North Sydney Boys High School

MATHEMATICS (EXTENSION 2)

2018 HSC Course Assessment Task 1

Wednesday December 6, 2017

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided.

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Ms Lee ◯ 12M4B – Dr Jomaa ◯ 12M4C – Mr Ireland

Marker's use only.

QUESTION	1-3	4	5	6	Total	%
MARKS	$\overline{3}$	$\overline{16}$	$\overline{14}$	$\overline{9}$	$\overline{42}$	

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

◯ 12M4A – Ms Lee

◯ 12M4B – Dr Jomaa

◯ 12M4C – Mr Ireland

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

Section I

3 marks
Attempt Question 1 to 3
Mark your answers on the answer grid provided.

Questions **Marks**

1. In which quadrant is the complex number $(-3+3i)^3$ located on the Argand plane? **1**

(A) the first quadrant

(B) the second quadrant

(C) the third quadrant

(D) the fourth quadrant

2. The equation of the polynomial equation $x^3 - 3x^2 + 2 = 0$ has roots α, β and γ. What is the value of $\alpha^3 + \beta^3 + \gamma^3$? **1**

(A) 9

(B) 13

(C) 21

(D) 25

3. The polynomial $P(z)$ has real coefficients and $P(0) = -1$. The imaginary number α and the real number β satisfy: **1**

$$P(\alpha) = 0, \quad P(\beta) = 0, \quad \text{and} \quad P'(\beta) = 0.$$

The degree of $P(z)$ is at least:

(A) 2

(B) 3

(C) 4

(D) 5

Please Turn Over

Section II

39 marks
Attempt Questions 4 to 6

Write your answers in the writing booklets supplied. Additional writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 4 (16 Marks) Commence a NEW page. **Marks**

(a) Find $\sqrt{5-12i}$. **2**

(b) Given that $z = -1 + i$ and $\omega = \sqrt{3} + i$.

i. Find, in the form $x + iy$, $z\bar{\omega}$. **2**

ii. By expressing z and ω in the modulus argument form, show that **3**

$$\frac{z}{w} = \frac{1}{\sqrt{2}}(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$$

iii. Hence or otherwise, find the exact values of $\sin\frac{7\pi}{12}$. **2**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z+1-i| \leq 2 \quad \text{and} \quad 0 \leq \arg(z+1-i) \leq \frac{3\pi}{4} \quad \text{hold.}$$

(d)

i. Factorise the cubic polynomial $z^3 + 1$ over the field of real numbers. **1**

Let ω be one of the non-real roots of the equation $z^3 + 1 = 0$.

ii. Show that $\omega^2 = \omega - 1$. **2**

iii. Hence or otherwise, find the value of $(\omega^2 + \omega + 1)^6$. **1**

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Question 5 (14 Marks) Commence a NEW page. **Marks**

(a) Let the roots of the polynomial equation $x^3 + 5x - 3 = 0$, be α, β and γ.

i. Given that $f(x) = \dfrac{x^2}{1 - x^2}$, find the inverse of $f(x)$, where $0 < x < 1$. **1**

ii. Find a cubic polynomial equation which has roots **3**

$$\frac{\alpha^2}{1-\alpha^2}, \frac{\beta^2}{1-\beta^2} \text{ and } \frac{\gamma^2}{1-\gamma^2}.$$

iii. Hence or otherwise, find the value of **1**

$$\left(\frac{\alpha^2}{1-\alpha^2}\right)\left(\frac{\beta^2}{1-\beta^2}\right) + \left(\frac{\alpha^2}{1-\alpha^2}\right)\left(\frac{\gamma^2}{1-\gamma^2}\right) + \left(\frac{\beta^2}{1-\beta^2}\right)\left(\frac{\gamma^2}{1-\gamma^2}\right).$$

(b) If ω is a complex root of $z^7 - 1 = 0$.

i. Show that ω is a solution of $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$. **1**

ii. Find a quadratic equation with real coefficients having a root $\omega + \omega^2 + \omega^4$. **2**

iii. Prove that $\cos\dfrac{\pi}{7} - \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} = \dfrac{1}{2}$. **2**

(c) Let $P(x) = x^4 + ax^3 + 36x^2 - 35x + b$, where a and b are real constants.
Given that $x = 5$ and $x = \dfrac{1 - i\sqrt{3}}{2}$ are zeros of $P(x)$.

i. Explain why $x^2 - x + 1$ must be a factor of $P(x)$. **2**

ii. Find a and b. **2**

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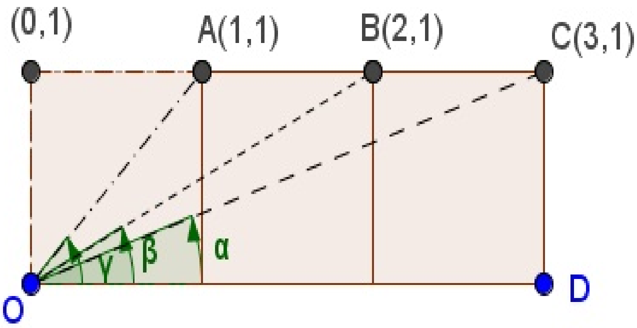
Question 6 (9 Marks) Commence a NEW page. **Marks**

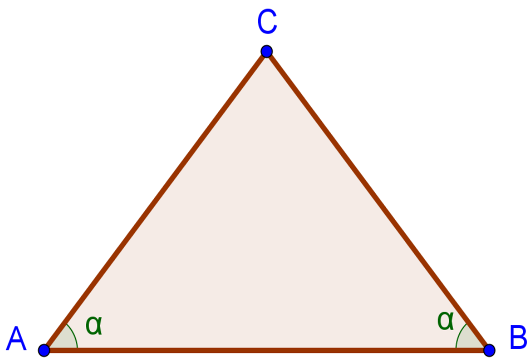
(a)

i. Prove that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. **2**



ii. A, B and C are three points in the Argand diagram (see diagram). **2**
Let $\alpha = \angle DOC$, $\beta = \angle DOB$ and $\gamma = \angle DOA$.
By using the diagram above and part (i), show that $\alpha + \beta + \gamma = \frac{\pi}{2}$.
(Do Not find α, β or γ individually).

(b) In the Argand diagram, the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle is isosceles with base AB (see diagram).



i. Find an expression of $z_3 - z_2$ in terms of $z_2 - z_1$. **2**

ii. Hence or otherwise, show that **3**

$$z_3 = \frac{z_2 + z_1\bar{\omega}}{1 + \bar{\omega}},$$

where $\omega = \cos 2\alpha + i \sin 2\alpha$ and α is the base angle of the triangle.

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1. $(-3+3i)^3 = 3^3(-1+i)^3 = 3^3\left(\sqrt{2}\ \mathrm{cis}\,\frac{3\pi}{4}\right)^3$

$= 54\sqrt{2}\ \mathrm{cis}\,\frac{\pi}{4}$

in the first quadrant

A

2. $\alpha+\beta+\gamma = 3$, $\alpha\beta\gamma = -2$

$\alpha^3+\beta^3+\gamma^3 = 3(\alpha^2+\beta^2+\gamma^2) - 2\times 3$

$= 3\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)\right] - 6$

$= 3[9 - 2\times 0] - 6$

$= 27-6$

$= 21$

C

3. $p(z)$ has real coefficients

$p(\alpha)=0$ and α imaginary

$\therefore$ $p(\bar{\alpha})=0$ (complex conjugate also is a root.)

β is real and $p(\beta)=0$ and $p(\beta')=0$

$\therefore$ β is a double roots

Therefore the degree of $p(z)$ is at least 4.

C

4. (16 marks)

(a) $\sqrt{5-12i} = \sqrt{3^2-(-2)^2+2\times3\times(-2)\times i}$ 1

$= \pm(3-2i)$ 1

(b) $z=-1+i$, $w=\sqrt{3}+i$

$\bar{w}=\sqrt{3}-i$ 1

(i) $z\bar{w}=(-1+i)(\sqrt{3}-i)=-\sqrt{3}+i+i\sqrt{3}+1$

$=1-\sqrt{3}+(1+\sqrt{3})i$ 1

(ii) $z=-1+i=\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$ 1

$\bar{w}=2\operatorname{cis}(-\pi/6)$, $|w|^2=(\sqrt{3})^2+1^2=4$ 1

$$\frac{z}{w}=\frac{z\bar{w}}{|w|^2}=\frac{1-\sqrt{3}+(1+\sqrt{3})i}{4}$$

$$\frac{z}{w}=\frac{z\bar{w}}{|w|^2}=\frac{\sqrt{2}\operatorname{cis}(3\pi/4)\times2\operatorname{cis}(-\pi/6)}{4}$$

$$=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)$$

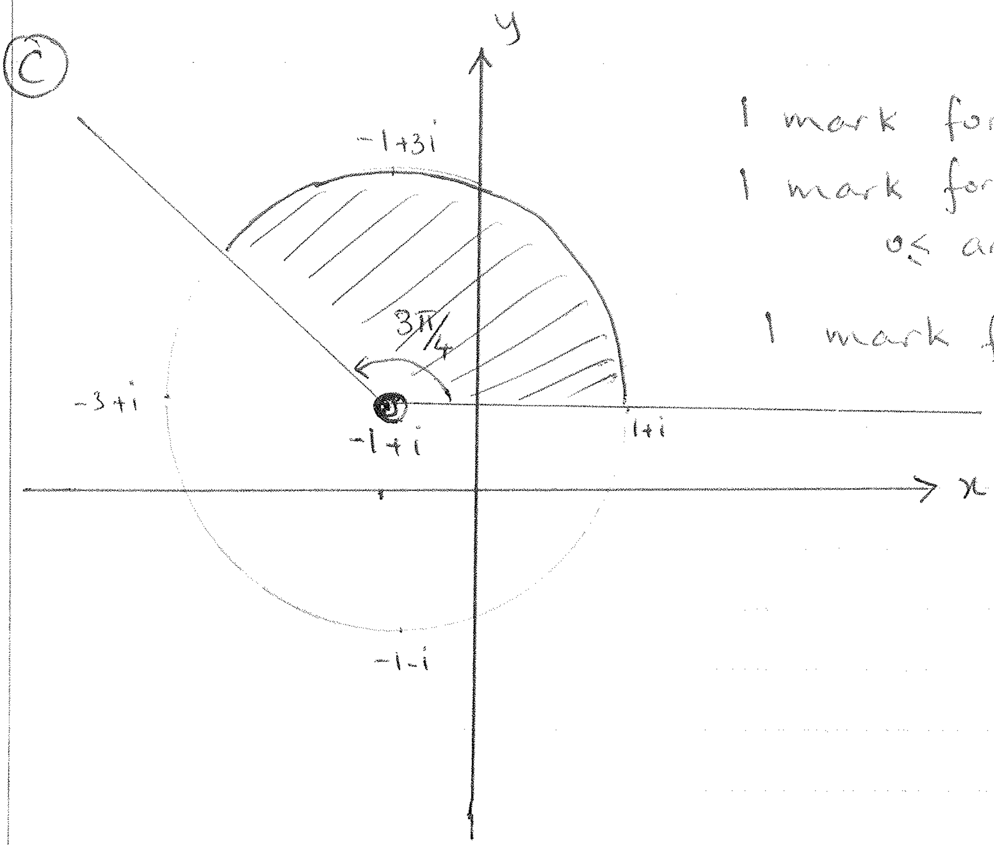
$$=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$$ 1

(iii) $\frac{1-\sqrt{3}}{4}+\frac{1+\sqrt{3}}{4}i=\frac{\sqrt{2}}{2}\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)$ 1

$\therefore\ \frac{\sqrt{2}}{2}\sin\frac{7\pi}{12}=\frac{1+\sqrt{3}}{4}$

$\sin\frac{7\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{4}$. 1

(c)



1 mark for $|z+1-i| \leq 2$

1 mark for $0 \leq \arg(z+1-i) \leq \frac{3\pi}{4}$

1 mark for correct region.

(d)

(i) $z^3+1 = (z+1)(z^2-z+1)$ 1

(ii) ω is the non real root

$\therefore \omega^2-\omega+1=0$ 1

$\therefore \omega^2 = \omega - 1$. 1

(iii) $(\omega^2+\omega+1)^6 = (2\omega)^6 = 2^6\omega^6 = 2^6 = 64$. 1

5. (14 marks)

(a) (i) $f(x) = \dfrac{x^2}{1-x^2}$

let $y = \dfrac{x^2}{1-x^2}$ $\therefore$ $y(1-x^2) = x^2$

$\therefore$ $x^2(1+y) = y$

$\therefore$ $x^2 = \dfrac{y}{1+y}$

$\therefore$ $x = \sqrt{\dfrac{y}{1+y}}$

$\therefore$ $f^{-1}(x) = \sqrt{\dfrac{x}{1+x}}$ 1

(ii) $\left(\sqrt{\dfrac{x}{1+x}}\right)^3 + 5\sqrt{\dfrac{x}{1+x}} - 3 = 0$ 1

$\left(\dfrac{x}{1+x}\right)\sqrt{\dfrac{x}{1+x}} + 5\sqrt{\dfrac{x}{1+x}} - 3 = 0$

$\left(\dfrac{x}{1+x} + 5\right)\sqrt{\dfrac{x}{1+x}} = 3$ 1

$\left(\dfrac{x}{1+x} + 5\right)^2 \dfrac{x}{1+x} = 9$

$(6x+5)^2 x = 9(1+x)^3$

$36x^3 + 60x^2 + 25x = 9(x^3 + 3x^2 + 3x + 1)$

$27x^3 + 33x^2 - 2x - 9 = 0$. 1

(iii) product of roots two at a time $= \dfrac{-2}{27}$. 1

(b)

(i) $z^7 - 1 = 0$

$\therefore (z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0$

Since $z=1$ is the real solution then the roots of $z^6+z^5+z^4+z^3+z^2+z+1=0$ are the non real roots of $z^7-1=0$.

(ii) Let $\alpha = \omega + \omega^2 + \omega^4$

$\bar{\alpha} = \bar{\omega} + \bar{\omega}^2 + \bar{\omega}^4 = \omega^6 + \omega^5 + \omega^3$

$\therefore \alpha + \bar{\alpha} = -1$

and

$$\begin{aligned}\alpha\bar{\alpha} &= (\omega+\omega^2+\omega^4)(\omega^3+\omega^5+\omega^6) \\ &= \omega^4+\omega^6+\omega^7+\omega^5+\omega^7+\omega^8+\omega^7+\omega^9+\omega^{10} \\ &= \omega^4+\omega^6+1+\omega^5+1+\omega+1+\omega^2+\omega^3 \\ &= 2\end{aligned}$$

$\therefore$ The quadratic equation with real coefficients is

$$z^2 + z + 2 = 0.$$

(iii) Let $\omega = \operatorname{cis}\frac{2\pi}{7}$

$\omega + \bar{\omega} = 2\cos\frac{2\pi}{7}$

$\omega^2 + \bar{\omega}^2 = 2\cos\frac{4\pi}{7} = -2\cos\frac{3\pi}{7}$

$\omega^4 + \bar{\omega}^4 = 2\cos\frac{8\pi}{7} = -2\cos\frac{\pi}{7}$

$\alpha + \bar{\alpha} = -1 \quad \therefore \quad 2\left(\cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\right) = -1$

$\therefore \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$

(c)

(i) $\alpha = \frac{1-i\sqrt{3}}{2}$ is a root $\therefore \bar{\alpha} = \frac{1+i\sqrt{3}}{2}$ is also a root

(polynomial with real coefficients).

$\alpha + \bar{\alpha} = 1$, $\alpha\bar{\alpha} = 1$

$\therefore$ The quadratic equation with α and $\bar{\alpha}$ are the zeros is $x^2 - x + 1 = 0$

$\therefore$ $x^2 - x + 1$ is a factor of $p(x)$.

(ii) $p(5) = 0$ $\therefore$

$$625 + 125a + 900 - 175 + b = 0$$

$$\boxed{125a + b + 1350 = 0} \quad (1)$$

$\alpha = \frac{1-i\sqrt{3}}{2}$, $\alpha^2 = \left(\frac{1-i\sqrt{3}}{2}\right)^2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

$\alpha^3 = -1$ and $\alpha^4 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$

$p(\alpha) = 0$

$\therefore$ $\frac{-1}{2} + \frac{i\sqrt{3}}{2} - a + 36\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) - 35\left(\frac{1-i\sqrt{3}}{2}\right) + b = 0$

$\therefore$ $\boxed{b - a = 36}$ (2) $\therefore$ $b = 36 + a$ (3)

(1) and (3) $\therefore$ $126a = -1386$ $\therefore$ $\boxed{a = -11}$

Use (3) $\therefore$ $b = 36 - 11 = \boxed{25}$.

(6) (9 marks)

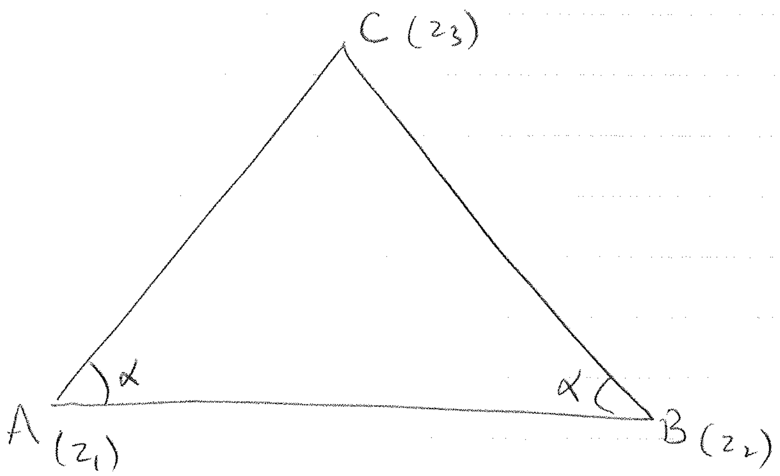
(a) (i) Let $z_1 = r_1 \operatorname{cis}\alpha$ and $z_2 = r_2 \operatorname{cis}\beta$

$$z_1 z_2 = r_1 r_2 \operatorname{cis}\alpha \operatorname{cis}\beta$$
$$= r_1 r_2 (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \qquad 1$$
$$= r_1 r_2 (\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta))$$
$$= r_1 r_2 (\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$
$$\arg(z_1 z_2) = \alpha + \beta = \arg(z_1) + \arg(z_2). \qquad 1$$

(ii) $z_1 = 1+i$, $z_2 = 2+i$, $z_3 = 3+i$

$$z_1 z_2 z_3 = (1+i)(2+i)(3+i)$$
$$= (1+i)(6-1+5i)$$
$$= 5(1+i)^2 = 10i = 10\operatorname{cis}\left(\frac{\pi}{2}\right) \qquad 1$$
$$\arg(z_1 z_2 z_3) = \pi/2$$
$$\therefore \arg(z_1) + \arg(z_2) + \arg(z_3) = \pi/2$$
$$\therefore \alpha + \beta + \gamma = \pi/2. \qquad 1$$

(b)



Method 1: (i) $\overrightarrow{BC} = k\,\overrightarrow{AB}\,\text{cis}(180-\alpha)$, k is a positive constant

$$z_3 - z_2 = k(z_2 - z_1)\,\text{cis}(180-\alpha)$$

(ii) Similarly $\overrightarrow{AB} = \frac{1}{k}\,\overrightarrow{CA}\,\text{cis}(180-\alpha)$

$$z_2 - z_1 = \frac{1}{k}(z_1 - z_3)\,\text{cis}(180-\alpha)$$

$$\therefore\ z_3 - z_2 = (z_1 - z_3)\,\text{cis}(180-\alpha)\,\text{cis}(180-\alpha)$$

$$z_3 - z_2 = (z_1 - z_3)(\cos 2\alpha - i\sin 2\alpha)$$

$$z_3 - z_2 = (z_1 - z_3)\,\overline{\omega}$$

$$z_3(1+\overline{\omega}) = z_2 + z_1\overline{\omega}$$

$$z_3 = \frac{z_2 + z_1\overline{\omega}}{1+\overline{\omega}}.$$

OR

Method 2: (i) $\overrightarrow{CA} = \overrightarrow{BC}\,\text{cis}\,2\alpha$

$$\therefore\ \overrightarrow{BC} = \overrightarrow{CA}\,\text{cis}(-2\alpha) = \overrightarrow{CA}\,\overline{\omega}$$

$$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$$

$$\overrightarrow{BC} = (\overrightarrow{CB} + \overrightarrow{BA})\,\overline{\omega}$$

$$\overrightarrow{BC}(1+\overline{\omega}) = \overrightarrow{BA}\,\overline{\omega}$$

$$\overrightarrow{BC} = \overrightarrow{BA}\,\frac{\overline{\omega}}{1+\overline{\omega}} \quad \therefore\ z_3 - z_2 = \frac{\overline{\omega}}{1+\overline{\omega}}(z_1 - z_2)$$

(ii) $$z_3 = z_2 + \frac{\overline{\omega}}{1+\overline{\omega}}(z_1 - z_2) = \frac{z_2 + \overline{\omega}z_1}{1+\overline{\omega}}.$$
