Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2023
ASSESSMENT TASK 1

Mathematics Extension 2

General Instructions

- **Working time – 60 minutes**
  Reading time – 5 minutes
- Write on both sides of the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators only
- All necessary working should be shown in every question
- Each new question is to be started on a **new booklet.**
- Attempt all questions.

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Mr Umakanthan
- O Ms Lee

Student Number:

(To be used by the exam markers only.)

Question	**1**	**2**	**3**	**Total**	**Percent**
Mark	$\overline{}$ 14	$\overline{}$ 11	$\overline{}$ 12	$\overline{}$ 37	$\overline{}$ 100

Question 1 (14 Marks)

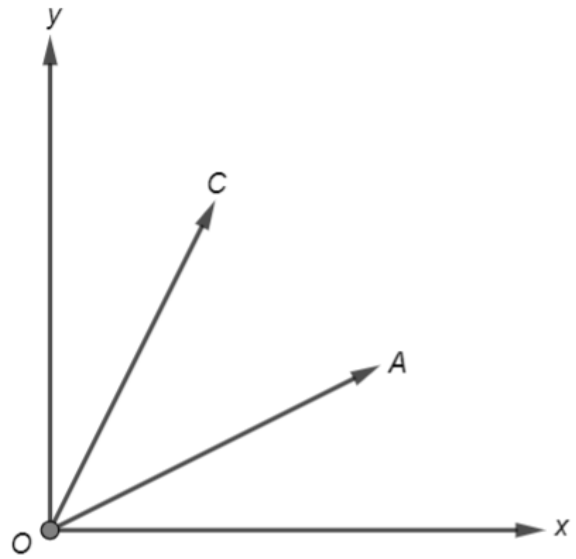
(a) Let $z = 4 + 2i$. Find:

(i) $z\bar{z}$ **1**

(ii) $z - \bar{z}$ **1**

(b) Write $\frac{3+3i}{2+i}$ in the form $x + iy$ where x and y are real numbers **1**

(c) If O is the origin and $\overrightarrow{OA} = 2 + i$ and $\overrightarrow{OC} = 1 + 2i$



(i) Find B such that $OABC$ is a rhombus **1**

(ii) Show that $\tan \angle AOB = \frac{1}{3}$ **2**

(iii) Hence show that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$ **2**

(d) Given that $\frac{z^2}{z-1}$ is a real number, sketch the locus of z on the Argand diagram. **3**

(e) Solve the equation $z + 1 + 8i = |z|(1 + i)$ where $z \in \mathbb{C}$ **3**

Question 2 (11 Marks)

(a) $y = f(x)$ is a function. Consider the following 2 statements

p: $f''(a) = 0$

q: There is a point of inflection at $(a, f(a))$

Which of the following statements are true **2**

1. *p* is a necessary condition for *q*
2. *q* is a necessary condition for *p*
3. *p* is a sufficient condition for *q*
4. *q* is a sufficient condition for *p*

(b) Consider the following statement:

If I study Extension 2 Mathematics, then I am super-cool.

Write down the:

(i) Converse **1**

(ii) Negation **1**

(iii) Contrapositive **1**

(c) Prove that $a^2 - 1$ is divisible by 3 if and only if a is not divisible by 3 where $a \in \mathbb{Z}$ **3**

(d) (i) Prove that if $a \mid b$ and $a \mid c$ then $a \mid b - c$ where a, b and c are integers **1**

(ii) Hence or otherwise prove that $n! + 1$ and $(n+1)! + 1$ are relatively prime where n is a positive integer. **2**

* $a \mid b$ means 'a is divides b' so a is a factor of b where a and b are both integers

** two numbers are relatively prime if they have no common integer factors other than 1

Question 3 (12 Marks)

(a) If w is a non-real root of the equation $z^3 = 1$

(i) Show that $1 + w + w^2 = 0$ **1**

(ii) Find the value of $(1 - w)(1 - w^2)(1 - w^4)(1 - w^8)$ **2**

(b) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^7$ or otherwise:

Write $\dfrac{\sin 7\theta}{\sin\theta}$ in terms of $\sin\theta$. **2**

(ii) Hence or otherwise find the roots of the equation $64x^3 - 112x^2 + 56x - 7 = 0$ **2**

(iii) Hence find $\text{cosec}^2\frac{\pi}{7} + \text{cosec}^2\frac{2\pi}{7} + \text{cosec}^2\frac{3\pi}{7}$ **2**

(c) Show that if m and n are positive integers there are no complex numbers z , such that:

$z^n = 1 + i$ and

$z^m = 2 - i$ **3**

Question 1

(a) (i) $z\bar{z} = (4+2i)(4-2i)$

$= 4^2 - (2i)^2$

$= 16 + 4$

$= 20$

(a) (ii) $z - \bar{z} = (4+2i) - (4-2i)$

$= \cancel{4} + 2i - \cancel{4} + 2i$

$= 4i$

(b) $\dfrac{3+3i}{2+i} = \dfrac{3+3i}{2+i} \times \dfrac{(2-i)}{(2-i)}$

$= \dfrac{(3+3i)(2-i)}{2^2 - i^2}$

$= \dfrac{6 - 3i + 6i + 3}{4+1}$

$= \dfrac{9+3i}{5}$

$= \dfrac{9}{5} + \dfrac{3}{5}i$

(c) (i) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= (2+i) + (1+2i) \quad (\text{as } \overrightarrow{AB} = \overrightarrow{OC})$

$= 3 + 3i$

(ii) $\tan\angle AOB = \tan(\angle xOB - \angle xOA)$

$= \tan(\operatorname{Arg}\overrightarrow{OB} - \operatorname{Arg}\overrightarrow{OA})$

$= \tan\left(\operatorname{Arg}\left(\frac{\overrightarrow{OB}}{\overrightarrow{OA}}\right)\right)$

$= \tan\left(\operatorname{Arg}\left(\frac{3+3i}{2+i}\right)\right)$

$= \tan\left(\operatorname{Arg}\left(\frac{9}{5} + \frac{3}{5}i\right)\right) \quad (\text{from b})$

$= \frac{3/5}{9/5} \quad \left(\text{as } \tan\operatorname{Arg}(z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)$

$= 3/9$

$= 1/3$

(iii) $\angle AOB = \tan^{-1}\left(\frac{1}{3}\right) \quad (\text{from (ii)})$

$\angle xOA = \tan^{-1}\left(\frac{1}{2}\right)$

$\angle xOB = \tan^{-1}\left(\frac{3}{3}\right)$

$= \tan^{-1}(1)$

$= \pi/4$

$\angle AOB + \angle xOA = \angle xOB$

$\therefore \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4}$

(d) $\dfrac{z^2}{z-1} = \dfrac{(x+iy)^2}{x+iy-1}$ (letting $z = x+iy$)

$$= \frac{x^2-y^2+2xyi}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

numerator $= (x^2-y^2+2xyi)(x-1-iy)$

$\mathrm{Im}(\text{numerator}) = 2x^2y - 2xy - x^2y + y^3$ ✓

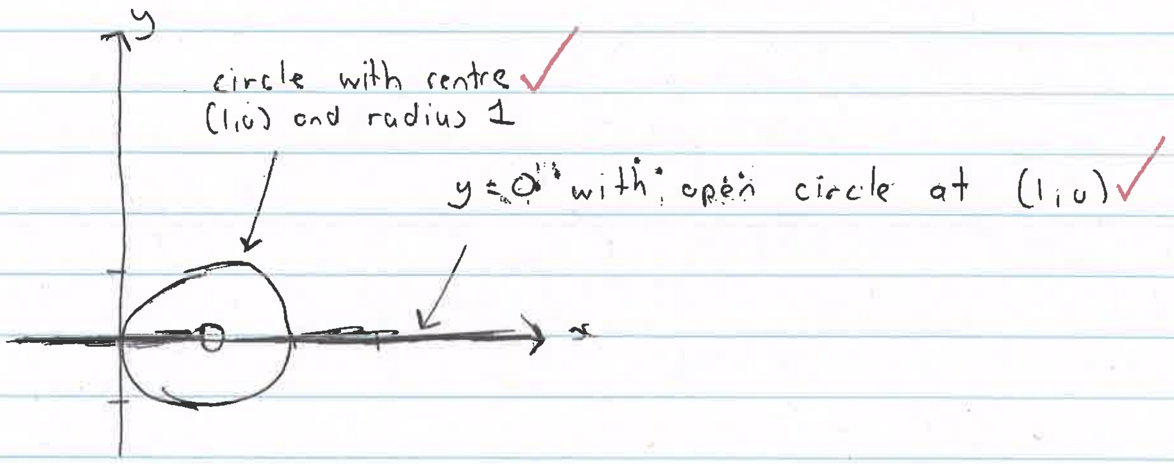
$= y^3 + x^2y - 2xy$

$= y(y^2+x^2-2x)$

$\therefore y = 0$ or $y^2+x^2-2x = 0$

$y^2+x^2-2x+1 = 1$

$y^2+(x-1)^2 = 1$



(e) $z + 1 + 8i = |z|(1 + i)$

$z + 1 + 8i = |z| + |z|i$

$z = |z| + |z|i - 1 - 8i$

$z = (|z| - 1) + (|z| - 8)i$

$\therefore |z|^2 = (|z| - 1)^2 + (|z| - 8)^2$

$|z|^2 = |z|^2 - 2|z| + 1 + |z|^2 - 16|z| + 64$

$\therefore |z|^2 - 18|z| + 65 = 0$

$(|z| - 5)(|z) - 13) = 0$

$|z| = 5$ or $|z| = 13$

$\therefore z = 4 - 3i$ or $z = 12 + 5i$

Question 2

(a) 1 & 4 ✓✓

(b) (i) If I am super-cool then I study Extension 2 Mathematics ✓

(b) (ii) I study Extension 2 Mathematics and I am not super-cool ✓

(b) (iii) If I am not super-cool then I do not study Extension 2 Mathematics ✓

(c) a^2-1 is divisible by 3 $\Rightarrow$ a is not divisible by 3

take the contrapositive:

a is divisible by 3 $\Rightarrow$ a^2-1 is not divisible by 3

let $a = 3k$, $k \in \mathbb{Z}$

$$\begin{aligned} \therefore\ a^2-1 &= (3k)^2-1 \\ &= 9k^2-1 \\ &= 3(3k)-1 \end{aligned}$$

not divisible by 3

$\therefore$ a^2-1 is divisible by 3 $\Rightarrow$ a is not divisible by 3 is true by contrapositive proof ✓

a is not divisible by 3 $\Rightarrow$ a^2-1 is divisible by 3

let $a = 3k \pm 1$ $k \in \mathbb{Z}$

$$\begin{aligned} a^2-1 &= (3k\pm1)^2-1 \\ &= 9k^2 \pm 6k + \cancel{1} \cancel{-1} \\ &= 9k^2 \pm 6k \\ &= 3(3k^2 \pm 2) \end{aligned}$$

divisibly by 3

✓ attempting both ways

$\therefore$ a is not divisible by 3 $\Rightarrow$ a^2-1 is divisible by 3 ✓

$\therefore$ a^2-1 is divisible by 3 $\Leftrightarrow$ a is not divisible by 3.

(d)(i) $a | b \Rightarrow b = pa \quad p \in \mathbb{Z} \quad p \neq 0$

$a | c \Rightarrow c = qa \quad q \in \mathbb{Z} \quad q \neq 0$

$$b - c = pa - qa$$
$$= a(p - q)$$

$\therefore a \mid b - c$

(d)(ii) assume the negation

i.e. assume $n! + 1$ and $(n+1)! + 1$ have a common factor.

by (i) this common factor must be shared by:

$$(n+1)! + 1 - (n! + 1) = (n+1)n! \cancel{+1} - n! \cancel{-1}$$
$$= n!(n \cancel{+1} \cancel{-1})$$
$$= n \times n!$$

but this is a contradiction as $n! + 1$ and $n \times n!$ clearly have no common factors. (other than 1)

$\therefore n! + 1$ and $(n+1)! + 1$ also have no common factors other than 1 and hence are ~~cop~~ relatively prime.

Question 3

(a) (i) $z^3 = 1$

$\therefore \; w^3 = 1$ (w is a solution to the equation)

$w^3 - 1 = 0$

$(w-1)(w^2+w+1) = 0$

$\therefore \; w - 1 = 0$ or $w^2 + w + 1 = 0$

but w is non real $\therefore \; w - 1 \neq 0$

$\therefore \; w^2 + w + 1 = 0$ ✓

(a) (ii) $(1-w)(1-w^2)(1-w^4)(1-w^8)$

$= (1-w)(1-w^2)(1-w)(1-w^2)$ $\begin{pmatrix} w^4 = w \\ w^8 = w^2 \end{pmatrix}$ ✓

$= (1-w)^2(1-w^2)^2$

$= (1-2w+w^2)(1-2w^2+w^4)$

$= (1-2w^2+w^2)(1-2w^2+w)$

$= 1 - 2w^2 + w - 2w + 4w^3 - 2w^2 + w^2 - 2w^4 + w^3$

$= \underline{1} - 2w^2 + \underline{\underline{w}} - \underline{\underline{2w}} + \underline{4} - 2w^2 + w^2 - \underline{\underline{2w}} + \underline{1}$

$= 6 - 3w - 3w^2$

$= 6 - 3(w + w^2)$

$= 6 - 3(-1)$ as $(w^2 + w = -1$ from (i))

$= 9$ ✓

(b)(i) $(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$ (De Muivre's Theorem)

$$(\cos\theta + i\sin\theta)^7 = \binom{7}{0}\cos^7\theta + \binom{7}{1}\cos^6\theta\, i\sin\theta - \binom{7}{2}\cos^5\theta\sin^2\theta + \binom{7}{3}\cos^4\theta\, i\sin^3\theta$$
$$+ \binom{7}{4}\cos^3\theta\sin^4\theta + \binom{7}{5}\cos^2\theta\, i\sin^5\theta - \binom{7}{6}\cos\theta\sin^6\theta + \binom{7}{7} i\sin^7\theta$$

equate imaginary parts

$$\sin 7\theta = 7\cos^6\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$\therefore \frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$= 7(1-\sin^2\theta)^3 - 35\sin^2\theta(1-\sin^2\theta)^2 + 21\sin^4\theta(1-\sin^2\theta) - \sin^6\theta$$
$$= 7(1 - 3\sin^2\theta + 3\sin^4\theta + \sin^6\theta) - 35\sin^2\theta(1 - 2\sin^2\theta + \sin^4\theta)$$
$$+ 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta + 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta$$
$$+ 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

(b)(ii) Letting $\sin 7\theta = 0$ and $x = \sin^2\theta$ where $\sin\theta \neq 0$

$$0 = 7 - 56x + 112x^2 - 64x^2$$
$$\therefore 64x^3 - 112x^2 + 56x - 7 = 0$$

as $\sin 7\theta = 0$

$7\theta = k\pi \qquad -7 < k \leq 7$

$\theta = k\pi/7 \qquad (k \neq 0$ as $\sin\theta \neq 0)$

$\therefore \theta = \pm\pi/7, \pm 2\pi/7, \pm 3\pi/7$ (other values of k will give duplicates)

$\therefore x = \sin^2\frac{\pi}{7}, \sin^2\frac{2\pi}{7}$ or $\sin^2\frac{3\pi}{7}$ (negative values of sin become positive) (b)(iii) $\operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7}$

$$= \frac{1}{\sin^2 \pi/7} + \frac{1}{\sin^2 2\pi/7} + \frac{1}{\sin^2 3\pi/7}$$

$$= \frac{\sin^2 \pi/7 \sin^2 2\pi/7 + \sin^2 \pi/7 \sin^2 3\pi/7 + \sin^2 2\pi/7 \sin^2 3\pi/7}{\sin^2 \pi/7 \times \sin^2 2\pi/7 \times \sin^2 3\pi/7}$$

$$= \frac{56/64}{-(-7)/64}$$

$$= 8$$

(c) Proof by contradiction.

assume $\exists$ n & m such that $m, n \in \mathbb{Z}$

let $z = r\operatorname{cis}\theta$

$\therefore z^n = r^n \operatorname{cis} n\theta$

$z^m = r^m \operatorname{cis} m\theta$

$\therefore r^n \operatorname{cis} n\theta = 1+i$ $\quad$ $r^m \operatorname{cis} m\theta = 2-i$

$\therefore r^n = \sqrt{2}$ $\quad$ $r^m = \sqrt{5}$

$r^n = 2^{1/2}$ $\quad$ $r^m = 5^{1/2}$

$r = 2^{1/2n}$ $\quad$ $r = 5^{1/2m}$

$\therefore 2^{1/2n} = 5^{1/2m}$ $\leftarrow$ both sides to power of $2mn$

$\therefore 2^m = 5^n$

contradiction

$\therefore$ no such integers exist.