Question 1 (15 Marks)

(a) If $z = 1 + \sqrt{3}i$ and $\omega = -1 - i$, find

(i) $|z|$ **1**

(ii) $\arg \omega$ **1**

(iii) $\omega\bar{z}$ **2**

(b) (i) Find the square roots of $-3 - 4i$ **2**

(ii) Hence, solve $z^2 - 3z + (3 + i) = 0$ **2**

(c) Suppose $P(x) = x^3 - 3x^2 + 4x - 2$

(i) Show that $P(1 - i) = 0$ **1**

(ii) Explain why $x = 1 + i$ is also a root of $P(x)$ **1**

(iii) Hence write $P(x)$ as a product of a linear and quadratic factor over $\mathbb{R}$ **2**

(d) Sketch the region represented by the following inequations: **3**

$-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$ and $\text{Re}\,(z) \leq 2$

Question 2 (10 Marks)

(a) For the following statement:

"For every integer n, if n is divisible by 3 then n^2 is divisible by 3."

Write down its

(i) Converse. **1**

(ii) Contrapositive **1**

(iii) Negation **1**

(b) Prove that the following is true for all positive integers n: n is even if and only if $3n^2 + 8$ is even **4**

(c) Using a proof by contradiction, prove that the equation $2x^3 + 6x + 1 = 0$ has no integer solutions. **3**

Question 3 (16 marks)

(a) The complex number z is represented by the point C in the Argand diagram. Find and sketch the curve that C is on if **4**

$$\mathrm{Re}\left(\frac{z-1}{z-i}\right)=0$$

(b) The points A and C represent the complex number $z_1 = 1+i$ and $z_3 = 7+3i$ respectively and B represents the complex number z_2. Find the complex number(s) z_2 in the form of $a+ib$ where a and b are real such that ΔABC is an isosceles triangle that is right angled at B. **4**

(c) Consider the equation $z^7 = 1$

(i) Show that the roots of the equation are given by **2**

$$\alpha_k = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right),\ k = 1,2,3,4,5,6,7$$

(ii) By using the factorisation $z^7 - 1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$, or otherwise **2**

Show $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_6) = 7$

(iii) Show that $1 - a_k = 2\sin\left(\frac{k\pi}{7}\right)\left[\cos\left(\frac{\pi}{14}(2k-7)\right) + \mathrm{i}\sin\left(\frac{\pi}{14}(2k-7)\right)\right]$ **2**

(iv) Hence show that **2**

$$\sin\left(\frac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{3\pi}{7}\right)\ldots\sin\left(\frac{6\pi}{7}\right) = \frac{7}{2^6}$$

End of Examination

Question 1

a) i) $|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= \sqrt{4}$

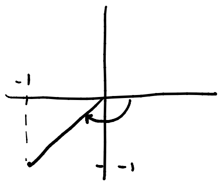
$= 2$

iii) $w\bar{z} = (-1-i)(1-\sqrt{3}i)$ [1]

$= -1 + \sqrt{3}i - i - \sqrt{3}$

$= (-1-\sqrt{3}) + i(\sqrt{3}-1)$ [1]

ii)



$\arg(w) = -\pi + \tan^{-1}(1)$

$= -\pi + \frac{\pi}{4}$

$= \frac{-3\pi}{4}$

b) i) let $\omega^2 = -3-4i$

where $\omega = a + ib$

$\therefore \quad a^2 - b^2 = -3$

$2ab = -4$

$ab = -2$

$\therefore$ by inspection

$a = 1 \quad b = -2 \quad$ or $\quad a = -1, \ b = 2$

$\therefore \omega = 1 - 2i \quad$ or $\quad \omega = -1 + 2i$

ii) $z^2 - 3z + (3+i) = 0$

$\Delta = 9 - 4 \times 1 \times (3+i)$

$= 9 - 12 - 4i$

$= -3 - 4i$

$\therefore \quad z = \dfrac{3 + \omega}{2}$ (from i)

$\therefore \quad z = \frac{1}{2}(3 + (1-2i))$ or $\frac{1}{2}(3 + (-1+2i))$

$= 2 - i$ or $1 + i$

c) i) $p(x) = x^3 - 3x^2 + 4x - 2$

$P(1-i) = (1-i)^3 - 3(1-i)^2 + 4(1-i) - 2$

$= (-2i)(1-i) - 3(-2i) + 4 - 4i - 2$

$= -2i - 2 + 6i + 4 - 4i - 2$

$= 0$

$\therefore \quad x = 1 - i$ is a zero of $P(x)$

$(1-i)(1-i) = 1 - 2i + i^2$

$= 1 - 2i - 1$

$= -2i$

ii) As the coefficient of all terms in $P(x)$ are real

then by the conjugate root theorem $x = 1 + i$ is also a zero

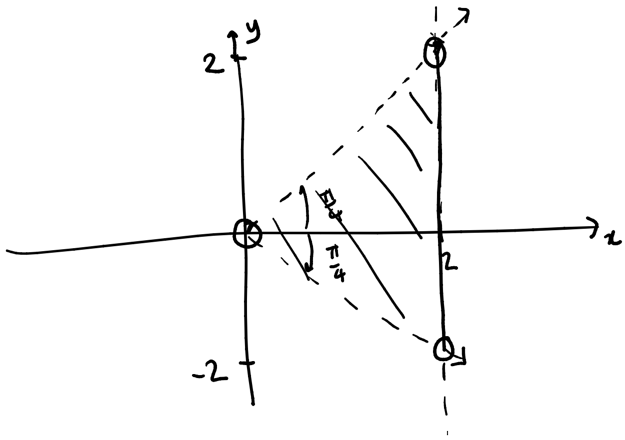
(ii) $P(x) = (x-(1-i))(x-(1+i))(x-\alpha)$ [1]

$= (x^2-2x+2)(x-\alpha)$

$\therefore$ by inspection

$p(x) = (x^2-2x+2)(x-1)$ [1]

d)



1 mark for correct rays and line (need to show gradient of -1 or 1, by either marking angle or points)

1 mark for correct region

1 mark for open circle at intersection points at region.

Question 2

a) For every integer n, if $\underbrace{n \text{ is divisible by } 3}_{A}$ then $\underbrace{n^2 \text{ is divisible by } 3}_{B}$

i) For every integer n, if n^2 is divisible by 3 then n is divisible by 3

ii) For every integer n, if n^2 is not divisible by 3 then n is not divisible by 3

iii) There exists an integer n such that n is divisible by 3 and n^2 is not divisible by 3

b) RTP: n is even if and only if $3n^2+8$ is even

Need to prove the following statements

1) if n is even $\Rightarrow$ $3n^2+8$ is even

2) if $3n^2+8$ is even $\Rightarrow$ n is even

To prove statement 1, we use a direct proof (2 marks)

1 mark for correct method

2 marks for correct proof

let $n = 2k$, $k \in \mathbb{Z}$

$$\therefore \quad 3n^2+8 = 3(2k)^2+8$$
$$= 3(4k^2)+8$$
$$= 12k^2+8$$
$$= 2(6k^2+4) \quad (6k^2+4 \text{ is an integer})$$

$\therefore$ $3n^2+8$ is even

Now to prove statement 2, we can prove the contrapositive. (2 marks)

if n is not even then $3n^2+8$ is not even

- 1 mark for correct method

- 2 mark for correct proof

let $n = 2m+1$ where $m \in \mathbb{Z}$

$$3(2m+1)^2+8$$
$$= 3(4m^2+4m+1)+8$$
$$= 12m^2+12m+3+8$$
$$= 12m^2+12m+11$$
$$= 2(6m^2+6m+5)+1$$
$$= 2M+1 \quad \text{where } M \text{ is an integer}$$

$\therefore$ $3n^2+8$ is not even

$\therefore$ n is even iff $3n^2+8$ is even

c) Suppose that $2x^3+6x+1=0$ has integer solutions, ie. x is an integer

3 marks

- full correct proof

2 marks

Now $2x^3+6x+1=0$

$\therefore\ 1=-2x^3-6x$

$= 2(-x^3-3)$

as x is an integer then $-x^3-3$ is also an integer. –

$\therefore\ 2(-x^3-3)$ is even

as 1 is not even then a contradiction occurs

$\therefore\ 2x^3+6x+1=0$ has no integer solutions.

Question 3

$$Re\left(\frac{z-1}{z-i}\right)=0$$

let $z = x+iy$

$$\therefore \quad \frac{z-1}{z-i} = \frac{x+iy-1}{x+iy-i}$$

$$= \frac{(x-1)+iy}{x+i(y-1)}$$

$$= \frac{[(x-1)+iy][x-i(y-1)]}{x^2-(y-1)^2}$$

$$= \frac{x(x-1) - i(x-1)(y-1) + ixy + y(y-1)}{x^2-(y-1)^2}$$

$$\therefore \quad Re\left(\frac{z-1}{z-i}\right) = \frac{x(x-1)+y(y-1)}{x^2-(y-1)^2}$$

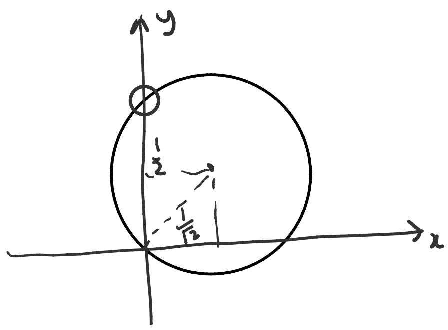
$$\therefore \quad Re\left(\frac{z-1}{z-i}\right) = 0 \Rightarrow \quad x(x-1)+y(y-1)=0$$

$$x^2-x+y^2-y=0$$

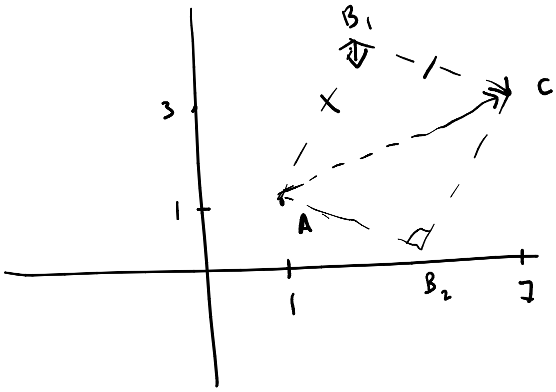
$$(x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2=\frac{1}{2}$$

$\therefore$ circle centre $(\frac{1}{2},\frac{1}{2})$ radius $\frac{1}{\sqrt{2}}$

But $z \neq i$



b)



$\vec{OB_1} = \vec{OA} + \vec{AB_1}$

Now $\quad B_1C = \vec{AB_1} \times -i$

$\vec{AC} = \vec{AB} + \vec{B_1C}$

$(7+3i) - (1+i) = \vec{AB_1} - \vec{AB_1}i$

$= \vec{AB_1}(1-i)$

$6+2i = \vec{AB_1}(1-i)$

$\therefore \vec{AB_1} = \dfrac{6+2i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{6+8i-2}{2}$

$= \dfrac{4+8i}{2}$

$= 2+4i$

$\therefore \quad \vec{OB_1} = (1+i) + (2+4i)$

$= 3+5i$

Now $\quad \vec{OB_2} = \vec{OA} + \vec{AB_2}$

$= \vec{OA} + \vec{AB_1} \times -i$

$= (1+i) + (2+4i)-i$

$= 1+i - 2i + 4$

$= 5-i$

$\therefore z_2 = 3+5i \quad$ or $\quad 5-i$

c) $\quad z^7 = 1$

i) let $\quad z = \cos\theta + i\sin\theta$

$\therefore z^7 = 1$

$z^7 = \cos(2k\pi) + i\sin(2k\pi) \quad , k \in \mathbb{Z}$

$z = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{7}}$

$= \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$ by De Moivre's Thm, for $k = 1,2,3,4,5,6,7$

$\therefore \alpha_k = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$

ii) as α_k are the roots of $z^7=1$ ie $z^7-1=0$

and $z^7-1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

then α_k are the roots of $(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

as $\alpha_7=1$ then α_7 is the root of $z-1$

$\therefore$ $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ are the roots of

$z^6+z^5+z^4+z^3+z^2+z+1=0$

$\therefore$ $z^6+z^5+z^4+z^3+z^2+z+1 = (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_6)$

let $z=1$

$\therefore$ $1+1+1+1+1+1+1 = (1-\alpha_1)(1-\alpha_2)\dots(1-\alpha_6)$

$\therefore$ $(1-\alpha_1)(1-\alpha_2)\dots(1-\alpha_6) = 7$

iii) $1-\alpha_k = 1-\left(\cos\left(\frac{2k\pi}{7}\right)+i\sin\left(\frac{2k\pi}{7}\right)\right)$

$= 1-\left(1-2\sin^2\left(\frac{k\pi}{7}\right)+2i\sin\left(\frac{k\pi}{7}\right)\cos\left(\frac{k\pi}{7}\right)\right)$ using double angle identity

$= 2\sin^2\left(\frac{k\pi}{7}\right)-2i\sin\left(\frac{k\pi}{7}\right)\cos\left(\frac{k\pi}{7}\right)$

$= 2\sin\left(\frac{k\pi}{7}\right)\left[\sin\left(\frac{k\pi}{7}\right)-i\cos\left(\frac{k\pi}{7}\right)\right]$

$= 2\sin\left(\frac{k\pi}{7}\right)\left[\cos\left(\frac{\pi}{2}-\frac{k\pi}{7}\right)-i\sin\left(\frac{\pi}{2}-\frac{k\pi}{7}\right)\right]$

$= 2\sin\left(\frac{k\pi}{7}\right)\left[\cos\left(\frac{7\pi-2k\pi}{14}\right)-i\sin\left(\frac{7\pi-2k\pi}{14}\right)\right]$

$= 2\sin\left(\frac{k\pi}{7}\right)\left(\cos\left(\frac{\pi}{14}(7-2k)\right)-i\sin\left(\frac{\pi}{14}(7-2k)\right)\right)$

$= 2\sin\left(\frac{k\pi}{7}\right)\left(\cos\left(-\left(\frac{\pi}{14}(2k-7)\right)\right)-i\sin\left(-\frac{\pi}{14}(2k-7)\right)\right)$

$= 2\sin\left(\frac{k\pi}{7}\right)\left(\cos\left(\frac{\pi}{14}(2k-7)\right)+i\sin\left(\frac{\pi}{14}(2k-7)\right)\right)$

as $\cos(-\theta)=\cos\theta$
$\sin(-\theta)=-\sin\theta$

iv) From (iii)

$$1 - a_1 = 2\sin\frac{\pi}{7}\left(\cos\left(\frac{-5\pi}{14}\right) + i\sin\left(\frac{-5\pi}{14}\right)\right)$$

$$\therefore \left|1 - a_1\right| = \left|2\sin\frac{\pi}{7}\left(\cos\left(-\frac{5\pi}{14}\right) + i\sin\left(-\frac{5\pi}{14}\right)\right)\right|$$

$$= \left|2\sin\frac{\pi}{7}\right|\left|\cos\left(-\frac{5\pi}{14}\right) + i\sin\left(-\frac{5\pi}{14}\right)\right|$$

$$= 2\sin\frac{\pi}{7}$$ as $\sin\frac{k\pi}{7} > 0$ for $k = 1, 2, 3, 4, 5, 6.$

and $\left|\cos\frac{\pi}{14}(2k-7) + i\sin\left(\frac{\pi}{14}(2k-7)\right)\right| = 1$

Similarly $|1 - a_2| = 2\sin\frac{2\pi}{7}$ etc

$$\therefore \left|1 - a_1\right| \cdot \left|1 - a_2\right| \cdot \dots \left|1 - a_6\right| = 2\left(\sin\frac{\pi}{7}\right) \cdot 2\left(\sin\frac{2\pi}{7}\right) \dots \left(2\sin\frac{6\pi}{7}\right)$$

$$\left|(1 - a_1)(1 - a_2) \cdots (1 - a_6)\right| = 2^6\left(\sin\frac{\pi}{7}\right)\left(\sin\frac{2\pi}{7}\right) \dots \sin\frac{6\pi}{7}$$

$$|7| = 2^6\left(\sin\frac{\pi}{7}\right)\left(\sin\frac{2\pi}{7}\right) \dots \sin\frac{6\pi}{7} \quad \text{(From (ii))}$$

$$\therefore \left(\sin\frac{\pi}{7}\right)\left(\sin\frac{2\pi}{7}\right) \dots \left(\sin\frac{6\pi}{7}\right) = \frac{7}{2^6}$$