NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2021 HSC Course Assessment Task 1

13 December 2021

General instructions

- Working time – 1 hour. (plus 5 minutes reading time)
- Write your answers in the answer booklet provided.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.

Student Number

Class (please tick)

◯ 12MX2 A4 Mr Berry

◯ 12MX2 B6 Ms Lee

◯ 12MX2 C5 Mr Umakanthan

Marker's use only.

QUESTION	1	2	3	4	5	**Total**
MARKS	$\overline{}$ 6	$\overline{}$ 6	$\overline{}$ 7	$\overline{}$ 7	$\overline{}$ 9	$\overline{}$ 35

	MEX 12-4	MEX 12-2
Q1	/6	
Q2	/6	
Q3		/7
Q4	/7	
Q5	/9	
Total	/28	/7

Question 1. (6 Marks)

(a) Let $\omega = \dfrac{20+21i}{29}$ and $z = \dfrac{3+4i}{5}$

(i) Express ωz in $(x+iy)$ form. 1

(ii) Express $\omega \overline{z}$ in $(x+iy)$ form. 1

(iii) Hence find two distinct ways of writing 145^2 as the sum a^2+b^2 where a and b are integers such that $0 < a < b$ 1

(b) In an Argand diagram, ABCD is a rhombus with $\angle ABC = 60^0$. 3
The points A and B represent the complex numbers $(3+4i)$ and $(6+6i)$ respectively.
Find the complex numbers representing the points C and D.
(your answer in $(a+ib)$ form with exact values for a and b.

Question 2. (6 Marks)

(i) Find the square roots of $8+6i$ 2

(ii) Solve $z^2+2(2+3i)z-13+6i=0$ 2

(iii) Describe geometrically the locus of z, if $|z+5+4i|$ = $|z^2+2(2+3i)z-13+6i|$ 2

Question 3. (7 Marks)

(a) Write down the negation to the statement "For all $x \epsilon R$, if $f(x) = \dfrac{2x+3}{x+2}$, then $f(x) \neq 2$ 1

(b) Write down the converse statement to the conditional statement 1
"If $5x-1=9$ then $x=2$"

(c) Let a, b and n are integers. Prove by contrapositive the conditional statement:
"If n does not divide ab then n does not divide a and n does not divide b " 2

(d) Prove by contradiction that there are no rational solution to the equation
$x^3+x+1=0$. 3

Question 4. (7 Marks)

Let $|z| \le 2$ and $\arg\left(\dfrac{z+2}{\sqrt{3}+i}\right) \ge \dfrac{\pi}{12}$.

(i) Sketch on an Argand diagram the region representing z. 3

(ii) Hence determine the exact value of a and b such that $a \le arg(z+i) \le b$ 2

(iii) Find the minimum value of $|z+i|$ 2

Question 5. (9 Marks)

(i) Solve the equation $z^5 + i = 0$. Write down all the roots in mod-arg form. 2

(ii) Express $z^4 - iz^3 - z^2 + iz + 1$ as the product of two quadratic factors in z 2

(iii) Hence evaluate $\left(\sin\dfrac{\pi}{10}\sin\dfrac{3\pi}{10}\right)$ 1

(iv) Hence evaluate $\left(\sin\dfrac{3\pi}{10} - \sin\dfrac{\pi}{10}\right)$ 1

(v) Hence find the exact value of $\sin\left(\dfrac{\pi}{10}\right)$ 3

End Of Paper

Suggested Solutions

Supplementary Task 1 Ext 2 2022

Q1 (a) "If n^2+4n+1 is even, then n is odd"

Contrapositive statement:

"If n is even then n^2+4n+1 is odd."

(b) (i) "If n^3+5 is odd then n is even".

Negation:

both n^3+5 and n are odd.

Prove by Negation (ii) Let $n=2k+1$ and $n^3+5=2j+1$

$$2j+1 = n^3+5$$
$$= (2k+1)^3+5$$
$$2j+1 = 8k^3+12k^2+6k+6$$
$$\Rightarrow 2j = 8k^3+12k^2+6k+5$$

Dividing by 2 and rearranging, we have

$$j-4k^3-6k^2-3k=\frac{5}{2}$$

This, however is impossible: $\frac{5}{2}$ is a non-integer rational number, while $j-4k^3-6k^2-3k$ is an integer by the closure properties for integers

Therefore, it must be the case that our assumption that when n^3+5 is odd then n is odd is false,

So, n must be even.

Proof by contrapositive If n is odd then n^3+5 is even.

Let $n=2k+1$ $k\in\mathbb{Z}$.

$$n^3+5 = (2k+1)^3+5$$
$$= 2[4k^3+6k^2+3k+3]$$
$$= 2\times\text{integer}$$

$\therefore n^3+5$ is even.

Q2. (a) (i) $24-70i = (7-5i)^2$

$\therefore \sqrt{24-70i} = \pm(7-5i)$

$\underline{\underline{\text{or}}}$ $\underline{\text{Standard Method}}$

Let $\sqrt{24-70i} = a+ib$ where a, b are real.

$\Rightarrow 24-70i = (a+ib)^2$

$= (a^2-b^2) + 2iab$

Matching the real and imaginary parts,

$$\left.\begin{array}{l} a^2-b^2 = 24 \\ ab = -35 \end{array}\right\} \Rightarrow a^4 - 24a^2 - 35^2 = 0$$

$(a^2-49)(a^2+25) = 0$

$\Rightarrow a = \pm 7$ $\quad a^2+25 \neq 0$ since a is real

$$\left.\begin{array}{l} a = 7 \\ b = -5 \end{array}\right\} \quad \left.\begin{array}{l} a = -7 \\ b = 5 \end{array}\right\}$$

$\therefore \sqrt{24-70i} = \pm(7-5i)$

(ii) $z^2 - (5-i)z + 15i = 0$

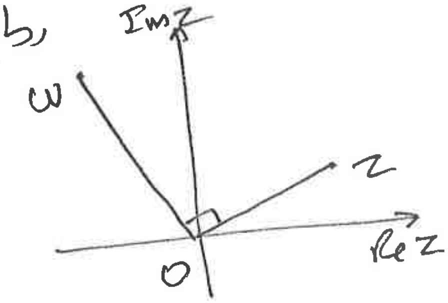
$$z = \frac{(5-i) \pm \sqrt{(5-i)^2 - 4\times 15i}}{2}$$

$$= \frac{(5-i) \pm \sqrt{24-70i}}{2}$$

$$= \frac{(5-i) \pm (7-5i)}{2}$$

$= \underline{\underline{(6-3i)}}$ or $\underline{\underline{(-1+2i)}}$

(b)



$w = 2iz$

$w^2 = (2iz)^2$

$= -4z^2$

$\Rightarrow \underline{\underline{w^2 + 4z^2 = 0}}$

Q3 (a) $z^5+1=0$

$z^5=-1$

$= cis(2k+1)\pi$

$z = cis(2k+1)\frac{\pi}{5}$

$\underline{k=0}$ $z_1 = cis\frac{\pi}{5}$

$\underline{k=-1}$ $z_2 = cis(-\pi/5)$

$\underline{k=1}$ $z_3 = cis(\frac{3\pi}{5})$

$k=-2$ $z_4 = \cancel{cis(\ )}\ cis(-\frac{3\pi}{5})$

$\cancel{k=2}$ $\cancel{z_5 = cis(\pi) = -1}$

$z^5+1 = (z+1)(z^4-z^3+z^2-z+1) = (z+1)[(z-z_1)(z-z_2)][(z-z_3)(z-z_4)]$

$\therefore z^4-z^3+z^2-z+1 = [z^2-(z_1+z_2)z+z_1z_2][z^2-(z_3+z_4)z+z_3z_4]$

$= [z^2-(2\cos\frac{\pi}{5})z+1][z^2-(2\cos\frac{3\pi}{5})z+1]$

<u>Matching the coefficients of z^3:</u>

$-2\cos\frac{\pi}{5}-2\cos\frac{3\pi}{5}=-1$

$\Rightarrow \cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$

<u>Matching the coefficients of z^2</u>

$2+4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=1$

$\Rightarrow \cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$

The equation whose roots are $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ is

$x^2-\frac{1}{2}x-\frac{1}{4}=0$

$4x^2-2x-1=0 \Rightarrow x=\frac{1\pm\sqrt{5}}{4}$

as $\cos\frac{3\pi}{5}<0$

$\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$ //