NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2017 HSC Course Assessment Task 1

Friday December 9, 2016

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided.

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Dr Jomaa ◯ 12M4B – Ms Lee ◯ 12M4C – Mr Lin

Marker's use only.

QUESTION	1-5	6	7	8	Total	%
MARKS	$\overline{5}$	$\overline{11}$	$\overline{14}$	$\overline{9}$	$\overline{39}$	

Section I

5 marks
Attempt Question 1 to 5
Mark your answers on the answer grid provided.

Questions **Marks**

1. Which of the following is true for all complex numbers z? **1**

(A) $Im(z) = \dfrac{z - \bar{z}}{2}$

(B) $Im(z) = \dfrac{z + \bar{z}}{2}$

(C) $Im(z) = \dfrac{z - \bar{z}}{2i}$

(D) $Im(z) = \dfrac{z + \bar{z}}{2i}$

2. Let the point R represent the complex number z on an Argand diagram. Which of the following describes the locus of R specified by $2|z| = z + \bar{z} + 4$. **1**

(A) Circle with centre $(0, 0)$ and radius 4.

(B) Parabola with vertex $(-1, 0)$, axis $y = 0$

(C) Parabola with vertex $(-1, 0)$, axis $x = 0$

(D) Perpendicular bisector of $(0, 0)$ and $(0, 4)$.

3. The polynomial $P(z) = z^3 + (1 + i)z^2 + (1 + i)z + 1$ has a real zero $z = -1$ and a complex zero $z = \alpha$. By considering the sum and the product of the roots, the third root is: **1**

(A) $\dfrac{1}{\alpha}$

(B) $\bar{\alpha}$

(C) $-\alpha$

(D) $1 - \alpha$

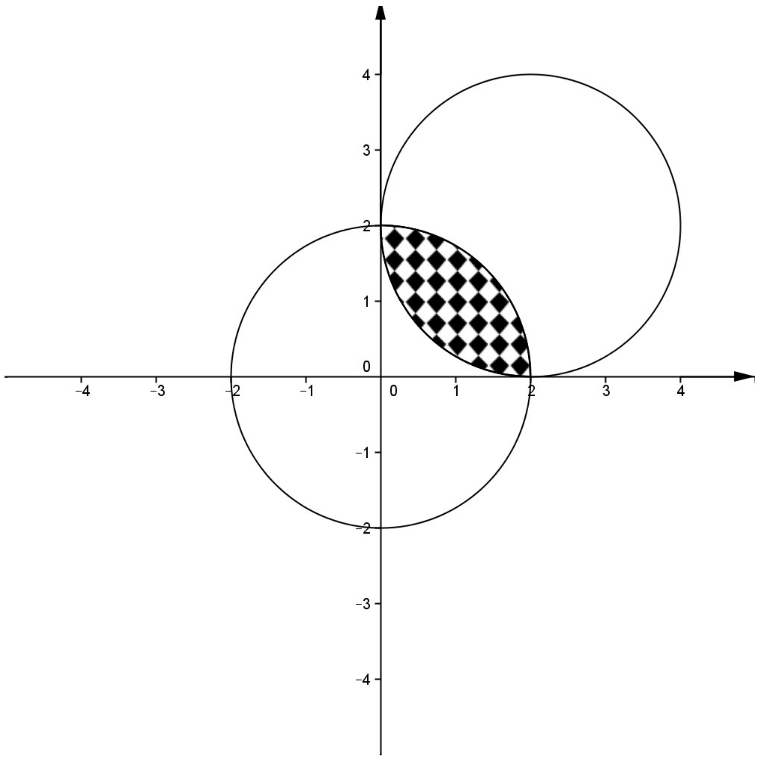
4. The shaded area in the Argand diagram below can be represented by the inequalities: **1**

(A) $|z| \le 4$ and $|z - 2 + 2i| \le 4$

(B) $|z| \le 2$ and $|z - 2 - 2i| \le 2$

(C) $|z| \le 4$ and $|z - 2 - 2i| \le 4$

(D) $|z| \le 2$ and $|z - 2 + 2i| \le 2$



5. If ω is a non-real sixth root of -1 and ϕ is a non-real fifth root of 1, consider the following two statements: **1**
(I) $1 - \omega + \omega^2 - \omega^3 + \omega^4 - \omega^5 = 0$.
(II) $1 + \phi + \phi^2 + \phi^3 + \phi^4 = 0$

(A) Both (I) and (II) are orrect.

(B) Only (I) is correct.

(C) Only (II) is correct.

(D) Neither is correct.

Please Turn Over

Section II

34 marks
Attempt Questions 6 to 7

Write your answers in the writing booklets supplied. Additional writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 Marks) Commence a NEW page. **Marks**

(a) Let $z_1 = 1 - i$ and $z_2 = -1 + i\sqrt{3}$.

i. Express $z = z_1 z_2$ in the form $a + ib$, where a and b are real. **2**

ii. By expressing z_1 and z_2 in the modulus argument form, show that **2**

$$z = \sqrt{8}\text{cis}\frac{5\pi}{12}.$$

iii. Hence, write down the exact values of $\cos\dfrac{5\pi}{12}$ and $\sin\dfrac{5\pi}{12}$. **2**

(b) The complex number z is a solution of the equation $\sqrt{z} = \dfrac{4}{1+i} + 7 - 2i$. Express z in the form $a + ib$, where a and b are integers. **2**

(c) If ω is a non-real cube root of unity, show the following:

i. **1**

$$(1+\omega)^3 = -1$$

ii. **2**

$$(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega) = 3$$

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Question 7 (14 Marks) Commence a NEW page. **Marks**

(a) The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has double root at $x = 2$. What are the values of a and b? **2**

(b) The equation $x^3 - x - 1 = 0$ has roots α, β and γ. Find an equation whose roots are $\dfrac{1+\alpha}{1-\alpha}, \dfrac{1+\beta}{1-\beta}, \dfrac{1+\gamma}{1-\gamma}$ in the form $Ax^3 + Bx^2 + Cx + D = 0$. **3**

(c) The Polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$, where a and b are real and $b \neq 0$.

i. By evaluating a and b, find all roots of $P(x)$. **3**

ii. Hence, or otherwise, find the quadratic polynomials with real coefficients that are factors of $P(x)$. **1**

(d) i. Find real numbers a and b, such that **2**

$$z^4 - z^3 + z^2 - z + 1 = (z^2 + az + 1)(z^2 + bz + 1).$$

ii. Given that $z = \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}$ is a solution of $z^4 - z^3 + z^2 - z + 1 = 0$, find the exact value of $\cos\dfrac{\pi}{5}$. **3**

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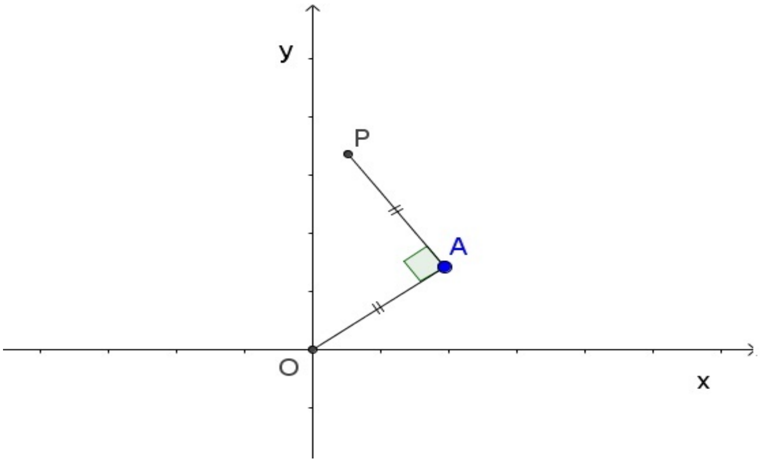
Question 8 (9 Marks) Commence a NEW page. **Marks**

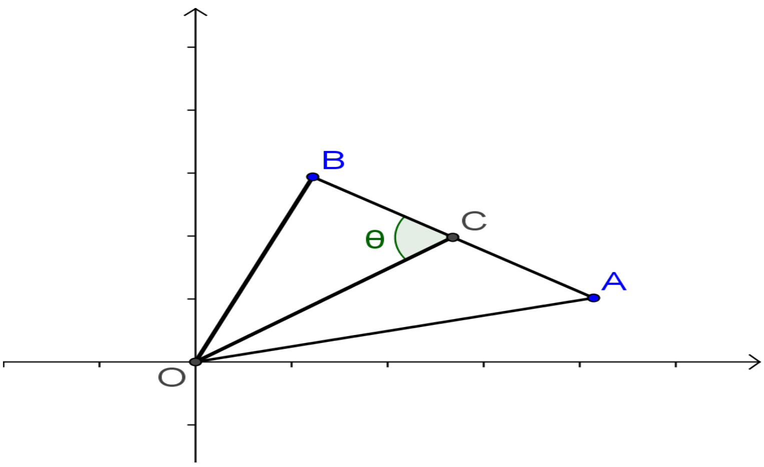
(a) The point A in the complex plane corresponds to complex numbers z. The triangle OAP is a right-angled isosceles triangle.



i. What complex number corresponds to the point P? **1**

ii. Let M is the midpoint of OP. What complex number corresponds to M? **1**

(b)



A and B are the points which represent the complex numbers z and w respectively. Let C be the midpoint of AB. By taking $\angle BCO = \theta$ and the cosine rule, prove that **3**

$$|z-w|^2 + |z+w|^2 = 2|z|^2 + 2|w|^2.$$

(c) Sketch the region in the Argand diagram consisting of those points z for which $|z| > 1$ intersect with $\dfrac{\pi}{4} \leq \arg(\dfrac{z-1}{z+1}) \leq \pi$. Show all important features. **4**

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: . . . .

Class (please ✔)

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1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

2017 Extension 2 Task 1 solutions

Section I:

1. C
2. B
3. A
4. B
5. C

Section II:

6.

a) $z_1 = 1 - i, z_2 = -1 + i\sqrt{3}$

i. $z = z_1 z_2 = (1 - i)(-1 + i\sqrt{3}) = -1 + \sqrt{3} + i(\sqrt{3} + 1)$

ii. $z_1 = \sqrt{2}\text{cis}(\frac{-\pi}{4}), z_2 = 2cis(\frac{2\pi}{3})$

$$z = z_1 z_2 = 2\sqrt{2}\, cis\left(\frac{-\pi}{4} + \frac{2\pi}{3}\right) = \sqrt{8}\, cis\left(\frac{5\pi}{12}\right).$$

iii. $-1 + \sqrt{3} + i(\sqrt{3} + 1) = \sqrt{8}\, cis\left(\frac{5\pi}{12}\right)$

$\cos\frac{5\pi}{12} = \frac{-1+\sqrt{3}}{\sqrt{8}}$, and $\sin\frac{5\pi}{12} = \frac{1+\sqrt{3}}{\sqrt{8}}$.

b) $\sqrt{z} = \frac{4}{1+i} + 7 - 2i = \frac{4(1-i)}{(1+i)(1-i)} + 7 - 2i = 2(1 - i) + 7 - 2i = 9 - 4i.$

$$z = (9 - 4i)^2 = 81 - 72i + 16i^2 = 81 - 16 - 72i = 65 - 72i.$$

c) $\omega^3 = 1 \therefore (\omega - 1)(\omega^2 + \omega + 1) = 0 \therefore 1 + \omega + \omega^2 = 0$. Since ω is a non-real cube root of unity.

i. $(1 + \omega)^3 = 1 + 3\omega + 3\omega^2 + \omega^3 = 3 + 3\omega + 3\omega^2 + \omega^3 - 2 =$
$3(1 + \omega + \omega^2) + \omega^3 - 2 = 3 \times 0 + 1 - 2 = -1$. Since $\omega^3 = 1$.

ii. $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega) = (\omega + 2\omega^2)(\omega^2 + 2\omega) =$
$$(\omega^2 - 1)(\omega - 1) = 1 - (\omega + \omega^2) + 1 = 2 - (-1) = 3.$$

7.

a) $P(x) = x^4 + ax^2 + bx + 28$ has double root at $x = 2$.

$$\therefore P(2) = 0\ and\ P'(2) = 0.$$
$$P'(x) = 4x^3 + 2ax + b$$

$P(2) = 0 \therefore 16 + 4a + 2b + 28 = 0$ and

$$P'(2) = 0\ \therefore 32 + 4a + b = 0.$$

Now $2a + b = -22$ and $4a + b = -32$

Solving simultaneously $a = -5,\ b = -12$.

b) $x^3 - x - 1 = 0$ has roots α, β and γ. Let $y = \frac{1+\alpha}{1-\alpha}$ and makes α the subject.

$\alpha = \frac{y-1}{y+1}$. Since α is a root of $x^3 - x - 1 = 0 \therefore \left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) - 1 = 0$

$$(y - 1)^3 - (y - 1)(y + 1)^2 - (y + 1)^3 = 0$$
$$y^3 - 3y^2 + 3y - 1 - (y - 1)(y^2 + 2y + 1) - (y^3 + 3y^2 + 3y + 1) = 0$$
$$-6y^2 - 2 - (y^3 + 2y^2 + y - y^2 - 2y - 1) = 0$$

$$-y^3 - 7y^2 + y - 1 = 0$$

Hence $x^3 + 7x^2 - x + 1 = 0$ has roots $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}$ and $\frac{1+\gamma}{1-\gamma}$.

c)

$P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$.

i. By the complex conjugate theorem $a - ib$ and $a - 2ib$ also are roots.

Sum of the roots $= 4 = 4a$ $\therefore a = 1$.

Product of roots$=10 = (a + ib)(a - ib)(a + 2ib)(a - 2ib) = (a^2 + b^2)(a^2 + (2b)^2)$

Now using $a = 1$ $\therefore (b^2 + 1)(4b^2 + 1) = 10$

$$4b^4 + 5b^2 - 9 = 0$$

$$\therefore\ b^2 = 1 \; or \; b^2 = \frac{-9}{4}$$

$$\therefore b^2 = 1 \; since \; b \; is \; real. and \; b = \pm 1$$

So the roots are $1 + i, 1 - i, 1 + 2i$ and $1 - 2i$.

ii. The quadratic polynomials with real coefficients are $x^2 - 2ax + a^2 + b^2$ and $x^2 - 2ax + a^2 + 4b^2$

Hence $x^4 - 4x^3 + 11x^2 - 14x + 10 = (x^2 - 2x + 2)(x^2 - 2x + 5)$

d)

i. $z^4 - z^3 + z^2 - z + 1 = (z^2 + az + 1)(z^2 + bz + 1)$

Expand the right hand side, we obtain:

$$z^4 - z^3 + z^2 - z + 1 = z^4 + (a + b)z^3 + (2 + ab)z^2 + (a + b)z + 1$$

Hence $a + b = -1$ and $2 + ab = 1$. Now $a + b = -1$ and $ab = -1$.

Solving simultaneously, we obtain $a^2 + a - 1 = 0$ where $a = \frac{-1\pm\sqrt{5}}{2}$.

So $a = \frac{-1+\sqrt{5}}{2}$ and $b = \frac{-1-\sqrt{5}}{2}$.

ii. $z^5 = -1$. Let $z = r(\cos\theta + i\ \sin\theta)$, applying De Moivre's theorem, we obtain:

$$r^5 cis5\theta = cis\pi, so \; r = 1 \; and \; 5\theta = \pi + 2k\pi = (2k + 1)\pi$$

We need $-\pi < \theta \leq \pi$. $-\pi < \frac{(2k+1)\pi}{5} \leq \pi$ and $-3 < k \leq 2$.

Hence $k = -2, -1, 0, 1, 2$. The roots are:

$cis\frac{-3\pi}{5}, cis\frac{-\pi}{5}, cis\frac{\pi}{5}, cis\frac{3\pi}{5}, cis\frac{5\pi}{5} = -1$

Now $z = \cos\frac{\pi}{5} + i\ \sin\frac{\pi}{5}$ corresponds to $k = 0$, and its conjugate corresponds to $k = -1$. Sum of the roots$=cis\frac{\pi}{5} + cis\frac{-\pi}{5} = 2\cos\frac{\pi}{5} = -b$, $(a > 0)$.

Hence $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.

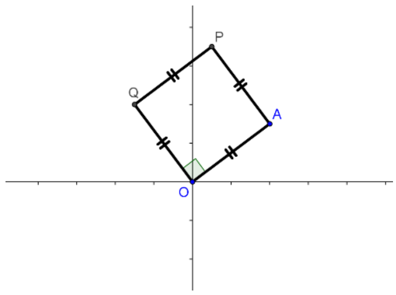
8.

a)

i. The point A corresponds to the complex number z. Q corresponds to . $\overrightarrow{AP} = iz$.

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = z + iz = (1+i)z$$

Hence P corresponds to the complex number $(1+i)z$.



ii. M is the midpoint of OP and the corresponding complex number is $\left(\frac{1+i}{2}\right) z$.

b) In triangle OBC,

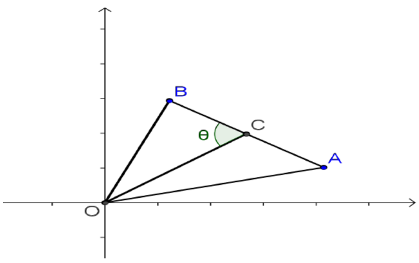
$OB^2 = OC^2 + BC^2 - 2\, OC\ BC\ \cos\theta$ (1)

Similarly in triangle OAC,

$OA^2 = OC^2 + AC^2 - 2\, OC\ AC\ \cos(180-\theta)$ (2)

But $\cos(180-\theta) = -\cos\theta$ *and*

$AC = BC$ (C *is the midpoint of* AB)



Now (1)+(2) gives:

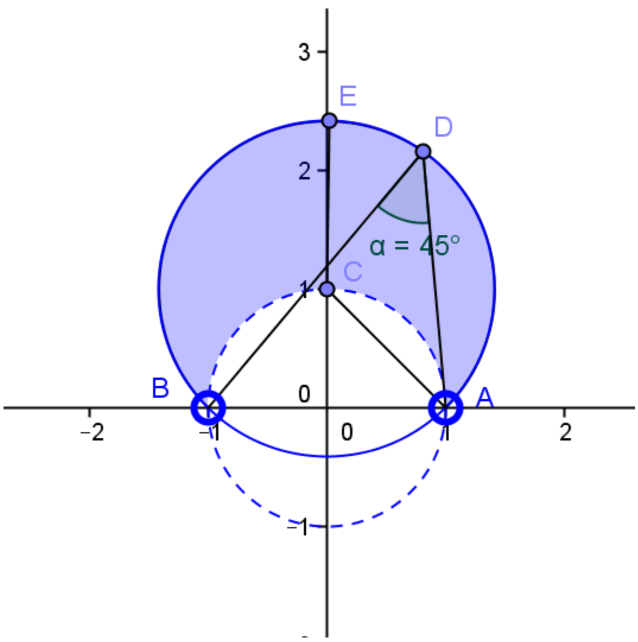
$$OA^2 + OB^2 = 2(OC^2 + BC^2)$$

$$OA = |z|, OB = |\omega|, OC = \frac{|z+\omega|}{2} \ and\ BC = \frac{|z-\omega|}{2}$$

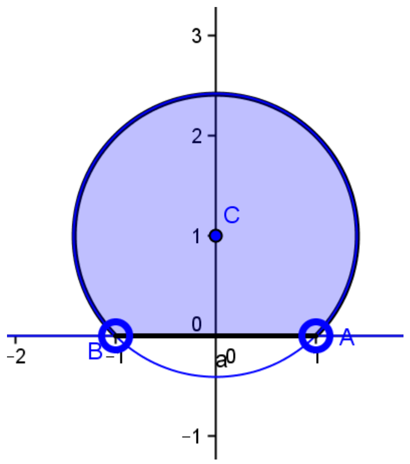
$$|z|^2 + |\omega|^2 = 2\left(\frac{|z+\omega|^2}{4} + \frac{|z-\omega|^2}{4}\right)$$

Hence $2(|z|^2 + |\omega|^2) = |z+\omega|^2 + |z-\omega|^2$.

c)



C(0,1) and the radius is $\sqrt{2}$.



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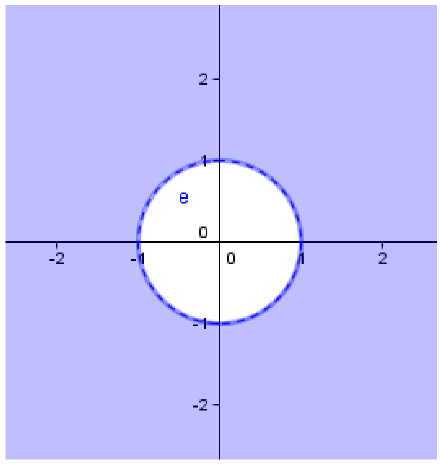

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