NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2022 Year 12 Course Assessment Task 3

Thursday June 9th, 2022

General instructions

- Working time – 1 hour.
  (plus 5 minutes reading time)
- Write your answers in the space provided.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle any additional writing booklets used in the correct order within this paper and hand to examination supervisors.
- A NESA Reference Sheet is provided.

Class (please ✔)

◯ 12MXX.1 – Ms Ham

◯ 12MXX.2 – Mr Sekaran

◯ 12MXX.3 – Mr Ho

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Marker's use only.

QUESTION	1	2	3	**Total**	%
MARKS	$\overline{17}$	$\overline{14}$	$\overline{9}$	$\overline{40}$	

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 1 (17 Marks) **Marks**

(a) i. Using partial fractions, or otherwise, find **3**

$$\int_1^3 \frac{1}{x(4-x)}\,dx$$

ii. Let $I = \displaystyle\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$. **2**

Use the substitution $u = 4 - x$ to show that

$$I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du$$

iii. Hence, find the value of I. **1**

(b) Find **3**

$$\int \sqrt{\frac{3-x}{x}}\,dx$$

(c) Use a suitable substitution to find **4**

$$\int_2^4 \frac{dx}{x^2\sqrt{x^2-4}}$$

(d) Let $I = \displaystyle\int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx$, where $k \in \mathbb{Z}$. Show that **4**

$$I = \frac{(-1)^{k-1}e^{k\pi}(e^\pi + 1)}{2e^\pi}$$

Examination continues overleaf...

Question 2 (14 Marks) **Marks**

(a) Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}$, where m is a positive integer.

The projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $-\dfrac{11}{18}(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k})$.

i. Find the value of m. **2**

ii. Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. **1**

(b) The line ℓ_1 has vector equation $\underset{\sim}{r_1} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda_1\begin{pmatrix}2\\2\\-3\end{pmatrix}$ and the line ℓ_2 has vector equation $\underset{\sim}{r_2} = \begin{pmatrix}4\\-2\\9\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

i. Show that the distance d between a point on ℓ_1 and a point on ℓ_2 is given by **1**

$$d = \sqrt{(3-2\lambda_1+\lambda_2)^2 + (-2-2\lambda_1+2\lambda_2)^2 + (7+3\lambda_1-2\lambda_2)^2}$$

It is given that

$$\begin{aligned}(3-2\lambda_1+\lambda_2)^2 &+ (-2-2\lambda_1+2\lambda_2)^2 + (7+3\lambda_1-2\lambda_2)^2 \\ &= (3\lambda_2-4\lambda_1-5)^2 + (\lambda_1-1)^2 + 36\end{aligned}$$

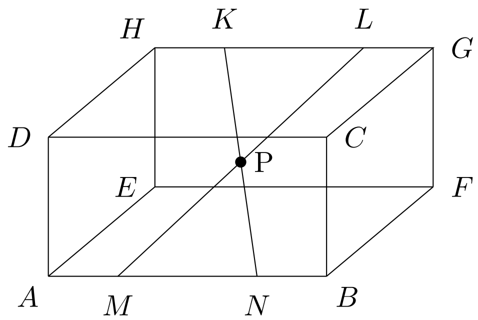
(DO NOT PROVE THIS)

ii. Hence, or otherwise, determine the minimum distance between ℓ_1 and ℓ_2. **1**

iii. Find the coordinates of the points on the two lines that are the minimum distance apart. **2**

(c) The diagram shows a rectangular prism whose vertices are A, B, C, D, E, F, G and H. The points M and N lie on AB such that $AM : MN : NB = 1 : 2 : 1$. The points K and L lie on HG such that $HK : KL : LG = 1 : 2 : 1$. P is the intersection of KN and ML.

Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{AE} = \underset{\sim}{e}$ and $\overrightarrow{AD} = \underset{\sim}{d}$.



i. Show that $\overrightarrow{ML} = \dfrac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{e}$ **2**

ii. Find a similar expression for $\overrightarrow{NK}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{d}$ and $\underset{\sim}{e}$ **1**

iii. By using $\overrightarrow{AP}$, prove that KN and ML bisect each other at P. **4**

Question 3 (9 Marks) **Marks**

(a) It is given that $a, b, c \in \mathbb{R}^+$ and $a + b + c = 1$.

i. Prove $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$. **1**

ii. Prove $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq 9$. **2**

iii. Hence, or otherwise, prove $(1-a)(1-b)(1-c) \geq 8abc$. **2**

(b) A function is defined to be bounded if $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$.

It is given that $g(x)$ and $h(x)$ are continuous and bounded functions.

i. By considering $\displaystyle\int_0^a \Big[\lambda g(x) + h(x)\Big]^2 \, dx$ for $a > 0$ and $\lambda \in \mathbb{R}$ as a quadratic function in λ, show that **2**

$$\left[\int_0^a g(x)h(x)\, dx\right]^2 \leq \int_0^a \Big[g(x)\Big]^2 dx \times \int_0^a \Big[h(x)\Big]^2 dx$$

ii. Hence show that **1**

$$\left[\int_0^1 g(x)\, dx\right]^2 \leq \int_0^1 \Big[g(x)\Big]^2 dx$$

iii. Deduce that **1**

$$\left[\int_0^1 g(x)\, dx\right]^4 \leq \int_0^1 \Big[g(x)\Big]^4 dx$$

End of paper.

Sample Band E4 Responses

Question 1 (Sekaran and Ham(d))

(a) i. (3 marks)

✓ [1] for A and B

✓ [1] for correct integration

✓ [1] for final answer

Let $\dfrac{1}{x(4-x)} = \dfrac{A}{x} + \dfrac{B}{4-x}$

$$\begin{aligned} 1 &= A(4-x) + Bx \\ A &= \frac{1}{4}, B = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \int_1^3 \frac{1}{x(4-x)}\,dx &= \frac{1}{4}\int_1^3 \frac{1}{x} + \frac{1}{4-x}\,dx \\ &= \frac{1}{4}\left[\ln|x| - \ln|4-x|\right]_1^3 \\ &= \frac{1}{4}\left(\ln 3 - \ln 1 - (\ln 1 - \ln 3)\right) \\ &= \frac{\ln 3}{2} \end{aligned}$$

ii. (2 marks)

✓ [1] for $du = -dx$ and the new limits

✓ [1] for final result

Let $u = 4-x, du = -dx$

When $x = 1, u = 3$ and $x = 3, u = 1$.

$$\begin{aligned} I &= \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx \\ &= -\int_3^1 \frac{\cos^2\left[\frac{\pi}{8}(4-u)\right]}{(4-u)u}\,du \\ &= \int_1^3 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}u\right)}{u(4-u)}\,du \\ &= \int_1^3 \frac{\sin^2 \frac{\pi}{8}u}{u(4-u)}\,du \end{aligned}$$

iii. (1 mark)

From (ii), $I = \int_1^3 \dfrac{\sin^2 \frac{\pi}{8}x}{x(4-x)}\,dx$

$$\begin{aligned} 2I &= \int_1^3 \frac{\cos^2 \frac{\pi}{8}x}{x(4-x)} + \frac{\sin^2 \frac{\pi}{8}x}{x(4-x)}\,dx \\ &= \int_1^3 \frac{1}{x(4-x)}\,dx \\ &= \frac{\ln 3}{2} \quad \ldots\text{from part (i)} \\ \therefore I &= \frac{\ln 3}{4} \end{aligned}$$

(b) (3 marks)

- ✓ [1] for correctly completing the square.
- ✓ [1] for substantial progress to final answer
- ✓ [1] for final answer

$$\begin{aligned}
\int \sqrt{\frac{3-x}{x}}\,dx &= \int \frac{3-x}{x\sqrt{3-x}}\,dx \\
&= \int \frac{3-x}{\sqrt{-x^2+3x}}\,dx \\
&= \int \frac{3-x}{\sqrt{-\left(x-\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}}\,dx \\
&= \int \frac{\frac{1}{2}(3-2x)+\frac{3}{2}}{\sqrt{-\left(x-\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}}\,dx \\
&= \int \frac{\frac{1}{2}(3-2x)+\frac{3}{2}}{\sqrt{-\left(x-\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}}\,dx \\
&= \frac{1}{2}\int \frac{3-2x}{\sqrt{3x-x^2}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}\,dx \\
&= \frac{1}{2}\sqrt{3x-x^2}\times 2 + \frac{3}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{3}{2}}\right)+C \\
&= \sqrt{3x-x^2}+\frac{3}{2}\sin^{-1}\left(\frac{2x-3}{3}\right)+C
\end{aligned}$$

(c) (4 marks)

- ✓ [1] for transforming the differential.
- ✓ [1] for transforming both limits.
- ✓ [1] for transforming integrand to an integrable form.
- ✓ [1] for final answer.

$$\begin{aligned}
x &= 2\sec\theta \\
\therefore dx &= 2\sec\theta\tan\theta\,d\theta
\end{aligned}$$

Transforming the limits,

$$\begin{aligned}
x = 4 &\quad \theta = \frac{\pi}{3} \\
x = 2 &\quad \theta = 0
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \frac{1}{4\sec^2\theta\sqrt{4\sec^2\theta - 4}} 2\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sec\theta \times 2\tan\theta}\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta \qquad\qquad = \frac{1}{4}\Big[\sin\theta\Big]_0^{\frac{\pi}{3}} \\
&= \frac{1}{4} \times \frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{3}}{8}
\end{aligned}$$

(d) (4 marks)

- ✓ [1] for one application of integrating by parts
- ✓ [1] for another application of integrating by parts
- ✓ [1] for obtaining $2I$
- ✓ [1] for evaluating $\cos(k\pi)$ and $\cos((k-1)\pi)$ and getting the final result

$$I = \int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx$$

$$\left|\begin{aligned} u &= \sin x & v &= e^x \\ du &= \cos x & dv &= e^x \end{aligned}\right.$$

$$I = \Big[e^x \sin x\Big]_{(k-1)\pi}^{k\pi} - \int_{(k-1)\pi}^{k\pi} e^x \cos x\,dx$$

$$\left|\begin{aligned} u &= \cos x & v &= e^x \\ du &= -\sin x & dv &= e^x \end{aligned}\right.$$

$$\begin{aligned}
&= 0 - \left(\Big[e^x \cos x\Big]_{(k-1)\pi}^{k\pi} + \int_{(k-1)\pi}^{k\pi} e^x \sin x\,dx\right) \\
\therefore 2I &= -e^{k\pi}\cos(k\pi) + e^{(k-1)\pi}\cos(k-1)\pi \\
&= -e^{k\pi}(-1)^k + e^{(k-1)\pi}(-1)^{k-1} \\
&= e^{k\pi}(-1)^{k+1} + e^{(k-1)\pi}(-1)^{k-1} \\
&= e^{(k-1)\pi}(-1)^{k-1}\left[(-1)^2 e^{\pi} + 1\right] \\
\therefore I &= \frac{(-1)^{k-1}e^{k\pi}(e^{\pi}+1)}{2e^{\pi}}
\end{aligned}$$

Question 2 (Ho)

(a) i. (2 marks)

✓ [1] for $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$

✓ [1] for final answer

$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= \frac{2-3m-1}{1^2+m^2+1^2}\left(\underset{\sim}{i}+m\underset{\sim}{j}-\underset{\sim}{k}\right)\\ &= \frac{1-3m}{2+m^2}\left(\underset{\sim}{i}+m\underset{\sim}{j}-\underset{\sim}{k}\right)\end{aligned}$$

$$\begin{aligned}\therefore \frac{1-3m}{2+m^2} &= -\frac{11}{18}\\ 18-54m &= -11m^2-22\\ 11m^2-54m+40 &= 0\\ (11m-10)(m-4) &= 0\\ \therefore m = \frac{10}{11} &\text{ or } m=4\end{aligned}$$

$$\text{But } \because m\in\mathbb{Z}^+,\ m=4.$$

ii. (1 mark)

$$2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}-\left(-\frac{11}{18}\right)\left(\underset{\sim}{i}+4\underset{\sim}{j}-\underset{\sim}{k}\right)=\frac{47}{18}\underset{\sim}{i}+\frac{-5}{9}\underset{\sim}{j}-\frac{7}{18}\underset{\sim}{k}$$

(b) i. (1 mark)

$$\begin{aligned}d &= \left|\begin{pmatrix}4+\lambda_2\\-2+2\lambda_2\\9-2\lambda_2\end{pmatrix}-\begin{pmatrix}1+2\lambda_1\\2\lambda_1\\2-3\lambda_1\end{pmatrix}\right|\\ &= \left|\begin{pmatrix}3-2\lambda_1+\lambda_2\\-2-2\lambda_1+2\lambda_2\\7+3\lambda_1-2\lambda_2\end{pmatrix}\right|\\ &= \sqrt{(3-2\lambda_1+\lambda_2)^2+(-2-2\lambda_1+2\lambda_2)^2+(7+3\lambda_1-2\lambda_2)^2}\end{aligned}$$

ii. (1 mark) Minimum $d=\sqrt{36}=6$

iii. (2 marks)

✓ [1] for λ_1 and λ_2.

✓ [1] for final answer

Minimum d when $3\lambda_2 - 4\lambda_1 - 5 = 0$ and $\lambda_1 - 1 = 0$

$$\therefore \lambda_1 = 1 \text{ and } \lambda_2 = 3$$

$$\text{When } \lambda_1 = 1,$$
$$\underset{\sim}{\mathrm{r}}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$\text{When } \lambda_2 = 3,$$
$$\underset{\sim}{\mathrm{r}}_2 = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$

$\therefore$ The coordinates are $(3, 2, -1)$ and $(7, 4, 3)$.

(c) i. (2 marks)

✓ [1] for (1) or equivalent

✓ [1] for final result

$$\begin{aligned} \overrightarrow{ML} &= \overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DH} + \overrightarrow{HL} \ \ldots (1) \\ &= -\frac{1}{4}\underset{\sim}{\mathrm{a}} + \underset{\sim}{\mathrm{d}} + \underset{\sim}{\mathrm{e}} + \frac{3}{4}\underset{\sim}{\mathrm{a}} \\ &= \frac{1}{2}\underset{\sim}{\mathrm{a}} + \underset{\sim}{\mathrm{d}} + \underset{\sim}{\mathrm{e}} \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} \overrightarrow{NK} &= \overrightarrow{NB} + \overrightarrow{BF} + \overrightarrow{FG} + \overrightarrow{GK} \\ &= \frac{1}{4}\underset{\sim}{\mathrm{a}} + \underset{\sim}{\mathrm{e}} + \underset{\sim}{\mathrm{d}} - \frac{3}{4}\underset{\sim}{\mathrm{a}} \\ &= -\frac{1}{2}\underset{\sim}{\mathrm{a}} + \underset{\sim}{\mathrm{d}} + \underset{\sim}{\mathrm{e}} \end{aligned}$$

iii. (4 marks)

✓ [1] for one of two expressions of $\overrightarrow{AP}$

✓ [1] for the other expression of $\overrightarrow{AP}$

✓ [1] for substantial progress to find λ and μ

✓ [1] for final result

$$\begin{aligned}
\overrightarrow{AP} &= \overrightarrow{AM} + \overrightarrow{MP} \\
&= \overrightarrow{AM} + \lambda\,\overrightarrow{ML} \quad \ldots\text{where } \lambda \in \mathbb{R} \\
&= \frac{1}{4}\underset{\sim}{a} + \lambda\left(\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{e}\right) \\
\overrightarrow{AP} &= \overrightarrow{AN} + \overrightarrow{NP} \\
&= \overrightarrow{AN} + \mu\,\overrightarrow{NK} \quad \ldots\text{where } \mu \in \mathbb{R} \\
&= \frac{3}{4}\underset{\sim}{a} + \mu\left(-\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{e}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{1}{4}\underset{\sim}{a} + \lambda\left(\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{e}\right) &= \frac{3}{4}\underset{\sim}{a} + \mu\left(-\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{d} + \underset{\sim}{e}\right) \\
\frac{1}{2}(1-\mu-\lambda)\underset{\sim}{a} + (\mu-\lambda)\underset{\sim}{d} + (\mu-\lambda)\underset{\sim}{e} &= \underset{\sim}{0}
\end{aligned}$$

$$\begin{aligned}
\therefore \mu = \lambda \text{ and } 1-\mu-\lambda &= 0 \\
1 - 2\lambda &= 0 \\
\lambda = \mu &= \frac{1}{2}
\end{aligned}$$

Hence KN and ML bisect each other at P.

Question 3 (Ham)

(a) (1 mark)

$$\begin{aligned}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2 &\geq 0\\ \frac{a}{b}-2+\frac{b}{a} &\geq 0\\ \therefore \frac{a}{b}+\frac{b}{a} &\geq 2\end{aligned}$$

(b) (2 marks)

✓ [1] for (1).

✓ [1] for final result.

$$\begin{aligned}\frac{1}{a}+\frac{1}{b}+\frac{1}{c} &= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad \ldots(1)\\ &= 1+1+1+\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}\\ &\geq 3+2+2+2\\ &= 9\end{aligned}$$

(c) (2 marks)

✓ [1] for (*).

✓ [1] for final result.

$$\begin{aligned}(1-a)(1-b)(1-c) &= (1-b-a+ab)(1-c)\\ &= 1-c-b+bc-a+ac+ab-abc\\ &= 1-(a+b+c)+ab+ac+bc-abc\\ &= 0+ab+ac+bc-abc \quad \ldots(\because a+b+c=1) \quad (*)\\ &\geq 9abc-abc \quad \ldots\left(\frac{ab+ac+bc}{abc}\geq 9,\ ab+ac+bc\geq 9abc \text{ from (ii)}\right)\\ &= 8abc\end{aligned}$$

OR

$$\begin{aligned}(1-a)(1-b)(1-c) &= (b+c)(c+a)(a+b) \quad \ldots(\because a+b+c=1)\\ &\geq 2\sqrt{bc}\times 2\sqrt{ac}\times 2\sqrt{ab} \quad \ldots\left(\because (\sqrt{a}-\sqrt{b})^2\geq 0,\ a+b\geq 2\sqrt{ab}\right)\\ &= 8\sqrt{a^2b^2c^2}\\ &= 8abc\end{aligned}$$

(d) i. (2 marks)

✓ [1] for expressing as a quadratic function in λ.

✓ [1] for final result.

$$\begin{aligned}\int_0^a \Big[\lambda g(x) + h(x)\Big]^2 dx &= \int_0^a \lambda^2 \Big[g(x)\Big]^2 + 2\lambda\, g(x)h(x) + \Big[h(x)\Big]^2 dx \\ &= \left(\int_0^a \Big[g(x)\Big]^2 dx\right)\lambda^2 + 2\left(\int_0^a g(x)h(x)\,dx\right)\lambda + \int_0^a \Big[h(x)\Big]^2 dx\end{aligned}$$

Since $\displaystyle\int_0^a \Big[\lambda g(x) + h(x)\Big]^2 dx \geq 0$

$$\begin{aligned}4\left(\int_0^a g(x)h(x)\,dx\right)^2 - 4\left(\int_0^a \Big[g(x)\Big]^2 dx\right)\left(\int_0^a \Big[h(x)\Big]^2 dx\right) &\leq 0 \\ \therefore \left[\int_0^a g(x)h(x)\,dx\right]^2 \leq \int_0^a \Big[g(x)\Big]^2 dx \times \int_0^a \Big[h(x)\Big]^2 dx\end{aligned}$$

ii. (1 mark)

Let $h(x) = 1$ and $a = 1$ in (i).

$$\left[\int_0^1 g(x)\,dx\right]^2 \leq \int_0^1 \Big[g(x)\Big]^2 dx \times \int_0^1 1\,dx = \int_0^1 \Big[g(x)\Big]^2 dx$$

iii. (1 mark)

Squaring both sides in (ii),

$$\left[\int_0^1 g(x)\,dx\right]^4 \leq \left[\int_0^1 \Big[g(x)\Big]^2 dx\right]^2$$

and replace $g(x)$ with $\Big[g(x)\Big]^2$ in (ii), then

$$\begin{aligned}\left[\int_0^1 \Big[g(x)\Big]^2 dx\right]^2 &\leq \int_0^1 \Big[g(x)\Big]^4 dx \\ \therefore \left[\int_0^1 g(x)\,dx\right]^4 \leq \left[\int_0^1 \Big[g(x)\Big]^2 dx\right]^2 &\leq \int_0^1 \Big[g(x)\Big]^4 dx\end{aligned}$$