NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2023 Year 12 Course Assessment Task 3

Wednesday, 31 May 2023

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- A NESA Reference Sheet is provided.
- Attempt **all** questions.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Commence each new question on a new booklet. Write on both sides of the paper.
- At the conclusion of the examination, arrange the booklets used in the correct order after this paper and hand to examination supervisors.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MXX.1 – Mr Ho ◯ 12MXX.2 – Mr Sekaran ◯ 12MXX.3 – Ms Ham

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	$\overline{13}$	$\overline{8}$	$\overline{15}$	$\overline{36}$	

Question 1 (13 Marks) Commence a NEW writing booklet **Marks**

(a) i. Find the real numbers A, B and C such that **3**

$$\frac{28}{(5+x^2)(3-x)} = \frac{Ax+B}{5+x^2} + \frac{C}{3-x}$$

ii. Hence find: $\displaystyle\int \frac{28}{(5+x^2)(3-x)}\,dx$. **3**

(b) Evaluate as an exact value: $\displaystyle\int_0^{\frac{1}{6}} \frac{\ln(1-3x)}{(1-3x)^3}\,dx$. **3**

(c) Use a suitable substitution to evaluate as an exact value: **4**

$$\int_4^8 \frac{1}{x^3\sqrt{x^2-16}}\,dx$$

Question 2 (8 Marks) Commence a NEW writing booklet **Marks**

(a) i. Show that, for $x > 1$, **1**

$$0 < \frac{1}{x\sqrt{x}} < \frac{1}{x} < \frac{1}{\sqrt[3]{x^2}}$$

ii. Hence show that, for $t \geq 1$, **2**

$$\frac{2\left(\sqrt{t}-1\right)}{\sqrt{t}} \leq \ln t \leq 3\left(\sqrt[3]{t}-1\right)$$

iii. Hence find $\displaystyle\lim_{t\to\infty} \frac{\ln t}{\sqrt{t}}$. **2**

(b) Consider the function $f(x) = \tan^{-1} x$.

i. Show that $f''(x) < 0$ for $x > 0$. **1**

ii. Hence by considering the concavity of $f(x)$, or otherwise, show that for $a > b > 0$: **2**

$$\tan^{-1}(a-b) + \tan^{-1}(a+b) < 2\tan^{-1} a$$

Question 3 (15 Marks) Commence a NEW writing booklet **Marks**

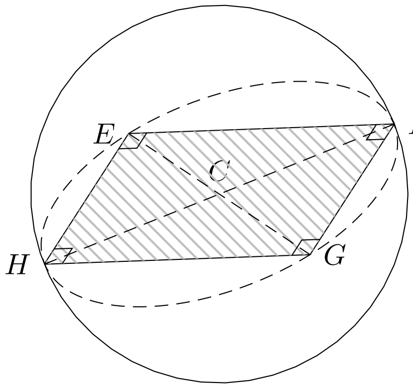
(a) The line ℓ_1 passes through the point $(0,2,1)$ and is parallel to the vector $\underset{\sim}{p} = -\underset{\sim}{i} - 4\underset{\sim}{j} + 9\underset{\sim}{k}$.

i. Find $\underset{\sim}{r}$, the vector equation of ℓ_1, in terms of $\lambda \in \mathbb{R}$. **1**

Let $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$ be the vector equation of the line ℓ_2, where $\mu \in \mathbb{R}$.

ii. Prove that ℓ_1 and ℓ_2 are skew. **4**

(b) The sphere S has its centre at point C. $EFGH$ is a square such that E, F, G and H are all points on the sphere S, and its diagonals intersect at C. The points $E(0,3,4)$, $F(-4,12,3)$ and $G(4,15,-2)$ are given.



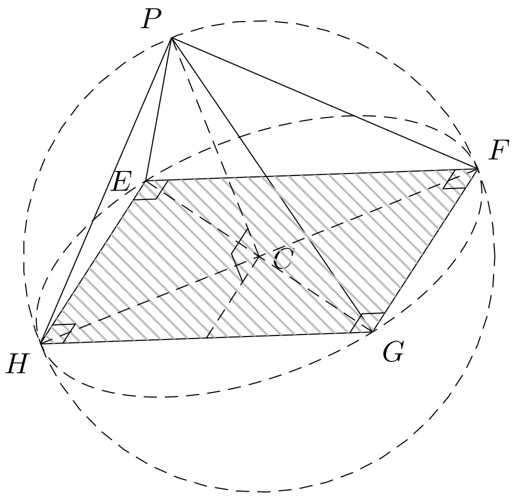
Let O be the origin.

i. Prove that $\overrightarrow{OC} = \dfrac{1}{2}\left(\overrightarrow{OE} + \overrightarrow{OG}\right)$ and show that C has coordinates $(2,9,1)$. **2**

ii. Hence find the vector equation of the sphere S. Let $\underset{\sim}{v}$ be the position vector of any point on the surface of the sphere. **2**

Examination continues overleaf...

$P(a, b, c)$ is a point on the sphere S such that $PEFGH$ is a right square pyramid inscribed in the sphere. PC is the perpendicular height of $PEFGH$, as shown in the diagram.



iii. Using vector methods, show that: **2**

$$-2a - 6b + 3c = -55$$
$$-6a + 3b + 2c = 17$$

iv. Hence find all possible coordinates for point P. **4**

End of paper.

Suggested Band E4 Responses

Question 1 (Sekaran & Ham)

(a) i. ✓ [1] for one correct value of A, B or C
✓ [1] for sufficient working out
✓ [1] for all correct values of A, B or C

$$\frac{28}{(5+x^2)(3-x)} = \frac{Ax+B}{5+x^2} + \frac{C}{3-x}$$
$$28 = (Ax+B)(3-x) + C(5+x^2)$$

Let $x = 3$:	$28 = 14C$	$\Rightarrow$	$C = 2$
Let $x = 0$:	$28 = 3B + 10$	$\Rightarrow$	$B = 6$
Let $x = 1$:	$28 = 2(A+6) + 12$	$\Rightarrow$	$A = 2$

$\therefore A = 2$, $B = 6$ and $C = 2$.

ii. ✓ [1] for correct $\ln\left(5+x^2\right)$ integral
✓ [1] for correct $\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right)$ integral
✓ [1] for correct $\ln|3-x|$ integral with absolute value

$$\begin{aligned}\int \frac{28}{(5+x^2)(3-x)}\,dx &= \int \frac{2x+6}{5+x^2} + \frac{2}{3-x}\,dx \\ &= \int \frac{2x}{5+x^2} + \frac{6}{5+x^2} + \frac{2}{3-x}\,dx \\ &= \ln\left(5+x^2\right) + \frac{6}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) - 2\ln|3-x| + C \qquad ✓✓✓\end{aligned}$$

(b) ✓ [1] for correct integration by parts
✓ [1] for correct subsequent integral
✓ [1] for correct exact value

$$\begin{aligned}&\text{Let} \quad u = \ln(1-3x) \quad \rightarrow \quad u' = \frac{-3}{1-3x} \\ &\text{and} \quad v' = \frac{1}{(1-3x)^3} \quad \rightarrow \quad v = \frac{1}{6(1-3x)^2}\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{6}} \frac{\ln(1-3x)}{(1-3x)^3}\,dx &= \left[\frac{\ln(1-3x)}{6(1-3x)^2}\right]_0^{\frac{1}{6}} - \int_0^{\frac{1}{6}} \frac{-1}{2(1-3x)^3}\,dx \qquad ✓ \\ &= \frac{1}{6}\left(4\ln\left(\frac{1}{2}\right) - 0\right) + \frac{1}{2}\left[\frac{1}{6(1-3x)^2}\right]_0^{\frac{1}{6}} \qquad ✓ \\ &= -\frac{2}{3}\ln 2 + \frac{1}{12}(4-1) \qquad ✓ \\ &= \frac{1}{4} - \frac{2}{3}\ln 2\end{aligned}$$

(c) ✓ [1] for correct substitution with bounds for integral in terms of θ
✓ [1] for correctly simplifying integral using substitution
✓ [1] for correct primitive
✓ [1] for correct exact value

Let $x = 4\sec\theta$.

$$\frac{dx}{d\theta} = -\frac{(-4\sin\theta)}{\cos^2\theta}$$
$$dx = 4\sec\theta\tan\theta\,d\theta$$

$$\theta = \cos^{-1}\left(\frac{4}{x}\right)$$

$$x = 8 \quad \rightarrow \quad \theta = \frac{\pi}{3}$$
$$x = 4 \quad \rightarrow \quad \theta = 0$$

$$\begin{aligned}
&\int_4^8 \frac{1}{x^3\sqrt{x^2-16}}\,dx \\
&= \int_0^{\frac{\pi}{3}} \frac{4\sec\theta\tan\theta}{64\sec^3\theta\sqrt{16\sec^2\theta-16}}\,d\theta \qquad \checkmark \\
&= \int_0^{\frac{\pi}{3}} \frac{\tan\theta}{64\sec^2\theta\tan\theta}\,d\theta \\
&= \frac{1}{64}\int_0^{\frac{\pi}{3}} \cos^2\theta\,d\theta \qquad \checkmark \\
&= \frac{1}{128}\int_0^{\frac{\pi}{3}} 1+\cos 2\theta\,d\theta \\
&= \frac{1}{128}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} \qquad \checkmark \\
&= \frac{1}{128}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}-0\right) \\
&= \frac{1}{128}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) \qquad \checkmark
\end{aligned}$$

Question 2 (Ham)

(a) i. ✓ [1] for correct proof with sufficient working

$$\begin{aligned} x^4 < x^6 &< x^9 \qquad \text{for } x > 1 \\ \sqrt[3]{x^2} < x &< \sqrt{x^3} \\ \frac{1}{\sqrt{x^3}} < \frac{1}{x} &< \frac{1}{\sqrt[3]{x^2}} \qquad (x > 1) \\ \therefore \quad 0 < \frac{1}{\sqrt{x^3}} < \frac{1}{x} &< \frac{1}{\sqrt[3]{x^2}} \qquad \text{as } x > 1 \qquad \checkmark \end{aligned}$$

ii. ✓ [1] for correct inequality (or equal) for primitives from (i)
✓ [1] for final result with correct working

Note that for $t = 1$, all integrals equal 0:

$$\int_1^1 \frac{1}{\sqrt{x^3}}\,dx = \int_1^1 \frac{1}{x}\,dx = \int_1^1 \frac{1}{\sqrt[3]{x^2}}\,dx = 0$$

So using part (i):

$$\begin{aligned} \int_1^t \frac{1}{\sqrt{x^3}}\,dx \le \int_1^t \frac{1}{x}\,dx &\le \int_1^t \frac{1}{\sqrt[3]{x^2}}\,dx \\ -2\left[\frac{1}{\sqrt{x}}\right]_1^t \le \Big[\ln x\Big]_1^t &\le 3\Big[\sqrt[3]{x}\Big]_1^t \qquad \checkmark \\ -2\left(\frac{1}{\sqrt{t}} - 1\right) \le \ln t &\le 3\left(\sqrt[3]{t} - 1\right) \\ \therefore \quad \frac{2\left(\sqrt{t} - 1\right)}{\sqrt{t}} \le \ln t &\le 3\left(\sqrt[3]{t} - 1\right) \qquad \checkmark \end{aligned}$$

iii. ✓ [1] for correct limit of $t \to \infty$ for both bounds
✓ [1] for correct limit with justification

Multiplying by $\dfrac{1}{\sqrt{t}}$ from (ii): (note: $\sqrt{t} > 0$)

$$\frac{2}{\sqrt{t}} - \frac{2}{t} \le \frac{\ln t}{\sqrt{t}} \le \frac{3}{\sqrt[6]{x}} - \frac{3}{\sqrt{t}}$$

$$\begin{aligned} \lim_{t\to\infty}\left(\frac{2}{\sqrt{t}} - \frac{2}{t}\right) &= 0 \\ \lim_{t\to\infty}\left(\frac{3}{\sqrt[6]{x}} - \frac{3}{\sqrt{t}}\right) &= 0 \qquad \checkmark \end{aligned}$$

As $t \to \infty$, $0 \le \dfrac{\ln t}{\sqrt{t}} \le 0$.

$$\therefore \quad \lim_{t\to\infty} \frac{\ln t}{\sqrt{t}} = 0 \qquad \checkmark$$

(b) i. ✓ [1] for correct proof with sufficient working

$$f'(x) = \frac{1}{1+x^2}$$
$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

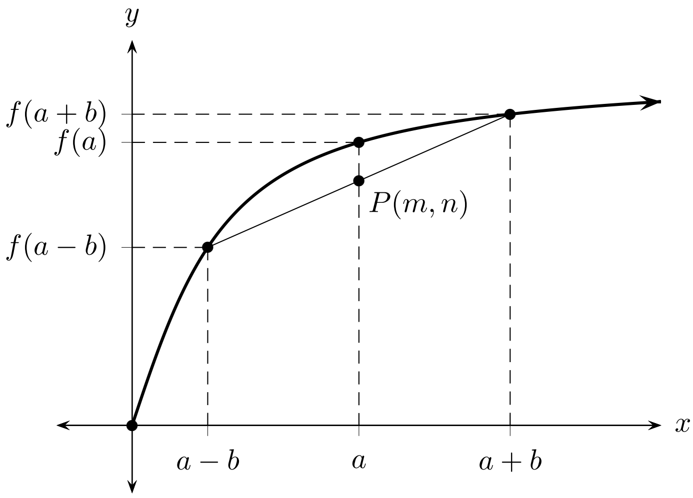
Since $(1+x^2)^2 > 0$ for $x > 0$,

$$\therefore \quad f''(x) > 0 \text{ for } x > 0 \qquad ✓$$

ii. ✓ [1] for establishing inequality with sufficient explanation
✓ [1] for correct proof with sufficient working

From part (i), $f(x)$ is concave down and increasing for $x > 0$.

Let $P(m,n)$ be the midpoint of the secant:



Note: diagram is not required.

$$\begin{aligned} n &< f(a) \qquad ✓ \\ \frac{\tan^{-1}(a-b) + \tan^{-1}(a+b)}{2} &< \tan^{-1}(a) \\ \therefore \quad \tan^{-1}(a-b) + \tan^{-1}(a+b) &< 2\tan^{-1}(a) \qquad ✓ \end{aligned}$$

Question 3 (Ho)

(a) i. ✓ [1] for correct answer

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -4 \\ 9 \end{pmatrix} \qquad \checkmark$$

ii. ✓ [1] for establishing correct set of equations from parametric forms
✓ [1] for proving that $\underset{\sim}{r}$ and $\underset{\sim}{s}$ do not intersect with correct conclusion
✓ [1] for proving that $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are not parallel with correct conclusion
✓ [1] for proving that ℓ_1 and ℓ_2 are skew with sufficient justification throughout

If ℓ_1 and ℓ_2 intersect, then $\underset{\sim}{r} = \underset{\sim}{s}$:

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

From the parametric equations:

$$\lambda + 5\mu = -3 \tag{3.1}$$
$$4\lambda + \mu = 4 \tag{3.2}$$
$$-9\lambda + 5\mu = 3 \qquad \checkmark \tag{3.3}$$

Equation (3.1) − (3.3):

$$10\lambda = -6 \qquad \Rightarrow \qquad \lambda = -\frac{3}{5}$$

Equation (3.1) − 5 × (3.2):

$$-19\lambda = 17 \qquad \Rightarrow \qquad \lambda = -\frac{17}{19}$$

$\therefore$ $\underset{\sim}{r}$ and $\underset{\sim}{s}$ do not intersect (no such $\lambda \in \mathbb{R}$ for which $\underset{\sim}{r} = \underset{\sim}{s}$). ✓

Considering the direction vectors for $\underset{\sim}{r}$ and $\underset{\sim}{s}$:

$$\begin{pmatrix} -1 \\ -4 \\ 9 \end{pmatrix} \neq \psi \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

$\therefore$ $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are not parallel (no such $\psi \in \mathbb{R}$ for which their direction vectors are scalar multiples of each other). ✓

Hence ℓ_1 and ℓ_2 are skew. ✓

(b) i. ✓ [1] for correct proof of $\overrightarrow{OC}$ sufficient working
✓ [1] for showing coordinates of C with sufficient working and conclusion

Diagonals of a square bisect each other.

$$\begin{aligned} \overrightarrow{EC} &= \frac{1}{2}\overrightarrow{EG} \\ \overrightarrow{OC} - \overrightarrow{OE} &= \frac{1}{2}\left(\overrightarrow{OG} - \overrightarrow{OE}\right) \\ \therefore \quad \overrightarrow{OC} &= \frac{1}{2}\left(\overrightarrow{OE} + \overrightarrow{OG}\right) \qquad \checkmark \\ &= \frac{1}{2}\left(\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 15 \\ -2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 2 \\ 9 \\ 1 \end{pmatrix} \end{aligned}$$

$\therefore \quad C(2, 9, 1) \qquad \checkmark$

ii. ✓ [1] for correct radius with working out
✓ [1] for correct vector equation for sphere S

$$\begin{aligned} \overrightarrow{CE} &= \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} \\ \left|\overrightarrow{CE}\right| &= 7 \end{aligned}$$

$\therefore \quad$ Sphere S has radius 7 units.

Hence the vector equation of sphere S is: $\left|\underset{\sim}{v} - \begin{pmatrix} 2 \\ 9 \\ 1 \end{pmatrix}\right| = 7. \qquad \checkmark$

iii. ✓ [1] for proving equation (3.1)
✓ [1] for proving equation (3.2)

$PC \perp CE$ and $PC \perp CF$, since PC is perpendicular to base $EFGH$.

$$\overrightarrow{CP} = \begin{pmatrix} a-2 \\ b-9 \\ c-1 \end{pmatrix} \quad , \quad \overrightarrow{CE} = \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} \quad , \quad \overrightarrow{CF} \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}$$

From $PC \perp CE$:

$$\begin{aligned} \begin{pmatrix} a-2 \\ b-9 \\ c-1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} &= 0 \\ -2a + 4 - 6b + 54 + 3c - 3 &= 0 \\ \therefore \quad -2a - 6b + 3c &= -55 \qquad \checkmark \end{aligned} \tag{3.1}$$

From $PC \perp CF$:

$$\begin{pmatrix} a-2 \\ b-9 \\ c-1 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = 0$$
$$-6a + 12 + 3b - 27 + 2c - 2 = 0$$
$$\therefore \quad -6a + 3b + 2c = 17 \qquad \checkmark \tag{3.2}$$

iv. ✓ [1] for a correct vector on ℓ

✓ [1] for one correct vector for $\overrightarrow{CP}$

✓ [1] for one correct vector for $\overrightarrow{OP}$

✓ [1] for both correct coordinates of P

Equations $(3.1) + 2 \times (3.2)$:

$$-14a + 7c = -21$$
$$2a - c = 3 \tag{3.3}$$

Let $a = 1$:

$$c = -1 \qquad \text{(from (3.3))}$$
$$\therefore \quad b = \frac{25}{3} \qquad \text{(from (3.2))}$$

So $M\left(1, \frac{25}{3}, -1\right)$ is a point on the line ℓ that is perpendicular to base $EFGH$.

$$\overrightarrow{CM} = \begin{pmatrix} -1 \\ -\frac{2}{3} \\ -2 \end{pmatrix} \qquad \checkmark$$
$$|\overrightarrow{CM}| = \frac{7}{3}$$

$|\overrightarrow{CP}| = 7$ and $CP \parallel CM$, since P is on sphere S and also on ℓ:

$$\overrightarrow{CP} = \pm 3 \times \overrightarrow{CM} \qquad \checkmark$$
$$\overrightarrow{CP_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad , \quad \overrightarrow{CP_2} = \begin{pmatrix} -3 \\ -2 \\ -6 \end{pmatrix}$$

$$\overrightarrow{OP} = \overrightarrow{OC} - \overrightarrow{CP}$$
$$\overrightarrow{OP_1} = \begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} \quad , \quad \overrightarrow{OP_2} = \begin{pmatrix} 5 \\ 11 \\ 7 \end{pmatrix} \qquad \checkmark$$

Hence P has coordinates $(-1, 7, -5)$ or $(5, 11, 7)$. ✓

iv. **Alternate marking scheme**

- ✓ [1] for correct $\overrightarrow{OP}$ in terms of one parameter only
- ✓ [1] for both correct values of the parameter
- ✓ [1] for one correct vector for $\overrightarrow{OP}$
- ✓ [1] for both correct coordinates of P

From (iii):

$$-2a - 6b + 3c = -55 \tag{3.1}$$
$$-6a + 3b + 2c = 17 \tag{3.2}$$

Equations $(3.1) + 2 \times (3.2)$:

$$\begin{aligned} -14a + 7c &= -21 \\ c &= 2a - 3 \end{aligned}$$

Equations $2 \times (3.1) - 3 \times (3.2)$:

$$\begin{aligned} 14a - 21b &= -161 \\ b &= \frac{2}{3}a + \frac{23}{3} \end{aligned}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} a \\ \frac{2}{3}a + \frac{23}{3} \\ 2a - 3 \end{pmatrix}$$

From (ii):

$$\left| \underset{\sim}{v} - \begin{pmatrix} 2 \\ 9 \\ 1 \end{pmatrix} \right| = 7 \tag{3.5}$$

Since P lies on the surface of the sphere, substitute $\overrightarrow{OP}$ into Equation (3.5):

$$\begin{aligned} \left| \begin{pmatrix} a-2 \\ \frac{2}{3}a - \frac{4}{3} \\ 2a - 4 \end{pmatrix} \right| &= 7 \\ \frac{1}{3}\sqrt{(3a-6)^2 + (2a-4)^2 + (6a-12)^2} &= 7 \\ 49a^2 - 196a + 196 &= 441 \\ a^2 - 4a - 5 &= 0 \\ (a+1)(a-5) &= 0 \\ a = -1 \quad \text{or} \quad a &= 5 \qquad \checkmark \end{aligned}$$

$$\therefore \quad \overrightarrow{OP_1} = \begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} \quad , \quad \overrightarrow{OP_2} = \begin{pmatrix} 5 \\ 11 \\ 7 \end{pmatrix} \qquad \checkmark$$

Hence P has coordinates $(-1, 7, -5)$ or $(5, 11, 7)$. ✓