NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2019 HSC Course Assessment Task 1

12th of December, 2018

General instructions

- Working time – 55 minutes (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.

QUESTIONS 1-12

- Commence each **new section** on a **new booklet**
- Commence each new question on a **new page**. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ Mr Ireland

◯ Dr Jomaa

◯ Ms Lee

Marker's use only.

SECTION	A	B			C		D		
QUESTION	$1-5$	6	7	8	9	10	11	12	Total
MARKS	$\overline{5}$	$\overline{4}$	$\overline{5}$	$\overline{3}$	$\overline{7}$	$\overline{5}$	$\overline{5}$	$\overline{7}$	$\overline{41}$

Section A

Mark your answers on the multiple choice section in the answer booklet. **Marks**

1. A quadratic expression with zeros of $4+i$ and $4-i$ is: **1**

(A) $x^2 - 8x + 17$

(B) $x^2 + 8x + 17$

(C) $x^2 - 8x - 17$

(D) $x^2 + 8x - 17$

2. Let $z = 4 + i$. What is the value of $\overline{iz}$? **1**

(A) $-1 - 4i$

(B) $-1 + 4i$

(C) $1 - 4i$

(D) $1 + 4i$

3. Let $z = 3 - i$. What is an expression for $\dfrac{1}{z}$? **1**

(A) $\dfrac{3}{4} + \dfrac{1}{4}i$

(B) $\dfrac{3}{8} + \dfrac{1}{8}i$

(C) $\dfrac{3}{10} + \dfrac{1}{10}i$

(D) $\dfrac{3}{2} + \dfrac{1}{2}i$

4. Given $3x^3 + 6x - 5 = 0$ has roots α, β and γ, what is the value of $\alpha^3 + \beta^3 + \gamma^3$? **1**

(A) 5

(B) 9

(C) 15

(D) -1

5. If ω is a complex cube root of unity, which of the following results is **NOT** correct? **1**

(A) ω^2 is the other complex root

(B) $\omega^2 + \omega = -1$

(C) $\omega^6 - \omega = \omega^2$

(D) $\overline{\omega}$ is also a root

End of Section A.
Examination continues overleaf.

Section B

Write your answers in the writing booklet supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (4 Marks) Commence a NEW page. **Marks**

(a) Express $\sqrt{24-70i}$ in the form $a+ib$. **2**

(b) Hence solve $z^2-(1-i)z-6+17i=0$. **2**

Question 7 (5 Marks) Commence a NEW page. **Marks**

(a) Given that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$ and $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$, prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ **1**

(b) Hence, or otherwise, find θ where $x^3-3x^2-3x+1=0$ and $x=\tan\theta$ if $0 \le \theta \le \pi$ **2**

(c) Hence prove that $\tan^2\dfrac{\pi}{12} + \tan^2\dfrac{5\pi}{12} = 14$. **2**

Question 8 (3 Marks) Commence a NEW page. **Marks**

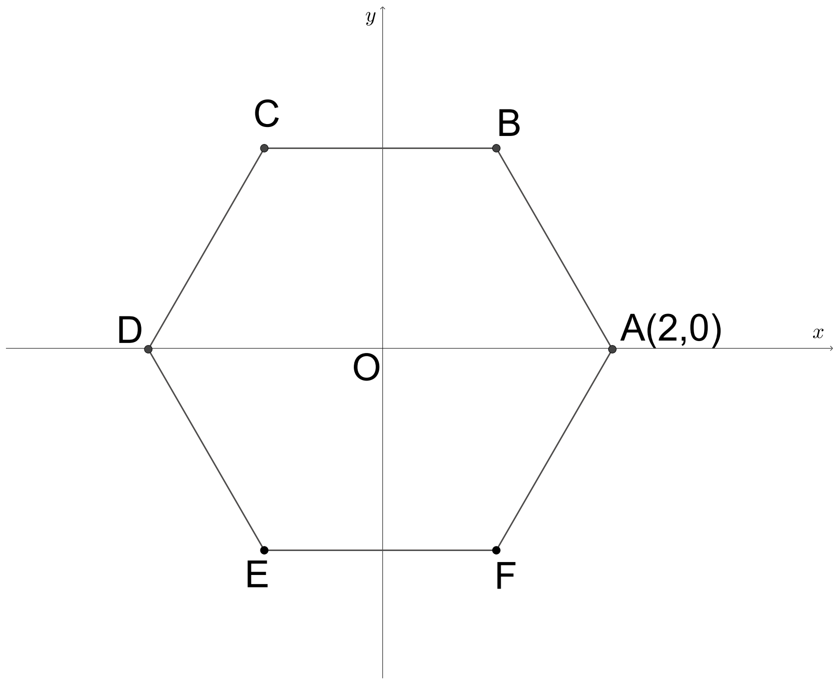
(a) If α, β, γ are zeros of the polynomial x^3-px^2+qx-r, prove $\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2 = q^2-2pr$ **3**

Section C

Write your answers in the writing booklet supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 9 (7 Marks) Commence a NEW page. **Marks**

$ABCDEF$ is a regular hexagon drawn on an Argand diagram with vertex A at the point $(2, 0)$. O is the centre of the hexagon.



(a) Sketch the region within the hexagon that satisfies $|z| \geq 1$ and $-\dfrac{\pi}{3} \leq \arg z \leq \dfrac{\pi}{3}$ **2**

(b) Find, in the form $|z - c| = R$, the equation of the circle through the points O, B and F. **1**

(c) Find the complex numbers represented by the points C and E. **2**

(d) The hexagon is rotated anticlockwise about the origin through an angle of $\dfrac{\pi}{2}$. Express in the form $r(\cos\theta + i\sin\theta)$ the complex numbers represented by the new positions of C and E. **2**

Question 10 (5 Marks) Commence a NEW page. **Marks**

If α is a zero of the polynomial $P(z) = az^2 + bz + c$, which has real coefficients a, b and c,

(a) Show that $\overline{\alpha}$ is a zero of $P(z)$ **2**

(b) Find the monic polynomial of least degree, with integer coefficients, having $1-i$ as a root of multiplicity 2. **3**

Section D

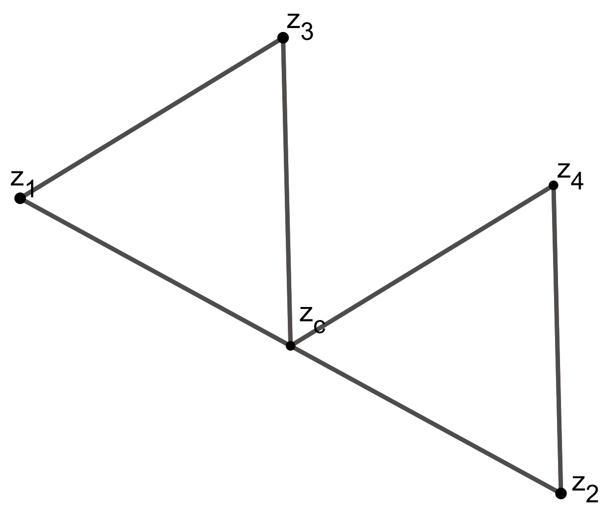
Write your answers in the writing booklet supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (5 Marks) Commence a NEW page. **Marks**

(a) Solve the equation $z^4 - 1 = 0$ where z is a complex number **2**

(b) Hence find all solutions of the equation $z^4 = (z-1)^4$. **3**

Question 12 (7 Marks) Commence a NEW page. **Marks**



Two complex numbers z_1 and z_2 are given on the Argand diagram above.
z_c is the complex number representing the midpoint of $z_1 z_2$.
z_3 is a complex number such that
$(z_3 - z_1)^2 - (z_3 - z_1)(z_c - z_1) + (z_c - z_1)^2 = 0$

(a) Find $\dfrac{z_3 - z_1}{z_c - z_1}$. Express in the form $r(\cos\theta + i\sin\theta)$ **2**

(b) Hence show that z_1, z_c, z_3 are the vertices of an equilateral triangle. **2**

(c) z_4 is the point such that z_c, z_2, z_4 are the vertices of an equilateral triangle **2**
Show that $(z_4 - z_3)^2 + (z_4 - z_3)(z_1 - z_3) + (z_1 - z_3)^2 = 0$

(d) What is the nature of triangle $z_1 z_4 z_3$? **1**

End of paper.

Blank page

Blank page

Answer sheet for Section A

Mark answers to Section A by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

- ◯ Mr Ireland
- ◯ Dr Jomaa
- ◯ Ms Lee

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

Section A 1. (A) 2. (A) 3. (C) 4. (A) 5. (C)

Section B

Question 6

2 a) Let $\sqrt{24-70i} = a + ib$

$a^2 - b^2 = 24$ ①
$a^2 + b^2 = \sqrt{24^2 + 70^2} = 74$ ② a, b opposite signs

① + ② $2a^2 = 98$
$a^2 = 49$
$a = \pm 7$
$b^2 = 49 - 24 = 25$
$b = \pm 5$

$\therefore \sqrt{24-70i} = \pm(7-5i)$

2 b) $z^2 - (1-i)z - 6 + 17i = 0$

$z = \dfrac{(1-i) \pm \sqrt{(1-i)^2 - 4(-6+17i)}}{2}$

$= \dfrac{(1-i) \pm \sqrt{-2i + 24 - 68i}}{2}$

$= \dfrac{(1-i) \pm \sqrt{24-70i}}{2}$

$= \dfrac{(1-i) \pm (7-5i)}{2}$

$= 4-3i, \ -3+2i$

Question 7

1 a) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$

$= \dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta} \times \dfrac{\frac{1}{\cos^3\theta}}{\frac{1}{\cos^3\theta}}$ $(\cos\theta \neq 0)$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

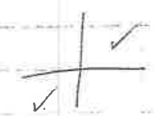
2 b) Let $x = \tan\theta$

$\tan 3\theta = \dfrac{3x - x^3}{1 - 3x^2}$

If $\tan 3\theta = 1$,
Then $1 = \dfrac{3x - x^3}{1 - 3x^2}$

$1 - 3x^2 = 3x - x^3$

i.e. $x^3 - 3x^2 - 3x + 1 = 0$



Now $\tan 3\theta = 1$
for $3\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}$ if $0 \le \theta \le \pi$
$\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$ $0 \le 3\theta \le 3\pi$

:) Sum of roots one at a time:

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} + \tan\frac{3\pi}{4} = 3$$

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4 \quad \text{since } \tan\frac{3\pi}{4} = -1$$

Product of roots:

$$\tan\frac{\pi}{12}\tan\frac{5\pi}{12}\tan\frac{3\pi}{4} = -1$$

$$\tan\frac{\pi}{12}\tan\frac{5\pi}{12} = +1$$

$$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} = \left(\tan\frac{\pi}{12} + \tan\frac{5\pi}{12}\right)^2 - 2\tan\frac{\pi}{12}\tan\frac{5\pi}{12}$$

$$= 4^2 - 2(+1)$$

$$= 16 - 2$$

$$= 14$$

(8) $x^3 - px^2 + qx - r = 0$

Replace x with $\sqrt{x}$ for eqn $\bar{c}$ root $\alpha^2, \beta^2, \gamma^2$.

$$x\sqrt{x} - px + q\sqrt{x} - r = 0$$

$$x\sqrt{x} + q\sqrt{x} = px + r$$

$$x(x+q)^2 = (px+r)^2$$

$$x(x^2 + 2qx + q^2) = p^2x^2 + 2prx + r^2$$

$$x^3 + 2qx^2 + q^2x = p^2x^2 + 2prx + r^2$$

$$x^3 + (2q - p^2)x^2 + (q^2 - 2pr)x - r^2 = 0$$

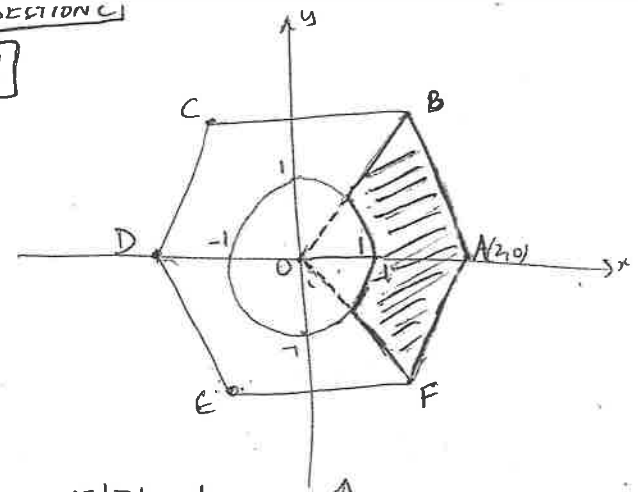
Sum of roots two at a time:

$$q^2 - 2pr$$

✓✓/

SECTION C

Q9



(a) $|z| \geq 1$ and
$-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$. #

(b) Since $AB = AO = AF = 2$
$\therefore$ A is the centre & radius $= 2$
$\therefore$ $|z-2| = 2$. #

(c) $\vec{OC} = \operatorname{cis}\frac{2\pi}{3} \times \vec{OA}$
$= \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(2+0i)$
$= -1+i\sqrt{3}$ $\quad\therefore$ C is $-1+i\sqrt{3}$.
(or $2\operatorname{cis}\frac{2\pi}{3}$)

Likewise,
$\vec{OE} = \operatorname{cis}\left(-\frac{2\pi}{3}\right) \times \vec{OA}$
$= \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(2+0i)$
$= -1-i\sqrt{3}$ $\quad\therefore$ E is $-1-i\sqrt{3}$. #
(or $2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$)

Q9 ctd.

(d) $\vec{OC'} = i\,\vec{OC}$
$= i\,(-1+i\sqrt{3})$
$= -\sqrt{3}-i$ $\quad\therefore$ $C' = -\sqrt{3}-i$
$= 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$ (or $2\operatorname{cis}\frac{7\pi}{6}$)

Also, $\vec{OE'} = i\,\vec{OE}$
$= i(-1-i\sqrt{3})$
$= \sqrt{3}-i$ $\quad\therefore$ $E' = \sqrt{3}-i$ #
$= 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$

7

SECTION C

Q10

(a) $P(z) = az^2 + bz + c$, (a, b, c all real)

Let α be a zero of $P(z)$, i.e. $P(\alpha) = 0$.

Now, $P(\bar{\alpha}) = a(\bar{\alpha})^2 + b\bar{\alpha} + c$

$= a\overline{\alpha^2} + b\bar{\alpha} + c$, as $(\bar{z})^n = \overline{z^n}$ ✓

$= \overline{a\alpha^2} + \overline{b\alpha} + \bar{c}$, as $c\bar{z} = \overline{cz}$ for real c. [ie as $a = \bar{a}$ if $a \in \mathbb{R}$]

$= \overline{a\alpha^2 + b\alpha + c}$, as $\bar{w} + \bar{z} = \overline{w+z}$

$= \overline{P(\alpha)}$

$= \bar{0}$

$= 0$ $\therefore$ $\bar{\alpha}$ is also a zero of $P(z)$. ✓ #

(b). Since $1-i$ is a root, so is $1+i$

(because our polynomial has integer, ie real, coefficients).

But $1-i$ is a root of multiplicity 2, and $P(z)$ is monic,

Hence:

$P(z) = [z-(1-i)]^2[z-(1+i)]^2$ ✓

$= [(z-1)+i]^2[(z-1)-i]^2$

$= [(z-1)+i][(z-1)-i] \times [(z-1)+i][(z-1)-i]$

$= [(z-1)^2+1][(z-1)^2+1]$ $= (z^2-2z+2)^2$

$= (z-1)^4 + 2(z-1)^2 + 1$

$= \cancel{z^4} - \cancel{4z^3} + 6z^2 - 4z + 1 + 2z^2 - 4z + 2 + 1$

$P(z) = z^4 - 4z^3 + 8z^2 - 8z + 4$. ✓✓ #

5

Section D

Question 11

a) $z^4 - 1 = 0$ Let $z = \text{cis}\,\theta$.

$z^4 = 1$ Then $z^4 = \text{cis}\,4\theta$ by De Moivre's Theorem

$\text{cis}\,4\theta = \text{cis}\,2k\pi$ where $k \in \mathbb{Z}$

$4\theta = 2k\pi$

$\theta = \frac{k\pi}{2}$

for principal arguments, $-\pi < \frac{k\pi}{2} \le \pi$

$-2 < k \le 2$

$k=-1$: $z_1 = \text{cis}\,\frac{-\pi}{2} = -i$

$k=0$: $z_2 = \text{cis}\,0 = 1$

$k=1$: $z_3 = \text{cis}\,\frac{\pi}{2} = i$

$k=2$: $z_4 = \text{cis}\,\pi = -1$

b) $z^4 = (z-1)^4$

$\left(\frac{z}{z-1}\right)^4 = 1$

Let $y = \frac{z}{z-1}$

$zy - y = z$

$z(y-1) = y$

$z = \frac{y}{y-1}$

$= \frac{-i}{-i-1}$, $\cancel{\frac{1}{1-1}}$ undefined, $\frac{i}{i-1}$, $\frac{-1}{-1-1}$

$= \frac{i}{i+1} \times \frac{i-1}{i-1}$, $\frac{i}{i-1} \times \frac{i+1}{i+1}$, $\frac{1}{2}$

$= \frac{-1-i}{-2}$, $\frac{-1+i}{-2}$, $\frac{1}{2}$

$= \frac{1+i}{2}$, $\frac{1-i}{2}$, $\frac{1}{2}$

Question 12

(a) $(z_3-z_1)^2-(z_3-z_1)(z_c-z_1)+(z_c-z_1)^2=0$

Multiply by $(z_3-2z_1+z_c)$, we obtain:

$(z_3-z_1+z_c-z_1)\left((z_3-z_1)^2-(z_3-z_1)(z_c-z_1)+(z_c-z_1)^2\right)=0$

$\therefore \quad (z_3-z_1)^3+(z_c-z_1)^3=0$

$\therefore \quad (z_3-z_1)^3=-(z_c-z_1)^3$

$\left(\frac{z_3-z_1}{z_c-z_1}\right)^3=-1=\operatorname{cis}\pi$

$\therefore \quad \frac{z_3-z_1}{z_c-z_1}=\operatorname{cis}\pi/3$

(b) $\left|\frac{z_3-z_1}{z_c-z_1}\right|=\left|\operatorname{cis}\pi/3\right|=1 \quad \therefore |z_3-z_1|=|z_c-z_1|$

$\therefore$ ΔABE is an isosceles triangle

$\angle ABE=\angle BEA=\frac{\pi-\frac{\pi}{3}}{2}=\frac{\pi}{3}$ (Equal angles opposite equal sides and angle sum of a triangle)

$\therefore$ ΔABE is an equilateral triangle.

(c) $\angle z_1 z_3 z_4=2\pi/3$

$\frac{z_4-z_3}{z_1-z_3}=\operatorname{cis}2\pi/3$

$\therefore \quad \left(\frac{z_4-z_3}{z_1-z_3}\right)^3=\operatorname{cis}2\pi=1$

$\therefore \quad (z_4-z_3)^3=(z_1-z_3)^3$

$\therefore \quad (z_4-z_3)^3-(z_1-z_3)^3=0$

$(z_4-z_3-(z_1-z_3))\left((z_4-z_3)^2+(z_4-z_3)(z_1-z_3)+(z_1-z_3)^2\right)=0$

Since $z_4-z_1\neq 0 \quad \therefore$

$(z_4-z_3)^2+(z_4-z_3)(z_1-z_3)+(z_1-z_3)^2=0$.