Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2016
ASSESSMENT TASK 1

Mathematics Extension 2

General Instructions

- **Working time – 60 minutes**
  Reading time – 5 minutes
- Write on both sides of the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators only
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.** (The multiple choice section is answered in a grid within the question booklet)
- Attempt all questions.

Class Teacher:
(Please tick or highlight)

○ Dr Jomaa

○ Mr Ireland

○ Mr Lin

Student Number:

(To be used by the exam markers only.)

Question	**1-5**	**6**	**7**	**8**	**Total**	**Percent**
Mark	/5	/13	/13	/14	/45	/100

Section I

5 Marks

Attempt Questions 1 to 5

Mark your answers on the answer sheet provided.

1. If $z = \frac{3+4i}{1+2i}$, the imaginary part of z is:

   A) -2

   B) $-\frac{2}{5}i$

   C) $-\frac{2}{5}$

   D) $-2i$

2. If $z = a + ib$ where a and b are real and non-zero. Which of the following is true?

   A) $z + \bar{z}$ is real

   B) $\frac{z}{\bar{z}}$ is non real

   C) $z^2 - (\bar{z})^2$ is real

   D) $z\bar{z}$ is real and positive

3. If $z = \sqrt{3} + i$ then the modulus/argument form $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. If $z^n + (\bar{z})^n$ is to be rational, then the integer $'n'$ cannot be:

   A) 2

   B) 3

   C) 5

   D) 6

4. The polynomial equation $P(x) = 0$ has real coefficients, and has roots which include $x = -2 + i$ and $x = 2$. What is the minimum degree of $P(x)$

   (A) 1

   (B) 2

   (C) 3

   (D) 4

5. The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero. What is the value of the double zero?

(A) -7

(B) -4

(C) 4

(D) 2

END OF SECTION I

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

1 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
2 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
3 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ
5 –	Ⓐ	Ⓑ	Ⓒ	Ⓓ

Section II

Total of 40 marks

Attempt Questions 6 to 8.

Write your answers in the writing book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks)

(a) Let $z = 2 + 2i$ and $w = 1 - i$. Find in the form of x+iy, where x and y are real,

(i) $z\overline{w}$ **1**

(ii) $|z|$ **1**

(iii) $arg(zw)$ **2**

(b) Sketch the intersection of the following loci: **3**

$|z - 2| = 2$ and $-\frac{\pi}{4} \leq Arg(z) \leq \frac{\pi}{4}$

(c) Find the square root of $21 - 20i$ **3**

(d) (i) Express $\frac{-1+i}{\sqrt{2}}$ in modulus-argument form **1**

(ii) Hence express $\left(\frac{-1+i}{\sqrt{2}}\right)^{1002}$ in Cartesian form **2**

Question 7 (13 marks)

(a) It is given that $1 + i$ is a root of $P(z) = 2z^3 - 3z^2 + rz + s$ where r and s are real.

(i) Explain why $1 - i$ is also a root of the equation. **1**

(ii) Factorise P(z) over the real field. **2**

(b) The equation $x^3 + px - 1 = 0$ has 3 non zero real roots α, β and γ

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Hence find the value of $\alpha^4 + \beta^4 + \gamma^4$ **2**

(iii) Show that p must be negative **1**

(iv) Find the monic equation with coefficients in terms of p whose roots are **2**

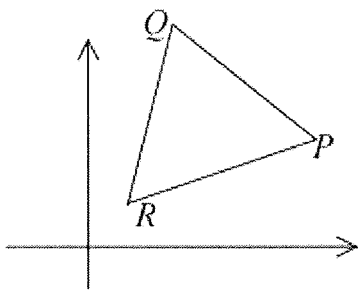
$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$$

(c) $P(x) = x^4 + 3x^3 - 6x^2 - 28x + c$ has a zero of multiplicity 3. **3**

Find the value of c

Question 8 (14 marks)

(a)



In the Argand diagram above, the points P, Q and R represent the complex numbers p, q and r.

(i) Given that the triangle PQR is equilateral, explain why **1**

$$r - q = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})(q - p)$$

(ii) Hence, or otherwise show that $2r = (p + q) + i\sqrt{3}(q - p)$ **1**

(b) (i) Find the roots of $z^5 + 1$, leaving your solutions in modulus-argument form **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$ **1**

(iii) Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = \cos\theta + i\sin\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$ **3**

(iv) Hence, find the exact value of $\sec\frac{3\pi}{5}$ **2**

(c) (i) By using $z\bar{z} = |z|^2$ to show that **2**

Show that $|1 - \bar{z}w|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2)$

(ii) Deduce that $|1 - \bar{z}w|^2 = |z - w|^2$ if either z or w lies on the unit circle **2**

Multiple Choice

1. C
2. C
3. C
4. C
5. D

Question 6

a) i) $4i$

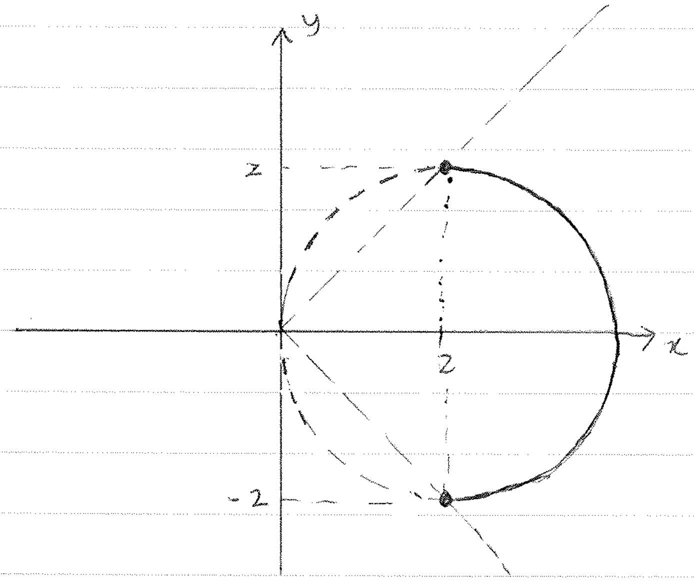
ii) $2\sqrt{2}$

iii) $\arg(zw) = \arg(z) + \arg(w)$ for correct method.

$$= \frac{\pi}{4} - \frac{\pi}{4}$$

$$= 0$$

b)



(intersection describes a semi-circle centre (2,0) and radius of 2.)

c) let $z^2 = 21 - 20i$

if $z = a + ib$

then $a^2 - b^2 = 2$

$ab = -10$

$a^2 + b^2 = 29$

$\therefore \quad 2a^2 = 50$

$a = \pm 5$ and $b = \mp 2$ $\therefore$ roots are $\pm(5 - 2i)$

d) i) $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$.

ii) $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{1002}$

$= \cos\left(\frac{1503\pi}{2}\right) + i\sin\left(\frac{1503\pi}{2}\right)$

$= 0 - i$

$= -i$

Question 7

a) i) Since $P(z)$ have all real coefficients then roots of $P(z)$ occurs in conjugate pairs.

$\therefore$ if $1+i$ is a zero then $1-i$ is a zero

ii) From (i) $1+i$ and $1-i$ are zeros. Let α be the third zero of $P(z)$

$\therefore$ Sum of zeros are $(1+i) + (1-i) + \alpha = \frac{3}{2}$

$\therefore \alpha = -\frac{1}{2}$.

$\therefore P(z) = 2(z-(1+i))(z-(1-i))(z+\frac{1}{2})$

$= (2z+1)(z^2-2z+2)$

b) i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

since $\alpha+\beta+\gamma = 0$ (sum of the roots)

and $\alpha\beta+\alpha\gamma+\beta\gamma = p$

$\alpha^2+\beta^2+\gamma^2 = 0 - 2p$

$= -2p$

7 ii) $x^3 = -px + 1$

$\therefore \; x^4 = -px^2 + x$

since α, β, γ are roots then

$\alpha^4 = -p\alpha^2 + \alpha$ — ①

$\beta^4 = -p\beta^2 + \beta$ — ②

$\gamma^4 = -p\gamma^2 + \gamma$ — ③

Now ① + ② + ③ yields

$\alpha^4 + \beta^4 + \gamma^4 = -p(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma)$

using part (i)

$\alpha^4 + \beta^4 + \gamma^4 = -p(-2p) + 0$

$= 2p^2$

iii) Since α, β and γ are non-zero and real

then $\alpha^2 + \beta^2 + \gamma^2 > 0$

But since $\alpha^2 + \beta^2 + \gamma^2 = -2p$ (from part (i))

Then $-2p > 0$

$\therefore \; p < 0$

iv) $\because \; \alpha\beta\gamma = 1$ (product of roots)

the roots $\dfrac{\alpha}{\beta\gamma}, \dfrac{\beta}{\alpha\gamma}, \dfrac{\gamma}{\alpha\beta}$

is equivalent to $\alpha^2, \beta^2, \gamma^2$.

Now replace x with $\sqrt{x}$

$x^{\frac{3}{2}} + p(x)^{\frac{1}{2}} = 1$

$x^{\frac{1}{2}}(x + p) = 1$ and squaring both sides,

$x^3 + 2px^2 + p^2x - 1 = 0$

c) $P(x) = x^4 + 3x^3 - 6x^2 - 28x + c$ has zero of multiplicity 3

then if α is the zero

then $P(\alpha) = 0$

$P'(\alpha) = 0$

$P''(\alpha) = 0$

$P'(x) = 4x^3 + 9x^2 - 12x - 28$

$P''(x) = 12x^2 + 18x - 12$

$P''(x) = 0 \Rightarrow 2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$\therefore$ $x = \frac{1}{2}$ or $x = -2$

$P'(-2) = 0$

$\therefore$ $x = -2$ is a triple zero.

$\therefore$ $P(-2) = 0 \Rightarrow (-2)^4 + 3(-2)^3 - 6(-2)^2 - 28(-2) + c = 0$

$24 + c = 0$

$c = -24$

Question 8

a) i) $r-q$ represents the vector $\overrightarrow{QR}$

since $\overrightarrow{QR}$ is equivalent to $\overrightarrow{PQ}$ rotated anticlockwise by $\frac{\pi}{3}$ twice.
(as ΔPQR is equilateral, and rotating $\overrightarrow{PQ}$ twice will form a rhombus; where opposite sides are equal and in same direction)

then $r-q = (q-p)\,\text{cis}\,\frac{2\pi}{3}$

ii) From (i) $r-q = (q-p)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$$2r-2q = -q+\sqrt{3}qi+p-\sqrt{3}pi$$

$$2r = (q+p)+\sqrt{3}(q-p)i$$

b) i) $z^5 = -1$

let $z = r(\cos\theta + i\sin\theta)$

then by De Moirve's Theorem

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$$

and $-1 = \cos(\pi + 2k\pi)$, $k = 0, 1, 2, 3, 4$

$\therefore 5\theta = \pi + 2k\pi$

$$\theta = \frac{\pi + 2k\pi}{5}$$

$\therefore z_1 = \text{cis}\,\frac{\pi}{5}$

$z_2 = \text{cis}\,\frac{3\pi}{5}$

$z_3 = -1$

$z_4 = \text{cis}\left(-\frac{3\pi}{5}\right)$

$z_5 = \text{cis}\left(-\frac{\pi}{5}\right)$

c) i) $|1-\bar{z}w|^2 - |z-w|^2$

$= (1-\bar{z}w)(\overline{1-\bar{z}w}) - (z-w)(\overline{z-w})$ as $|z|^2 = z\bar{z}$

$= (1-\bar{z}w)(1-z\bar{w}) - (z-w)(\bar{z}-\bar{w})$

$= (1 - z\bar{w} - \bar{z}w + z\bar{z}w\bar{w}) - (z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w})$

$= 1 - \cancel{z\bar{w}} - \cancel{\bar{z}w} + |z|^2|w|^2 - |z|^2 + \cancel{z\bar{w}} + \cancel{w\bar{z}} - |w|^2$

$= 1 - |z|^2|w|^2 - |z|^2 - |w|^2$

$= -|z|^2(1-|w|^2) + (1-|w|^2)$

$= (1-|w|^2)(1-|z|^2)$

ii) If $|1-\bar{z}w|^2 = |z-w|^2$

$\therefore |1-\bar{z}w|^2 - |z-w|^2 = 0$ then from part (i)

$(1-|w|^2)(1-|z|^2) = 0$

$\therefore |w|^2 = 1$ or $|z|^2 = 1$

$|w| = 1$, $|w| > 0$ and $|z| = 1$, $|w| > 0$

which lies on the unit circle.

ii) $z^5+1 = (z+1)(z^4-z^3+z^2-z+1) = 0$

since $z=-1$ is the real solution then the roots of

$z^4-z^3+z^2-z+1=0$ are the non real roots of

$z^5+1 = 0$

iii) Dividing through by z^2, we obtain

$(z^2+\frac{1}{z^2}) - (z+\frac{1}{z}) + 1 = 0$

since $z = \cos\theta + i\sin\theta$. $\frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$\therefore z + \frac{1}{z} = 2\cos\theta$

and similarly $z^2 + \frac{1}{z^2} = 2\cos 2\theta$.

$2(2\cos^2\theta - 1) - (2\cos\theta) + 1 = 0$.

$4\cos^2\theta - 2 - 2\cos\theta + 1 = 0$

$4\cos^2\theta - 2\cos\theta - 1 = 0$.

iv) let $x = \cos\theta$

$\therefore 4x^2 - 2x - 1 = 0$

$\therefore x = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$

$\therefore \cos\theta = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$ [1]

For $\theta = \frac{3\pi}{5}$, $\cos\theta < 0$ [1]

$\therefore \cos\theta = \frac{1-\sqrt{5}}{4}$

$\therefore \sec\theta = \frac{4}{1-\sqrt{5}}$