NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2020 HSC Course Assessment Task 1

11th of December, 2019

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.
- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Mr Lin

◯ 12M4B – Dr Jomaa

◯ 12M4C – Ms Ziaziaris

Marker's use only.

QUESTION	1	2	3	4	5	6	Total	%
MARKS	$\overline{10}$	$\overline{8}$	$\overline{7}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{43}$	

Write your answers in the writing booklet supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (10 Marks) Commence a NEW booklet. **Marks**

(a) Given $z = -1 + i\sqrt{3}$ and $\omega = 1 + i$.

i. Evaluate $|z|$. **1**

ii. Find $\dfrac{\omega}{z}$ in the $x + iy$ form, where x and y are real numbers. **2**

iii. Find $\omega\bar{z}$ in modulus argument form. **2**

(b) Given $z = x + iy$, if $|z - 3 - 4i| = 4$, find z such that $x^2 + y^2$ is minimum. **2**

(c) Solve for z, such that $z^2 + \dfrac{1}{z^2} = 1$. Express your answers in the $x + iy$ form. **3**

Question 2 (8 Marks) Commence a NEW booklet. **Marks**

(a) Write the converse of the following statement: If quadrilateral $ABCD$ is a square, then quadrilateral $ABCD$ is a rectangle. **1**

(b) Write the contrapositive of the following statement: If two distinct lines in the xy plane are parallel, then they have the same slope. **1**

(c) Negate the following statement: If Harold passes his maths course and finishes his statistics project, then he will graduate at the end of the year. **1**

(d) Write the following statement in symbolic form: There exists an even integer divisible by 5. **1**

(e) Prove that the following statement is false, giving a counter example: If a positive integer p is prime, then so is $2p + 1$. **1**

(f) Prove that if a positive integer n has remainder 1 when divided by 3, then n^3 has remainder 1 when divided by 3. **3**

Question 3 (7 Marks) Commence a NEW booklet. **Marks**

(a) A polynomial function is

$$P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10.$$

i. Show that $2 - i$ is a root of $P(x)$. **1**

ii. Hence, or otherwise, factorise $P(x)$ completely over $\mathbb{C}$. **3**

(b) Sketch the region in the Argand diagram consisting of those point z for which **3**

$$|z - 2 + i| > |z + 2 + 2i| \quad \text{intersect with} \quad 0 \leq \arg(z + 2 + 2i) \leq \frac{\pi}{4}.$$

Show all the important features.

Question 4 (5 Marks) Commence a NEW booklet. **Marks**

(a) Prove (without using induction) that for all integes $n \geq 2$ **2**

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\dots\left(1 - \frac{1}{(n-1)^2}\right)\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

(b) Prove by contradiction the following: **3**

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})[(3x + \sqrt{2}y = 0) \implies (x = 0 \quad \text{and} \quad y = 0)]$$

Question 5 (6 Marks) Commence a NEW booklet. **Marks**

Given that

$$(x + y)^5 - (x^5 + y^5) = 5xy(x + y)(x^2 + xy + y^2).$$

and

$$z^n + z^{-n} = 2\cos n\theta.$$

i. Show that **2**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

ii. Hence, show that the equation $16x^5 - 20x^3 + 5x - 1 = 0$ has roots 1, $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\cos\frac{6\pi}{5}$, and $\cos\frac{8\pi}{5}$. **2**

iii. Use (ii) to show that the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has roots $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\cos\frac{6\pi}{5}$, and $\cos\frac{8\pi}{5}$. **2**

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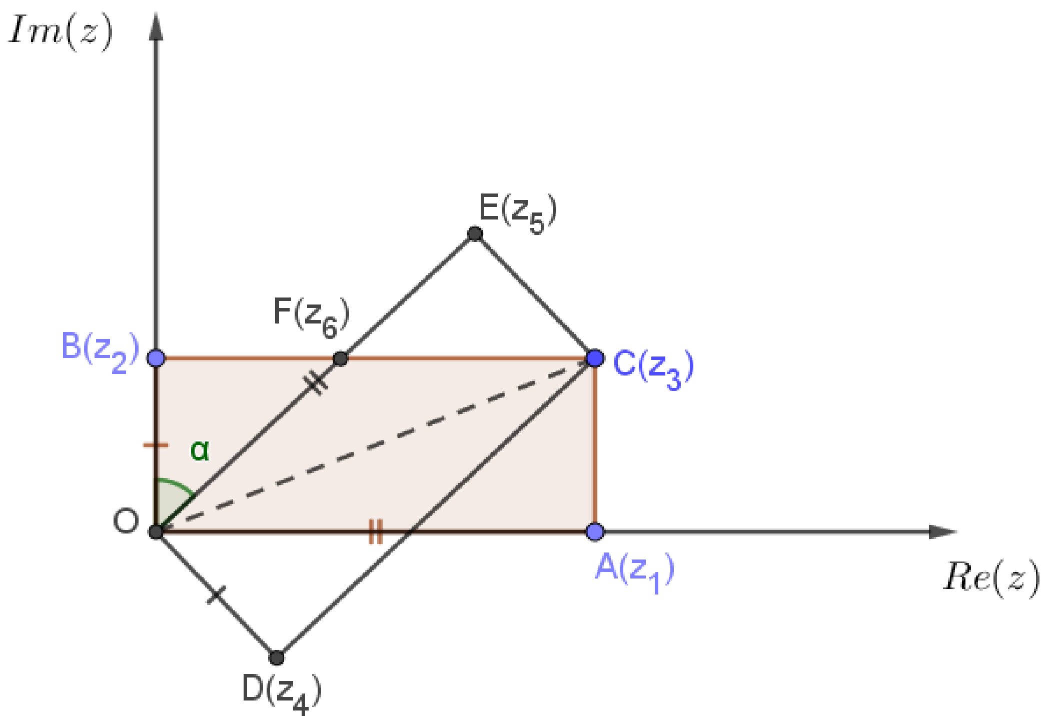

Given z_1 and z_2 two complex numbers in the Argand diagram (see diagram).

i. Find z_3 in terms of z_1 and z_2, such that $OACB$ formed a rectangle. **1**

ii. Find z_4 and z_5 in terms of z_1, z_2 and α, where $\alpha = \angle EOB$. **2**

iii. Hence, or otherwise show that **2**

$$z_3 = i(z_1 - z_2)\overline{cis\alpha}.$$

iv. The complex number z_6 represents the point F (the intersection of BC and OE). Show that **2**

$$z_6 = \frac{(z_1 + z_2)}{2}\left(1 + i\cot\left(\frac{\pi + 2\alpha}{4}\right)\right).$$

End of paper.

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North Sydney Boys High School

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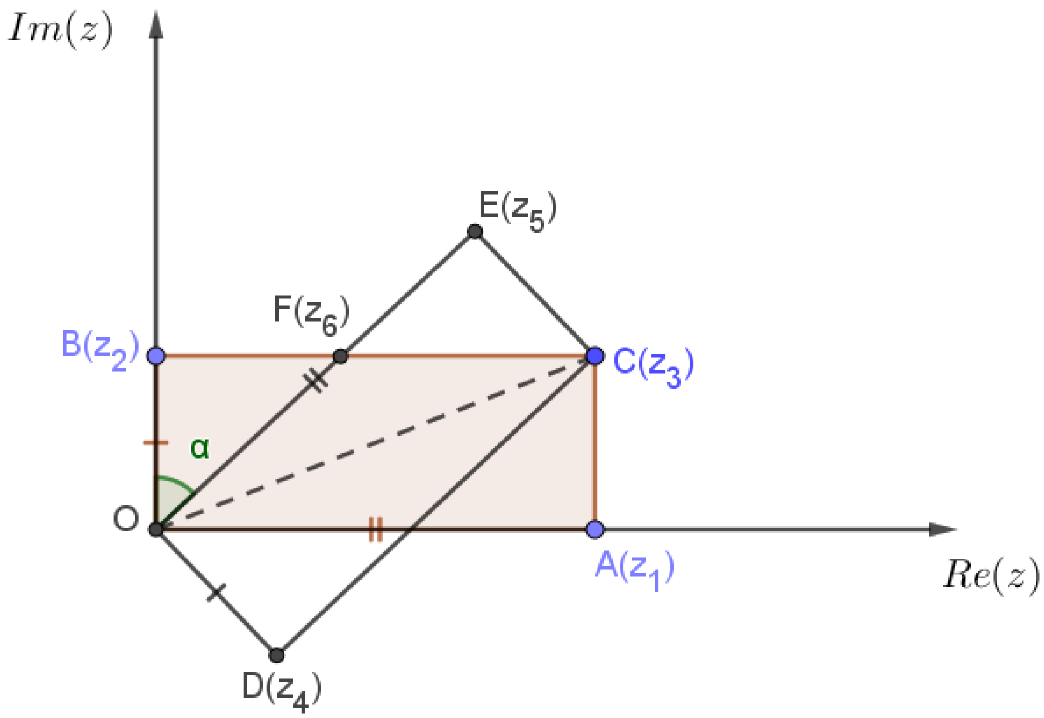

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End of paper.

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Suggested Solutions

Section II

Question 1

(a) $z = -1 + i\sqrt{3}$ and $\omega = 1 + i$.

i. $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$

ii. $\dfrac{\omega}{z} = \dfrac{1+i}{-1+i\sqrt{3}} = \dfrac{(1+i)(-1-i\sqrt{3})}{(-1+i\sqrt{3})(-1-i\sqrt{3})} = \dfrac{-1-i\sqrt{3}-i+\sqrt{3}}{4} = \dfrac{-1+\sqrt{3}}{4} - \dfrac{1+\sqrt{3}}{4}i.$

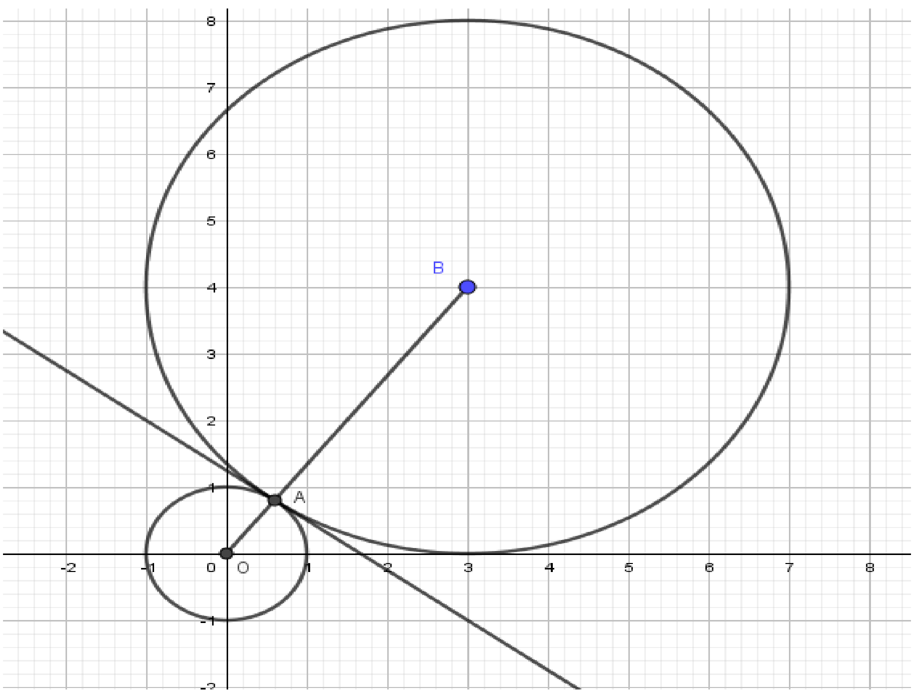
iii.

$$\omega = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}cis\frac{\pi}{4}.$$

$$\overline{z} = -1 - i\sqrt{3} = 2\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) = 2cis\frac{4\pi}{3}.$$

$$\omega\overline{z} = \left(\sqrt{2}cis\frac{\pi}{4}\right)\left(2cis\frac{4\pi}{3}\right) = 2\sqrt{2}cis\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) = 2\sqrt{2}cis\frac{19\pi}{12} = 2\sqrt{2}cis\left(\frac{-5\pi}{12}\right).$$

(b) The circle $(x-3)^2 + (y-4)^2 = 16$ centered at $(3,4)$ and has radius 4. The distance from the origin to the center equal 5 (by using Pythagoras theorem) and the shortest distance to the circle is $5-4=1$. The line through the origin and the center has equation $4x - 3y = 0$. The x-value of the points of intersection of this line and the circle are given by $(x-3)^2 + (\frac{4x}{3} - 4)^2 = 16$ which give $x = \frac{3}{5}$ and $x = \frac{27}{5}$. Hence $y = \frac{4}{5}$ and $y = \frac{36}{5}$. So $z = \frac{3}{5} + \frac{4}{5}i$ and $|z| = 1$.



(c) $\left(z+\dfrac{1}{z}\right)^2 = z^2+2+\dfrac{1}{z^2} = z^2+\dfrac{1}{z^2}+2 = 1+2 = 3.$

$z+\dfrac{1}{z} = \pm\sqrt{3}.$

z are the solutions of $z^2+\sqrt{3}z+1=0$ and $z^2-\sqrt{3}z+1=0.$

$b^2-4ac = 3-4 = -1 = i^2.$

Hence $z = \dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i,\quad \dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i,\quad \dfrac{-\sqrt{3}}{2}+\dfrac{1}{2}i,\quad \dfrac{-\sqrt{3}}{2}-\dfrac{1}{2}i.$

Question 2 solutions

(a) **Converse**: If Quadrilateral $ABCD$ is a rectangle, then the quadrilateral $ABCD$ is a square.

(b) **Contrapositive**: If two distinct lines in the xy plane do not have the same slope, then they are not parallel.

(c) **Negation** :Harold passes his math course and he finishes his statistics project, but he did not graduated at the end of the year.

(d) **Symbolic form**: $(\exists p \in \mathbb{Z})(\exists m \in \mathbb{Z})(2p = 5m)$.

(e) **Counter example**: 7 is a prime number but $15 = 2 \times 7 + 1$ is not prime.

(f) Let n be a positive integer which has remainder 1 when divided by 3, ie. $\exists m \in \mathbb{Z}, n = 3m+1$.
We need to prove that n^3 has remainder 1 when divided by 3.

$$n^3 = (3m+1)^3 = (3m)^3+3(3m)^2+3(3m)+1 = 27m^3+27m^2+9m+1 = 3(9m^3+9m^2+3m)+1 = 3k+1,$$

where $k = 9m^3 + 9m^2 + 3m \in \mathbb{Z}$. Hence the remainder of dividing n^3 by 3 is 1.

Question 3 solutions

(a) $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10.$

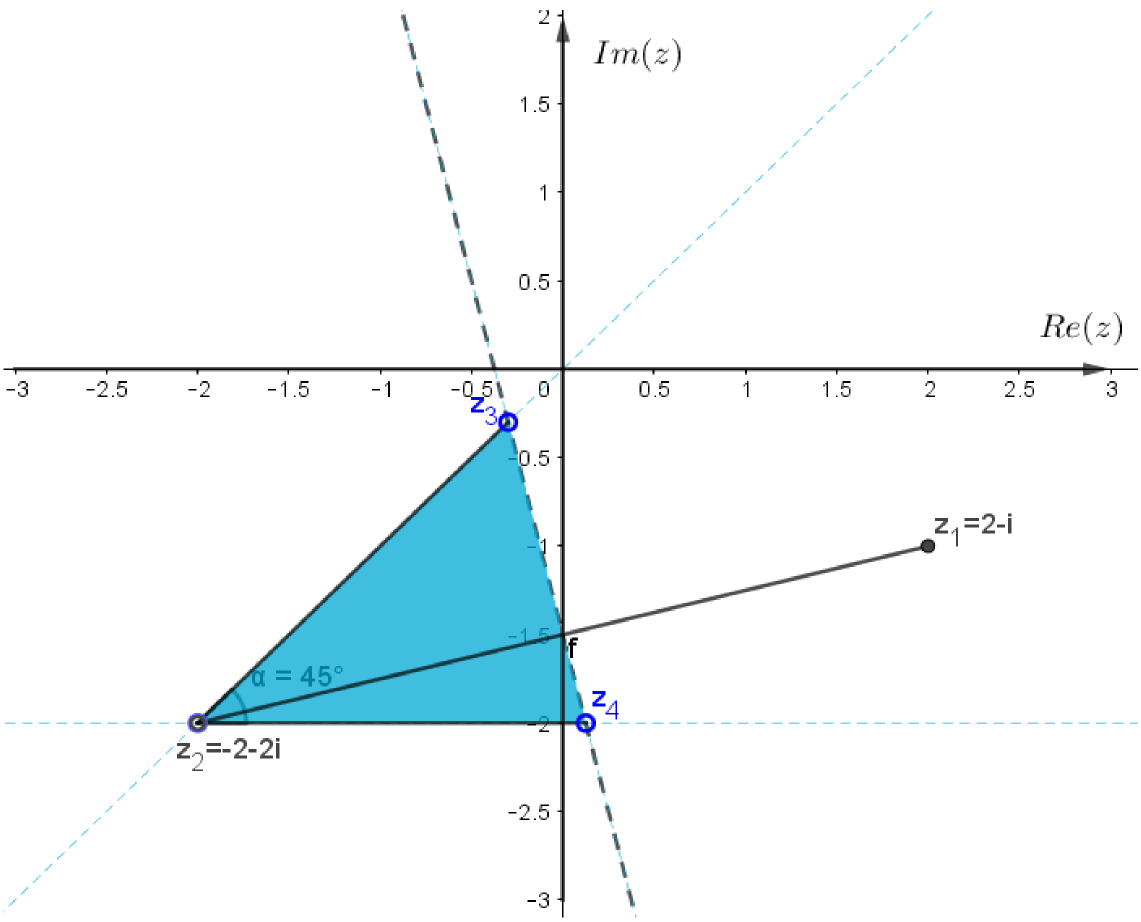
i.

$$(2-i)^2 = 4 - 4i + i^2 = 3 - 4i$$
$$(2-i)^3 = (3-4i)(2-i) = 6 - 3i - 8i + 4i^2 = 6 - 4 - 11i = 2 - 11i$$
$$(2-i)^4 = (2-11i)(2-i) = 4 - 2i - 22i + 11i^2 = 4 - 11 - 24i = -7 - 24i$$

$$\begin{aligned}&P(2-i) = (2-i)^4 - 5(2-i)^3 + 7(2-i)^2 + 3(2-i) - 10 = \\ &-7 - 24i - 5(2-11i) + 7(3-4i) + 3(2-i) - 10 = \\ &-7 - 24i - 10 + 55i + 21 - 28i + 6 - 3i - 10 = 0 + 0i = 0\end{aligned}$$

ii. $P(x)$ has integer coefficients and $2-i$ is a root implies that $2+i$ is also a root. So $(x-2+i)$ and $(x-2-i)$ are factors of $P(x)$. Hence $x^2-4x+5 = (x-2+i)(x-2-i)$ is a factor of $P(x)$. Using long division or alternative approach, you can show that $P(x) = (x^2-4x+5)(x^2-x-2) = (x-2+i)(x-2-i)(x-2)(x+1)$.

(b) $z_1 = 2 - i$ and $z_2 = -2 - 2i$



Question 4 solutions

(a)

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(n-1)^2}\right)\left(1-\frac{1}{n^2}\right)=$$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{(n-1)}\right)\left(1-\frac{1}{n}\right)\times$$

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{(n-1)}\right)\left(1+\frac{1}{n}\right)=$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{n-2}{n-1}\right)\left(\frac{n-1}{n}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{n}{n-1}\right)\left(\frac{n+1}{n}\right)=$$

$$\left(\frac{1}{n}\right)\left(\frac{n+1}{2}\right)=\frac{n+1}{2n}.$$

(b) **Proof by contradiction**:
If not, there exist integers x and y such that $3x+\sqrt{2}y=0$ and $(x\neq 0 \quad \text{or} \quad y\neq 0)$.
If $x=0$, then $\sqrt{2}y=0$ which implies that $y=0$.;
If $y=0$, then $3x=0$ which implies that $x=0$.;
Hence $x\neq 0$ and $y\neq 0$. But then $\sqrt{2}=\dfrac{-3x}{y}$, a quotient of integers, so $\sqrt{2}$ is rational.
proof for $\sqrt{2}$ is an irrational: $\sqrt{2}=\dfrac{a}{b}$, where $a,b\in\mathbb{Z}$ and HCF$(a,b)=1$ implies that $2b^2=a^2$, hence a^2 is even, hence a is even. There exists $k\in\mathbb{Z}, a=2k$.

$$2b^2=(2k)^2=4k^2 \quad \text{and} \quad b^2=2k^2,$$

so b^2 is even and hence b is even. Contradiction.
Therefore

$$(\forall x\in\mathbb{Z})(\forall y\in\mathbb{Z})[(3x+\sqrt{2}y=0)\implies(x=0 \quad \text{and} \quad y=0)].$$

Question 5 solutions

Given that

$$(x+y)^5-(x^5+y^5)=5xy(x+y)(x^2+xy+y^2).$$

and

$$z^n+z^{-n}=2\cos n\theta.$$

$$\left(z+\frac{1}{z}\right)^5-\left(z^5+\frac{1}{z^5}\right)=5z\frac{1}{z}\left(z+\frac{1}{z}\right)\left(z^2+z\frac{1}{z}+\frac{1}{z^2}\right).$$

$$\left(z+\frac{1}{z}\right)^5-\left(z^5+\frac{1}{z^5}\right)=5\left(z+\frac{1}{z}\right)\left(\left(z+\frac{1}{z}\right)^2-1\right)$$

$$\left(z+\frac{1}{z}\right)^5-\left(z^5+\frac{1}{z^5}\right)=5\left(z+\frac{1}{z}\right)^3-5\left(z+\frac{1}{z}\right).$$

$$(2\cos\theta)^5-2\cos 5\theta=5\left(2\cos\theta\right)^3-5(2\cos\theta)$$

$$32\cos^5\theta-2\cos 5\theta=40\cos^3\theta-10\cos\theta.$$

$$\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta.$$

$$16x^5-20x^3+5x-1=0.$$

Let $x=\cos\theta$, we obtain:

$$16\cos^5\theta-20\cos^3\theta+5\cos\theta-1=0.$$

which implies that

$$\cos 5\theta=1.$$

$$\theta=\frac{2k\pi}{5},k=0,1,2,3,4.$$

Hence

$$1,\ \cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5},\ \cos\frac{8\pi}{5}.$$

are the roots of $16x^5-20x^3+5x-1=0$.

$$(x-1)\left(16x^4+16x^3-4x^2-4x+1\right)=$$

$$16x^5+16x^4-4x^3-4x^2+x-16x^4-16x^3+4x^2+4x-1=$$

$$16x^5-20x^3+5x-1=0.$$

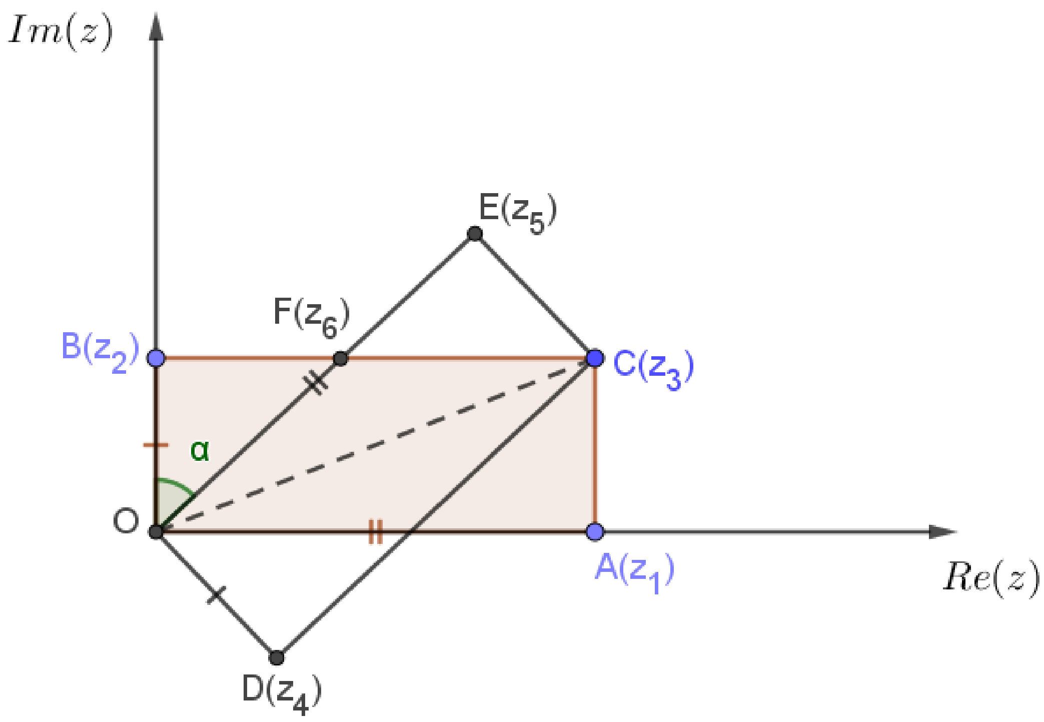
Hence

$$16x^4+16x^3-4x^2-4x+1=0$$

has roots

$$\cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5},\ \cos\frac{8\pi}{5}.$$

iii. Question 6 solutions



i. $z_3 = z_1 + z_2$.

ii.
$$|z_4| = |z_2| \quad \text{and} \quad \angle BOD = \frac{\pi}{2} + \alpha.$$
$$z_2 = z_4 cis\left(\frac{\pi}{2} + \alpha\right)$$
$$z_4 = z_2 \overline{cis\left(\frac{\pi}{2} + \alpha\right)} = z_2\left(-\sin\alpha - i\cos\alpha\right)$$
$$|z_5| = |z_1| \quad \text{and} \quad \angle AOE = \frac{\pi}{2} - \alpha.$$
$$z_5 = z_1 cis\left(\frac{\pi}{2} - \alpha\right) = z_1\left(\sin\alpha + i\cos\alpha\right).$$

iii.
$$z_3 = z_4 + z_5 = -z_2\left(\sin\alpha + i\cos\alpha\right) + z_1\left(\sin\alpha + i\cos\alpha\right) = (z_1 - z_2)\left(\sin\alpha + i\cos\alpha\right)$$
$$\overline{cis\alpha} = \overline{(\cos\alpha + i\sin\alpha)} = \cos\alpha - i\sin\alpha.$$
$$i\overline{(\cos\alpha + i\sin\alpha)} = \sin\alpha + i\cos\alpha.$$

Hence $z_3 = i(z_1 - z_2)\overline{cis\alpha}$

iv. $\Delta OBF \equiv \Delta CEF \quad (AAS)$

$$z_2 - z_6 = (z_5 - z_6)\, cis(\frac{\pi}{2} + \alpha) = \left(z_1 cis\left(\frac{\pi}{2} - \alpha\right) - z_6\right) cis(\frac{\pi}{2} + \alpha) = z_1 cis\pi - z_6 cis(\frac{\pi}{2} + \alpha)$$
$$z_2 - z_6 = -z_1 - z_6 cis(\frac{\pi}{2} + \alpha)$$

$$z_1 + z_2 = \left(1 - cis(\frac{\pi}{2} + \alpha)\right) z_6$$

$$z_6 = (z_1 + z_2)\frac{1}{1 - \cos(\frac{\pi}{2} + \alpha) - i\sin(\frac{\pi}{2} + \alpha)}$$

$$z_6 = (z_1+z_2)\frac{1}{1 + \sin\alpha - i\cos\alpha} = (z_1+z_2)\frac{1 + \sin\alpha + i\cos\alpha}{(1 + \sin\alpha)^2 + (\cos\alpha)^2} = (z_1+z_2)\frac{1 + \sin\alpha + i\cos\alpha}{2(1 + \sin\alpha)}$$

$$z_6 = \frac{z_1 + z_2}{2}\left(1 + i\frac{\cos\alpha}{1 + \sin\alpha}\right)$$

$$z_6 = \frac{z_1 + z_2}{2}\left(1 + i\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\right)^2}\right)$$

$$z_6 = \frac{z_1 + z_2}{2}\left(1 + i\frac{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}\right)$$

$$z_6 = \frac{z_1 + z_2}{2}\left(1 + i\frac{\cot\frac{\alpha}{2} - 1}{\cot\frac{\alpha}{2} + 1}\right)$$

$$z_6 = \frac{z_1 + z_2}{2}\left(1 + i\cot\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right) = \frac{z_1 + z_2}{2}\left(1 + i\cot\left(\frac{\pi + 2\alpha}{4}\right)\right)$$