2}} \\ &= 2\cos\frac{\theta}{2} \end{aligned}$$

Noting that

$$\begin{aligned} \cos^2\theta &= \frac{1}{2} + \frac{1}{2}\cos 2\theta \\ \cos^2\frac{\theta}{2} &= \frac{1}{2} + \frac{1}{2}\cos\theta \\ \therefore 2\cos^2\frac{\theta}{2} &= 1 + \cos\theta \end{aligned}$$

Alternatively for the modulus:

$$\begin{aligned} |z+1| &= |(\cos\theta + 1) + i\sin\theta| \\ &= \sqrt{(\cos\theta+1)^2 + \sin^2\theta} \\ &= \sqrt{\cos^2 + 2\cos\theta + 1 + \sin^2\theta} \\ &= \sqrt{2\cos\theta + 2} \end{aligned}$$

Question 7 (Ho)

(a) (3 marks)

- ✓ [1] for identifying $-2i$ being a root.
- ✓ [1] for identifying one other factor apart from the complex factors $(z-2i)$ and $(z+2i)$.
- ✓ [1] for full factorisation into real factors.

$$\begin{aligned} P(z) &= z^4 - 3z^3 + 6z^2 - 12z + 8 \\ P(2i) &= 0 \end{aligned}$$

Hence the complex conjugate is also a root, i.e.

$$P(-2i) = 0$$

Hence (z^2+4) is a factor

$$\begin{aligned} P(z) &= \left(z^2 + 0z + 4\right)\left(z^2 - 3x + 2\right) \\ &= \left(z^2+4\right)(z-2)(z-1) \end{aligned}$$

(b) i. (3 marks)

- ✓ [1] for correct expansion to
$$9 - \left(3e^{i2\theta} + 3e^{-i2\theta}\right) + e^0$$
- ✓ [1] for correct simplification of complex exponential terms to $6\cos 2\theta$
- ✓ [1] for recognition that the final expression does not contain any imaginary part.

$$\begin{aligned} &\left(3 - e^{i2\theta}\right)\left(3 - e^{-i2\theta}\right) \\ &= 9 - \left(3e^{i2\theta} + 3e^{-i2\theta}\right) + e^0 \\ &= 10 - 3\left(e^{i2\theta} + e^{-i2\theta}\right) \\ &= 10 - 3 \times 2\cos 2\theta \\ &= 10 - 6\cos 2\theta \\ &\mathrm{Im}(10 - 6\cos 2\theta) = 0 \end{aligned}$$

Hence, $\left(3 - e^{i2\theta}\right)\left(3 - e^{-i2\theta}\right)$ is purely real.

ii. (3 marks)

- ✓ [1] for recognising $|r| < 1$
- ✓ [1] for substituting into limiting sum formula and multiplying by the conjugate $\left(3 - e^{-i2\theta}\right)$
- ✓ [1] for showing the final result.

As $r = \frac{1}{3}e^{i2\theta}$, $|r| = \left|\frac{1}{3}e^{i2\theta}\right| < 1$ given $\left|e^{i2\theta}\right| = 1$

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{e^{i\theta}}{1 - \frac{1}{3}e^{i2\theta}} \frac{\times 3}{\times 3} \\ &= \frac{3e^{i\theta}}{3 - e^{i2\theta}} \frac{\times\left(3 - e^{-i2\theta}\right)}{\times\left(3 - e^{-i2\theta}\right)} \\ &= \frac{3e^{i\theta}\left(3 - e^{-i2\theta}\right)}{10 - 6\cos 2\theta} \\ &= \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos 2\theta} \end{aligned}$$

iii. (2 marks)

- ✓ [1] for recognising the imaginary part of the summation from the previous part is required.
- ✓ [1] for final required proof.

$$\begin{aligned} \sin\theta &= \mathrm{Im}\left(e^{i\theta}\right) \\ \sin 3\theta &= \mathrm{Im}\left(e^{i3\theta}\right) \end{aligned}$$

etc. Hence, the required summation is simply

$$\begin{aligned} &\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{3^2} + \cdots \\ &= \mathrm{Im}\left(\frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos 2\theta}\right) \\ &= \frac{3}{10 - 6\cos 2\theta}\left(3\sin\theta - \sin(-\theta)\right) \\ &= \frac{3\left(3\sin\theta + \sin\theta\right)}{10 - 6\cos 2\theta} \\ &= \frac{12\sin\theta}{10 - 6\cos 2\theta} \\ &= \frac{6\sin\theta}{5 - 3\cos 2\theta} \end{aligned}$$

Question 8 (Ho)

(a) (3 marks)

- ✓ [1] for correct usage of De Moivre's Theorem and expansion via the Binomial Theorem
- ✓ [1] for correct reduction of $\sin\theta$ terms into $\cos\theta$
- ✓ [1] for final result.

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5\\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta\\ &\quad + i^2 10\cos^3\theta\sin^2\theta\\ &\quad + i^3 10\cos^2\theta\sin^3\theta\\ &\quad + i^4 5\cos\theta\sin^4\theta\\ &\quad + i^5\sin^5\theta\end{aligned}$$

Equating real parts,

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right)\\ &\quad + 5\cos\theta\left(1-\cos^2\theta\right)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta\\ &\quad + 5\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta\\ &\quad + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

(b) (3 marks)

- ✓ [1] for obtaining quadratic in terms of m.
- ✓ [1] for obtaining $\cos\theta = \pm\sqrt{\frac{10\pm2\sqrt5}{16}}$.
- ✓ [1] for final result.

- $\cos(5\times18^\circ) = \cos 90^\circ = 0$
- Solve $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ to obtain the surdic expression to $\cos 18^\circ$, i.e.

$$\begin{aligned}16\cos^5\theta - 20\cos^3\theta + 5\cos\theta &= 0\\ \cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) &= 0\end{aligned}$$

As $\cos 18^\circ \neq 0$,

$$\therefore 16\cos^4\theta - 20\cos^2\theta + 5 = 0$$

Let $m = \cos^2\theta$,

$$16m^2 - 20m + 5 = 0$$

Applying the quadratic formula,

$$\begin{aligned}m &= \frac{20\pm\sqrt{20^2 - 4(16)(5)}}{2(16)}\\ &= \frac{20\pm\sqrt{80}}{32}\\ &= \frac{20\pm4\sqrt5}{32}\\ &= \frac{10\pm2\sqrt5}{16}\\ \therefore \cos^2\theta &= \frac{10\pm2\sqrt5}{16}\\ \cos\theta &= \pm\sqrt{\frac{10\pm2\sqrt5}{16}}\end{aligned}$$

As 18° is in the first quadrant and $\cos\theta > 0$ for first quadrant values of θ, the negative root is discarded.

$$\therefore \cos\theta = \frac{\sqrt{10\pm2\sqrt5}}{4}$$

Also, $\cos 18^\circ$ is closer to $\cos 0^\circ = 1$, take the value which is closer to 1:

$$\therefore \cos 18^\circ = \frac{\sqrt{10+2\sqrt5}}{4}$$

as $\frac{\sqrt{10-2\sqrt5}}{4}$ is closer to 0.

(c) (1 mark)

$$\cos(5\times54^\circ) = \cos 270^\circ = 0$$

No other acute angle θ would result in $\cos 5\theta = 0$, and from the graph of $y = \cos x$, $\cos 54^\circ < \cos 18^\circ$.

Question 9 (Sekaran)

(a) (3 marks)

- ✓ [1] for correct testing of the base case.
- ✓ [1] for correct usage of the inductive hypothesis in $P(k+1)$
- ✓ [1] for final result.

Let $P(n)$ be the proposition

$$u_n = \frac{1}{2}n(n+1)$$
$$\text{where } u_1 = 1, u_n = u_{n-1} + n$$

1. Base case: $P(2)$

$$u_2 = u_1 + 2 = 1 + 2 = 3$$

Also,

$$u_2 = \frac{1}{2}(2)(2+1) = 3$$

Hence $P(2)$ is true.

2. Inductive step: assume $P(k)$ is true, i.e.

$$u_k = \frac{1}{2}k(k+1)$$
$$\text{where } u_1 = 1, u_k = u_{k-1} + k$$

Examine $P(k+1)$:

$$\begin{aligned} u_{k+1} &= u_k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \frac{1}{2}(k+1)\big((k+1)+1\big) \end{aligned}$$

(b) (4 marks)

- ✓ [1] for correct testing of the base case.
- ✓ [1] for correct usage of the inductive hypothesis in $P(k+1)$
- ✓ [1] for further testing of $P(k+1)$
- ✓ [1] for final result.

Let $P(n)$ be the proposition

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \ge \frac{n^2}{s}$$

where $s = x_1 + x_2 + \cdots x_n$, $n \ge 2$.

1. Base case: $P(2)$

- Left side: $\dfrac{1}{x_1} + \dfrac{1}{x_2}$
- Right: $\dfrac{2^2}{x_1+x_2} = \dfrac{4}{x_1+x_2}$.

$P(2)$ is true by the identity provided, $\dfrac{1}{a} + \dfrac{1}{b} \ge \dfrac{4}{a+b}$.

2. Inductive step: assume $P(k)$ is true, i.e.

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \ge \frac{k^2}{x_1 + x_2 + \cdots + x_k}$$

Examine $P(k+1)$:

$$\begin{aligned} &\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} + \frac{1}{x_{k+1}} \\ &\ge \frac{k^2}{x_1 + x_2 + \cdots + x_k} + \frac{1}{x_{k+1}} \end{aligned}$$

Note: the requirement is to prove

$$\begin{aligned} &\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} + \frac{1}{x_{k+1}} \\ &\qquad \ge \frac{(k+1)^2}{x_1 + x_2 + \cdots + x_k + x_{k+1}} \end{aligned}$$

Now check for whether

$$\frac{k^2}{x_1 + x_2 + \cdots + x_k} + \frac{1}{x_{k+1}} \qquad (\dagger)$$

is greater than

$$\frac{(k+1)^2}{x_1 + x_2 + \cdots + x_k + x_{k+1}} \qquad (\ddagger)$$

or not: complete a subtraction of $(\ddagger)$ from $(\dagger)$ and check whether that is greater than zero.

$$\begin{aligned} &\frac{k^2}{x_1 + x_2 + \cdots + x_k} + \frac{1}{x_{k+1}} \\ &\qquad - \frac{(k+1)^2}{x_1 + x_2 + \cdots + x_k + x_{k+1}} \\ &= \frac{k^2}{a} + \frac{1}{b} - \frac{(k+1)^2}{a+b} \end{aligned}$$

where

$$\begin{aligned} a &= x_1 + x_2 + \cdots + x_k \\ b &= x_{k+1} \end{aligned}$$

$$\begin{aligned} &\frac{k^2}{a} + \frac{1}{b} - \frac{(k+1)^2}{a+b} \\ &= \frac{k^2 b(b+a) + a(a+b) - (k+1)^2 ab}{ab(a+b)} \\ &= \frac{k^2b^2 + k^2ab + a^2 + ab - k^2ab - 2kab - ab}{ab(a+b)} \\ &= \frac{k^2b^2 - 2kab + a^2}{ab(a+b)} \\ &= \frac{(kb-a)^2}{ab(a+b)} \\ &\ge 0 \end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}}\\ \geq \frac{k^2}{x_1+x_2+\cdots+x_k}+\frac{1}{x_{k+1}}\\ \geq \frac{(k+1)^2}{x_1+x_2+\cdots+x_k+x_{k+1}}\end{aligned}$$

which makes $P(k+1)$ true, and $P(n)$ true by induction.

Question 10 (Sekaran)

(a) (2 marks)

✓ [1] for setting up the contradiction.

✓ [1] for final proof.

Two of the techniques are presented here.

Method 1

- Assume $\log_2 6$ is rational, i.e.

$$\log_2 6 = \frac{p}{q}$$

where p, $q \in \mathbb{Z}^+$, and p and q are coprime such that $p \neq q$

- Then,

$$\begin{aligned}2^{\frac{p}{q}} &= 6\\ 2^p &= 6^q\\ 2^p &= 2^q 3^q\\ \frac{2^p}{2^q} &= 3^q\end{aligned}$$

- 3 divides $2^p 3^q$. Hence, 3 divides 2^p, which is a contradiction as 2^p is an even number.
- Hence $2^{\frac{p}{q}} \neq 6$, and $\log_2 6 \neq \frac{p}{q}$ and $\log_2 6$ is irrational.

Method 2

$$\frac{p}{q} = \log_2 6 > \log_2 4 > 2$$

$$\begin{aligned}\therefore \underset{-q}{p} &> \underset{-q}{2q}\\ p-q &> q\end{aligned}$$

Note also that $p-q \in \mathbb{Z}^+$ as $q \in \mathbb{Z}^+$. Hence,

$$\begin{aligned}2^p &= 2^q 3^q\\ \therefore 2^{p-q} &= 3^q\end{aligned}$$

which is a contradiction as 2^{p-q} is even and 3^q is odd.

(b) i. (2 marks) Let

- p be the statement *a pair of angles are alternate*
- q be the statement *stand on parallel lines*
- r be the statement *pair of angles are equal*

Then the statement becomes

$$(p \wedge q) \Rightarrow r$$

The converse is thus

$$r \Rightarrow (p \wedge q)$$

i.e.

> If the pair of angles are equal, then the angles are alternate *and* stand on parallel lines.

This is not necessarily true.

ii. (1 mark) The contrapositive is:

$$\neg r \Rightarrow \neg(p \wedge q)$$

Note that

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

i.e.

> If a pair of angles are not equal, then the angles are not alternate or not stand on parallel lines.

(c) (4 marks)

✓ [1] Factorises expression for p.

✓ [1] Realises the factors must be 1 and p.

✓ [1] for further substantial progress

✓ [1] for final proof.

- Let m, n and p be positive integers where p is prime, and $(3m+2)^2 = n^2 + p$.
- Then,

$$\begin{aligned}p &= (3m+2)^2 - n^2\\ &= (3m+2-n)(3m+2-n)\end{aligned}$$

- If m, n, and p are positive integers, then

$$0 < (3m + 2 - n) < (3m + 2 + n)$$

- Also given p is prime, then

$$(3m + 2 - n) = 1 \qquad (10.1)$$
$$(3m + 2 + n) = p \qquad (10.2)$$

as it contains only two factors.

- Adding (10.1) and (10.2):

$$6m + 4 = p + 1$$
$$\therefore p = 6m + 3 = 3(2m + 1)$$

which is a contradiction: p is meant to be prime, but the result indicates p is also divisible by 3. Hence for all integers n, p, where p is prime, there is no positive integer m such that

$$(3m + 2)^2 = n^2 + p$$