NORTH SYDNEY BOYS HIGH SCHOOL

2022 YEAR 12 ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- **Working time – 60 minutes**
- **Reading time – 5 minutes**
- Write on the lined paper in the booklet provided and label the question
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Use the Reference Sheet

Class Teacher: Please tick the circle.

- O Mr Berry
- O Ms Lee
- O Mr Umakanthan

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL
$\overline{9}$	$\overline{9}$	$\overline{9}$	$\overline{27}$

Question 1 (9 Marks) — Marks

a) Use the principle of mathematical induction to prove that: **3**

$$3^n > 1 + 2n \text{ for } n > 1$$

b) Prove by induction that $3^{2n+4} - 2^{2n}$ is divisible by 5 for all integers $n \geq 1$. **3**

c) A sequence $U_1, U_2, \ldots\ldots$ is defined by $U_1 = 1, U_2 = 5$ and $U_n = 5U_{n-1} - 6U_{n-2}$ for $n \geq 2$. **3**

Prove by mathematical induction, or otherwise, that $U_n = 3^n - 2^n$.

Question 2 (9 Marks) — Marks

a) For a, an integer when $a > 1$, show that:

i) $\frac{1}{a^2} < \frac{1}{a(a-1)}$. **1**

ii) Show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2} < 2$. **2**

b) A particle P of mass m kg is released from rest and falls under its own weight. The particle is subject to air resistance of magnitude kmv^2 N, where v ms^{-1} is the speed of the particle after it has travelled x m and k is a positive constant.

i) Show that $v^2 = \left(1 - e^{-2kx}\right)\frac{g}{k}$. **3**

ii) Hence determine the terminal velocity, V_T, of the particle. **1**

iii) Show that the particle attains a speed of $\frac{1}{2}V_T$ after it has fallen a distance of **2**

$\frac{1}{2k}\ln\left(\frac{4}{3}\right)$ metres.

Question 3 (9 Marks) **Marks**

A particle of mass 5 kg experiences a resistive force, in Newtons, equivalent to 20% of the square of its velocity (v metres per second) when it moves through the air. The particle is projected vertically upwards from a point P with velocity u metres per second. The highest point reached is Q, directly above P.
Take $g = 10 \text{ m/s}^2$, and upwards as the positive direction:

i) Show that the acceleration of the particle as it rises is given by $\ddot{x} = -\frac{250 + v^2}{25}$. **1**

ii) Show that the distance x metres of the particle from P as it rises is given by **2**

$$x = \frac{25}{2}\log_e\left(\frac{250 + u^2}{250 + v^2}\right).$$

iii) Show that the time t seconds that the particle takes to reach a velocity of v metres per second is given by **2**

$$t = \frac{\sqrt{10}}{2}\left(\tan^{-1}\left(\frac{u}{5\sqrt{10}}\right) - \tan^{-1}\left(\frac{v}{5\sqrt{10}}\right)\right).$$

iv) Now suppose that we take two of these 5kg particles described above. **4**
One of these particles is projected upwards from P with initial velocity $5\sqrt{10}$ m/s, then 1 second later, the other particle is projected upwards from P with an initial velocity $20\sqrt{10}$ m/s.
Will the second particle catch up to the first particle before the first particle reaches its maximum height? Justify your answer with working.

End of Examination

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Question 1 (9 Marks)

a) Step 1: Prove true for $n = 2$ 3

$$3^2 > 1 + 2 \times 2$$
$$9 > 5$$

$\therefore$ true for $n = 2$

Step 2: Assume true for $n = k$

$$3^k > 1 + 3k \quad (1)$$

Step 3: Prove true for $n = k + 1$

$$3^{k+1} > 1 + 3(k+1)$$

$$\begin{aligned}
\text{LHS} &= 3^{k+1} \\
&= 3 \times 3^k \\
&> 3(1 + 3k) \quad \text{From } (1) \\
&> 3(1 + k + 2k) \\
&> 3 \times 2k + 3(k+1) \quad \text{also } 3 \times 2k > 1 \\
&> 1 + 3(k+1) \\
&= \text{RHS}
\end{aligned}$$

$\therefore$ true for $n = k + 1$

$\therefore$ by the principle of mathematical induction, the statement is true for all integer $n \geq 2$

b) Step 1: Prove true for $n = 1$ 3

$$3^{2\times1+4} - 2^{2\times1} = 3^6 - 2^2 = 725$$

which is divisible by 5.

$\therefore$ true for $n = 1$

Step 2: Assume true for $n = k$ (where k is a positive integer)

$$3^{2k+4} - 2^{2k} = 5\text{P} \quad (1)$$

where P is an integer

Step 3: Prove the result true for $n = k + 1$

$3^{2(k+1)+4} - 2^{2(k+1)} = 5Q$ where Q is an integer

$$\begin{aligned}
\text{LHS} &= 3^{2k+6} - 2^{2k+2} \\
&= 9 \times 3^{2k+4} - 4 \times 2^{2k} \\
&= 9 \times 3^{2k+4} - 9 \times 2^{2k} + 5 \times 2^{2k} \\
&= 9(3^{2k+4} - 2^{2k}) + 5 \times 2^{2k} \\
&= 9(5P) + 5 \times 2^{2k} \text{ from } (1) \\
&= 5(9P) + 5 \times 2^{2k} \\
&= 5(9P + 2^{2k}) \\
&= 5Q \\
&= \text{RHS}
\end{aligned}$$

$\therefore$ true for $n = k + 1$

$\therefore$ by the principle of mathematical induction, the statement is true for all integer $n \geq 1$

c) If $n = 1,\ U_1 = 3^1 - 2^1 = 1$ 3

If $n = 2,\ U_2 = 3^2 - 2^2 = 5$

$\therefore$ result true for $n = 1, 2$.

Assume result is true for n = k and n = k + 1 (for $k > 1$)

$U_k = 3^k - 2^k$

$U_{k-1} = 3^{k-1} - 2^{k-1}$

Prove for n = k + 1

$$
\begin{aligned}
U_{k+1} &= 5U_k - 6U_{k-1} \\
&= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\
&= 5.3^k - 5.2^k - 6.3^{k-1} + 6.2^{k-1} \\
&= 5.3^k - 5.2^k - 2.3^k + 3.2^k \\
&= 3.3^k - 2.2^k \\
&= 3^{k+1} - 2^{k+1}
\end{aligned}
$$

∴ true for $n = k + 1$

∴ by the principle of mathematical induction, the statement is true for all integer $n \geq 2$

Question 2 (9 Marks)

ai) If $a > 1$ then $a > a - 1$ 1

$$\text{then} \quad a \times a > a(a-1)$$

$$\text{i.e.} \quad a^2 > a(a-1)$$

$$\therefore \; \frac{1}{a^2} < \frac{1}{a(a-1)}$$

aii) Also $\frac{1}{a(a-1)} = \frac{1}{a-1} - \frac{1}{a}$ 2

Hence $\frac{1}{a^2} < \frac{1}{a-1} - \frac{1}{a}$

$$\text{So:} \; \frac{1}{2^2} < \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}$$

$$\frac{1}{4^2} < \frac{1}{3} - \frac{1}{4}$$

And so on.

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}$$

$$< 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots\ldots\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$< 2 - \frac{1}{n} \qquad \text{as } n \to \infty, \frac{1}{n} \to 0$$

$$\therefore \; \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2} < 2$$

bi) 3

$$m\ddot{x} = mg - mkv^2$$

$$\therefore \quad \ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + c$$

$$x = 0, v = 0 \Rightarrow \qquad 0 = -\frac{1}{2k}\ln g + c$$

$$\therefore \quad x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln\left(g - kv^2\right) = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$\text{Finding } v^2 \qquad e^{2kx} = \frac{g}{g - kv^2}$$

$$g - kv^2 = ge^{-2kx}$$

$$\therefore \quad v^2 = \left(1 - e^{-2kx}\right)\frac{g}{k}$$

ii) $$\text{As } x \to \infty, v^2 \to \frac{g}{k} \quad \therefore V_T = \sqrt{\frac{g}{k}}$$ 1

iii) $$\text{When } v = \frac{1}{2}V_T = \frac{1}{2}\sqrt{\frac{g}{k}}: \quad \frac{g}{4k} = \left(1 - e^{-2kx}\right)\frac{g}{k}$$ 2

$$\frac{1}{4} = 1 - e^{-2kx}$$

$$e^{-2kx} = \frac{3}{4}$$

$$\Rightarrow \quad e^{2kx} = \frac{4}{3}$$

$$\Rightarrow \quad x = \frac{1}{2k}\ln\left(\frac{4}{3}\right)$$

Question 3 (9 Marks)

i) $F = -mg - 20\%$ of v^2 1

$$m\ddot{x} = -5(10) - \frac{v^2}{5}$$

$$5\ddot{x} = -5(10) - \frac{v^2}{5}$$

$$\ddot{x} = -(10) - \frac{v^2}{25}$$

$$= -\frac{250 - v^2}{25}$$

$$= -\left(\frac{250 + v^2}{25}\right)$$

ii) $$v\,\frac{dv}{dx} = -\left(\frac{250 + v^2}{25}\right)$$ 2

$$\frac{dv}{dx} = -\left(\frac{250 + v^2}{25v}\right)$$

$$\frac{dx}{dv} = -\left(\frac{25v}{250 + v^2}\right)$$

$$x = -\frac{25}{2}\int \frac{2v}{250 + v^2}\,dv$$

$$= -\frac{25}{2}\ln(250 + v^2) + c$$

$$\text{When } x = 0, v = u \qquad \therefore c = \frac{25}{2}\ln(250 + u^2)$$

$$\therefore x = \frac{25}{2}\ln(250 + u^2) - \frac{25}{2}\ln(250 + v^2)$$

$$x = \frac{25}{2}\ln\left(\frac{250 + u^2}{250 + v^2}\right)$$

iii) $$\frac{dv}{dt} = -\left(\frac{250 + v^2}{25}\right)$$ 2

$$\frac{dt}{dv} = -\left(\frac{25}{250 + v^2}\right)$$

$$t = -25\int \frac{1}{250 + v^2}\,dv$$

$$= -25\left(\frac{1}{5\sqrt{10}}\right)tan^{-1}\left(\frac{v}{5\sqrt{10}}\right) + c$$

when $t = 0, v = u$ $so\ c = \left(\frac{5}{\sqrt{10}}\right) tan^{-1}\left(\frac{u}{5\sqrt{10}}\right)$

$$\therefore t = \frac{\sqrt{10}}{2}\left(tan^{-1}\left(\frac{u}{5\sqrt{10}}\right) - tan^{-1}\left(\frac{v}{5\sqrt{10}}\right)\right)$$

iv) Firstly, find the distance AB and the time taken for the first particle. 4

When $u = 5\sqrt{10}$ and $v = 0$,

$$x = \frac{25}{2}\ln\left(\frac{250 + \left(5\sqrt{10}\right)^2}{250 + 0^2}\right)$$

$$x = \frac{25}{2}\ln 2 \text{ metres}$$

When $u = 5\sqrt{10}$ and $v = 0$

$$t = \frac{\sqrt{10}}{2}\left(tan^{-1}\left(\frac{5\sqrt{10}}{5\sqrt{10}}\right) - tan^{-1}\left(\frac{0}{5\sqrt{10}}\right)\right)$$

$$= \frac{\sqrt{10}}{2}\left(tan^{-1}(1) - tan^{-1}(0)\right)$$

$$= \frac{\sqrt{10}}{2} \times \frac{\pi}{4}$$

$$= \frac{\sqrt{10}\pi}{8}$$

Second particle, $u = 20\sqrt{10}$

$$\frac{25}{2}\ln 2 = \frac{25}{2}\ln\left(\frac{250 + \left(20\sqrt{10}\right)^2}{250 + v^2}\right)$$

$$2 = \frac{4\,250}{250 + v^2}$$

$$v^2 + 250 = 2125$$

$$v^2 = 1\,875$$

$$v = 25\sqrt{3}$$

$$t = \frac{\sqrt{10}}{2}\left(tan^{-1}\left(\frac{20\sqrt{10}}{5\sqrt{10}}\right) - tan^{-1}\left(\frac{25\sqrt{3}}{5\sqrt{10}}\right)\right)$$

$$= \frac{\sqrt{10}}{2}\left(tan^{-1}(4) - tan^{-1}\left(\frac{5\sqrt{3}}{\sqrt{10}}\right)\right)$$

$$= 0 \cdot 1662205631$$

Now,

$$\frac{\sqrt{10}}{2}\left(tan^{-1}(4) - tan^{-1}\left(\frac{5\sqrt{3}}{\sqrt{10}}\right)\right) + 1 < \frac{\sqrt{10}\pi}{8}$$

i.e. $1 \cdot 1662205631 < 1 \cdot 241823533$

Second particle reaches Q before the first particle. So the second particle overtakes the first particle while they are rising.