Question 1 (3 marks)

A sequence $\mu_1, \mu_2, \mu_3 \ldots \mu_n$ is such that any three consecutive terms are related by the equation $\mu_{n+3} = 6\mu_{n+2} - 5\mu_{n+1}$. It is given that $\mu_1 = 2$ and $\mu_2 = 6$. **3**

Use mathematical induction to prove that $\mu_n = 5^{n-1} + 1$.

Question 2 (10 marks)

Consider the equation $z^4 - z^3 + z^2 - z + 1 = 0$

(a) Show that the roots of this equations are $e^{\pm\frac{i\pi}{5}}$ and $e^{\pm\frac{i3\pi}{5}}$ **3**

(b) Explain why the roots satisfy $z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$ **2**

(c) By letting $x = z + \frac{1}{z}$, obtain a quadratic equation in x **2**

(d) Hence find the exact value of $\cos\left(\frac{3\pi}{5}\right)$ **3**

Question 3 (8 Marks)

A particle of mass m is thrown from a place h metres above ground level with an initial velocity u at an angle of α to the horizontal.

The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions are given respectively by

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}$$

Where k is a constant and the acceleration due to gravity is g (You are not required to show these)

(a) Show that $x = \frac{u\cos\alpha}{k}(1 - e^{-kt})$ **3**

(b) Show that the equation of path can be written as **5**

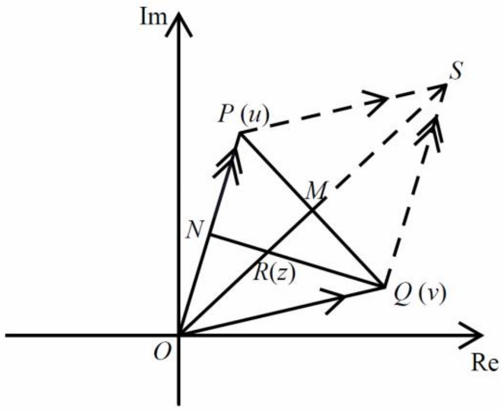
$$y = h + \frac{1}{k}(u\sin\alpha + g\ln\left(-\frac{kx}{u\cos\alpha} + 1\right) + \frac{1}{k^2}\left(g - (g + ku\sin\alpha)\left(-\frac{xk}{\cos\alpha} + 1\right)\right)$$

Question 4 (6 marks)

In the Argand diagram below, P and Q represent complex numbers u and v respectively. O, P and Q are not collinear.

In ΔOPQ, the line from O to the midpoint M of PQ, meets the line from Q to the midpoint N of OP in the point R. Let R represent the complex number z.

S is the point such that $OPSQ$ is a parallelogram.



(a) Explain why there are positive real numbers k and l such that **2**

$$kz = \tfrac{1}{2}(u+v) \text{ and } l(z-v) = \tfrac{1}{2}u - v$$

(b) Hence show $(3-2l)v = 2(k-l)z$. **1**

(c) Deduce that $z = \dfrac{1}{3}(u+v)$. **3**

Question 5 (6 marks)

Let $T_n(x) = \dfrac{{}^nC_0}{x} - \dfrac{{}^nC_1}{x+1} + \dfrac{{}^nC_2}{x+2} - \quad .... \quad + (-1)^n \dfrac{{}^nC_n}{x+n}$ for a given integer n and all real x

(a) If $S_k(x) = \dfrac{k!}{x(x+1)(x+2)....(x+k)}$ where k is an integer, show that **2**

$$S_k(x) - S_k(x+1) = S_{k+1}(x)$$

(b) Hence prove using mathematical induction that for $n \ge 1$ **4**

$$T_n(x) = \frac{n!}{x(x+1)(x+2)....(x+n)}$$

NOTE: you may use without proof the result ${}^{m+1}C_r = {}^mC_r + {}^mC_{r-1}$

End of Examination

Question 1

Test $n=1$

$\mu_1 = 5^0 + 1$

$= 2$

$\therefore$ true for $n=1$

Test $n=2$

$\mu_2 = 5^1 + 1$

$= 6$

$\therefore$ true for $n=2$

Assume true for $n=k+1$ and $n=k+2$

i.e. $\mu_{k+1} = 5^k + 1$

$\mu_{k+2} = 5^{k+1} + 1$ $\quad *$

Prove true for $n=k+3$

i.e. $\mu_{k+3} = 5^{k+2} + 1$

Now $\mu_{k+3} = 6\mu_{k+2} - 5\mu_{k+1}$

$= 6(5^{k+1}+1) - 5(5^k+1)$ [From assumption $*$]

$= 6 \cdot 5^{k+1} + 6 - 5^{k+1} - 5$

$= 5 \cdot 5^{k+1} + 1$

$= 5^{k+2} + 1$

$\therefore$ true for $n=k+3$

$\therefore$ true for all n, by principle of mathematical induction.

Question 2

a) consider $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$

The roots of $z^5+1=0$ are the fifth roots of -1

$z^5 = e^{i(\pi+2k\pi)}$, $k = 0, \pm1, \pm2$

$\therefore z = e^{\frac{i}{5}(\pi+2k\pi)}$

$\therefore z = e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, e^{i\frac{5\pi}{5}}, e^{i(\frac{-\pi}{5})}, e^{i(\frac{-3\pi}{5})}$ are the roots of $z^5+1=0$

$z^5+1=0 \Rightarrow (z+1)=0$ or $(z^4-z^3+z^2-z+1)=0$

$z = e^{i(\frac{5\pi}{5})} = -1$ corresponds to $z+1=0$

$\therefore$ the roots of $z^4-z^3+z^2-z+1=0$ are $z = e^{i(\pm\frac{\pi}{5})}, e^{i(\pm\frac{3\pi}{5})}$

b) Consider $z^4-z^3+z^2-z+1 = z^2\left(z^2-z+1-\frac{1}{z}+\frac{1}{z^2}\right)$

$z^4-z^3+z^2-z+1=0 \Rightarrow z=0$ or $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

$z=0$ is <u>not</u> a root of $z^4-z^3+z^2-z+1=0$

Thus the roots of $z^4-z^3+z^2-z+1=0$ are also the roots of $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

c) $x = z+\frac{1}{z}$

$x^2 = \left(z+\frac{1}{z}\right)^2 = z^2+\frac{1}{z^2}+2$

$z^2+\frac{1}{z^2} = \left(z+\frac{1}{z}\right)^2-2$

$= x^2-2$

substituting into equation $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

$\left(z^2+\frac{1}{z^2}\right)-\left(z+\frac{1}{z}\right)+1=0$

$x^2-2-x+1=0$

$x^2-x-1=0$

d) The roots of $x^2-x-1=0$ corresponds to $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$ where

$z+\frac{1}{z}=x$

$x = e^{i\theta}+\frac{1}{e^{i\theta}} = 2\cos\theta$

$\therefore x = 2\cos\frac{\pi}{5}, 2\cos\frac{3\pi}{5}$ as $\cos\theta$ is even

using quadratic formula

$x^2-x-1=0$ has roots $x = \frac{1\pm\sqrt{5}}{2}$

as $\frac{3\pi}{5}$ is in quadrant 2 $\cos\frac{3\pi}{5}<0$

$\therefore 2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2}$

$\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$

Question 3

a) using the given equation of motion

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}$$

solving these DEs with initial condition

$$x(0) = 0 \qquad y(0) = h$$
$$x'(0) = u\cos\alpha \qquad y'(0) = u\sin\alpha$$

using separation of variables

$$\frac{dv_x}{dt} = -kv_x$$

$$\int\frac{dv_x}{v_x} = \int -k\,dt$$

$$\ln v_x = -kt + C_1$$

$$v_x = C_2 e^{-kt}$$

when $t=0$ $v_x = u\cos\alpha$ $\therefore C_2 = u\cos\alpha$

$$v_x = u\cos\alpha\, e^{-kt}$$

$$\therefore x = -\frac{u\cos\alpha}{k}e^{-kt} + C_3$$

when $t=0$ $x=0$ $\therefore C_3 = \frac{u\cos\alpha}{k}$

$$\therefore x = \frac{u\cos\alpha}{k}(1 - e^{-kt})$$

b) Similarly for y,

$$\frac{dv_y}{dt} = -g - kv_y$$

$$\frac{dv_y}{-g-kv_y} = dt$$

$$\frac{1}{k}\ln(g + kv_y) = -t + D$$

$$g + kv_y = D_1 e^{-kt}$$

when $t=0$ $v_y = u\sin\alpha$

$$\therefore D_1 = g + k(u\sin\alpha)$$

$$\therefore g + kv_y = (g + ku\sin\alpha)e^{-kt}$$

$$\therefore \frac{dy}{dt} = \frac{1}{k}\left(-g + (g + ku\sin\alpha)e^{-kt}\right)$$

using the initial conditions $y(0) = h$

$$y = -\frac{g}{k}t - \frac{g e^{-kt}}{k^2} - \frac{u \sin\alpha \, e^{-kt}}{k} + D_2$$

$$D_2 = h + \frac{g}{k^2} + \frac{u \sin\alpha}{k}$$

$$\therefore \quad y = -\frac{g}{k}t - \frac{g e^{-kt}}{k^2} - \frac{u \sin\alpha \, e^{-kt}}{k} + h + \frac{g}{k^2} + \frac{u \sin\alpha}{k}$$

$$= h + \frac{1}{k}\left(u \sin\alpha - gt\right) + \frac{1}{k^2}\left(g - g e^{-kt} - uk \sin\alpha \, e^{-kt}\right)$$

$$= h + \frac{1}{k}\left(u \sin\alpha - gt\right) + \frac{1}{k^2}\left(g - e^{-kt}\left(g + ku \sin\alpha\right)\right)$$

Now $x = \frac{u \cos\alpha}{k}\left(1 - e^{-kt}\right)$

$$\therefore t = -\frac{1}{k} \ln\left(\frac{-xk}{u\cos\alpha} + 1\right)$$

$$\therefore \quad y = h + \frac{1}{k}\left(u \sin\alpha + g\left(\frac{-xk}{u\cos\alpha} + 1\right)\right) + \frac{1}{k^2}\left(g - \left(g + ku\sin\alpha\right)\left(-\frac{xk}{u\cos\alpha} + 1\right)\right)$$

Question 4

a) M corresponds to $\frac{1}{2}(u+v)$

$\vec{OR}$ and $\vec{OM}$ are coincident i.e. $\arg(\vec{OR}) = \arg(\vec{OM})$

So there are real numbers k such that $kz = \frac{1}{2}(u+v)$

Now N corresponds to the complex number $\frac{1}{2}u$ and so the $\vec{QN}$ corresponds to $\frac{1}{2}u - v$ and $\vec{QR}$ corresponds to $z - v$

as $\vec{QR}$ and $\vec{QN}$ are coincident i.e. $\arg(\vec{QR}) = \arg(\vec{QN})$

there is a real number l such that $l(z-v) = \frac{1}{2}u - v$

b) $kz = \frac{1}{2}u + \frac{1}{2}v$ ①

$lz - lv = \frac{1}{2}u - v$ ②

① - ②

$kz - lz + lv = \frac{3}{2}v$

$2kz - 2lz + 2lv = 3v$

$2(k-l)z = (3-2l)v$

c) From b) $(3-2l)v = 2(k-l)z$

As $3-2l$ and $2(k-l)$ are real numbers this implies that the vectors representing the complex numbers v and z are coincident.

This is impossible as the vectors $\vec{OR}$ and $\vec{OQ}$ are not coincident.

$\therefore$ $3 - 2l = 2(k-l) = 0$

$\therefore$ $k = l = \frac{3}{2}$

As $kz = \frac{1}{2}(u+v)$ then $\frac{3}{2}z = \frac{1}{2}(u+v)$

$\therefore$ $z = \frac{1}{3}(u+v)$

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Question 5

a) LHS $= S_k(x) - S_k(x+1)$

$$= \frac{k!}{x(x+1)(x+2)\dots(x+k)} - \frac{k!}{(x+1)(x+2)(x+3)\dots(x+k+1)}$$

$$= \frac{k!\,(x+k+1) - xk!}{x(x+1)(x+2)(x+3)\dots(x+k+1)}$$

$$= \frac{k!\,((x+k+1)-x)}{x(x+1)(x+2)(x+3)\dots(x+k+1)}$$

$$= \frac{k!\,(k+1)}{x(x+1)(x+2)\dots(x+k+1)}$$

$$= \frac{(k+1)!}{x(x+1)(x+2)\dots(x+k+1)}$$

$$= S_{k+1}(x)$$

b) Test $n=1$

LHS $= T_1(x) = \frac{{}^1C_0}{x} - \frac{{}^1C_1}{x+1}$

$$= \frac{1}{x} - \frac{1}{x+1}$$

$$= \frac{1}{x(x+1)}$$

RHS $= \frac{1!}{x(x+1)} = \frac{1}{x(x+1)}$

$\therefore$ true for $n=1$

Assume true for $n=k$ i.e. $T_k(x) = \frac{k!}{x(x+1)(x+2)\dots(x+k)}$

Prove true for $n=k+1$ i.e. $T_{k+1}(x) = \frac{(k+1)!}{x(x+1)(x+2)\dots(x+k+1)}$

LHS $= T_{k+1}(x)$

$$= \frac{{}^{k+1}C_0}{x} - \frac{{}^{k+1}C_1}{x+1} + \frac{{}^{k+1}C_2}{x+2} + \dots + (-1)^k \frac{{}^{k+1}C_k}{x+k} + (-1)^{k+1} \frac{{}^{k+1}C_{k+1}}{x+k+1}$$

$$= \frac{{}^kC_0}{x} - \frac{({}^kC_1 + {}^kC_0)}{x+1} + \frac{({}^kC_2 + {}^kC_1)}{x+2} + \dots + (-1)^k \frac{({}^kC_k + {}^kC_{k-1})}{x+k} + (-1)^{k+1} \frac{{}^kC_k}{(x+1)+k}$$

$$= \frac{{}^kC_0}{x} - \frac{{}^kC_1}{x+1} + \frac{{}^kC_2}{x+2} + \dots + (-1)^k\, {}^kC_k - \left[\frac{{}^kC_0}{x+1} - \frac{{}^kC_1}{x+2} + \dots + (-1)^{k-1} \frac{{}^kC_{k-1}}{x+k} + (-1)^k \frac{{}^kC_k}{x+1+k}\right]$$

$= T_k(x) - T_k(x+1)$

$= S_k(x) - S_k(x+1)$ [from assumption *]

$= S_{k+1}(x)$ [from (a)]

$\therefore$ true for $n = k+1$

$\therefore$ true for all $n \geq 1$ by principle of mathematical induction.