NORTH SYDNEY BOYS HIGH SCHOOL

EXTENSION 2 MATHEMATICS

HSC Course Assessment Task 1

December, 2023

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.

QUESTIONS 1-4

- Commence each new question on a NEW BOOKLET.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Teacher (please ✔)

- ◯ Mr Berry
- ◯ Ms Lee
- ◯ Mr Umakanthan
- ◯ Dr Vranesevic

Marker's use only.

QUESTION	1	2	3	4	**Total**
MARKS	$\overline{12}$	$\overline{10}$	$\overline{12}$	$\overline{13}$	$\overline{47}$

Question 1 (12 Marks) — Marks

a) If $(1+i)z = 3 - i$, find:

		Marks
i)	z in the form $x + iy$	**2**
ii)	$\lvert z \rvert$	**1**
iii)	$Arg\, z$	**1**
iv)	$\overline{z}$	**1**

b)

		Marks
i)	Find $\sqrt{8+6i}$	**2**
ii)	Solve $z^2 + (2+4i) - 11 - 2i = 0$	**2**

c) z is a complex number such that $z = k(\cos\theta + i\sin\theta)$ where $k \in \mathbb{R}$ **3**

Show $\arg(z+k) = \frac{\theta}{2}$

Question 2 (10 Marks) — Marks

a) Consider the statement "for all integers a and b, if $a+b$ is even, then a and b are even"

		Marks
i)	Write the contrapositive of the statement	**1**
ii)	Write the converse of the statement	**1**
iii)	Write the negation of the statement	**1**
iv)	Is the original statement true or false? Prove the statement is true/false	**1**

b) State the negation of **1**

$$(\forall x \in \mathbb{Z})(x = 3 \Rightarrow x^2 = 9)$$

c) Prove that if $n > 0$ and $4^n - 1$ is prime, then n is odd **3**

d) Prove that there are no integers x and y such that $x^2 = 4y + 2$ **2**

Question 3 (12 Marks)	Marks

a)

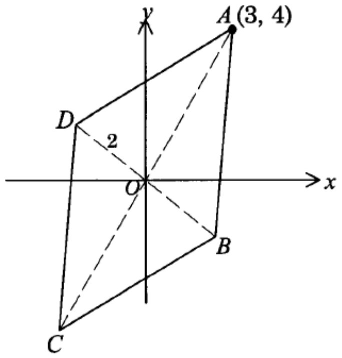

Diagram is not to scale

In the diagram, $ABCD$ is a rhombus whose diagonals meet at O, the origin. A represents the complex number $3 + 4i$ and $OD = 2$ units.
Find the complex number represented by

i) C **1**

ii) D **2**

b) The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

Where p and q are real, has roots α, β and γ.
It is given that $\alpha = 2 + 2\sqrt{3}i$

i) Write down another root, β, of the equation **1**

ii) Find the third root, γ **2**

iii) Find the values of p and q **2**

iv) By expressing α in modulus-argument form, show that **2**

$$\left(2 + 2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$$

v) Hence, show that **2**

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$$

where n is an integer

Question 4 (13 Marks)	Marks

a) Find the five roots of unity, and then express the complex roots in terms of ω, where ω is the complex root with the smallest positive argument. **2**

b) Prove that $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$ **2**

c) Show that $(1-\omega)(1-\omega^4) = 2 - 2\cos\frac{2\pi}{5}$ **2**

d) Hence or otherwise, show that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$ **3**

e) Suppose P_0, P_1, P_2, P_3, P_4 are the corresponding points of $1, \omega, \omega^2, \omega^3$ and ω^4 in the Argand plane

i) Show that $\left|\overrightarrow{P_0P_1}\right| = 2\sin\frac{\pi}{5}$ **2**

ii) Hence find the value of **2**

$$\left|\overrightarrow{P_0P_1}\right| \times \left|\overrightarrow{P_0P_2}\right| \times \left|\overrightarrow{P_0P_3}\right| \times \left|\overrightarrow{P_0P_4}\right|$$

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Q1

a) If $(1+i)z = 3-i$ find :

i) z in the form $x+iy$

$$z = \frac{3-i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-i-3i-1}{1+1}$$

$$= \frac{2-4i}{2}$$

$$= 1-2i \quad ✓$$

alternately:

$(1+i)z = 3-i$

$z + iz = 3-i$

where $z = x+iy$

$x+iy+ix-y = 3-i$

$x-y = 3$

$y+x = -1$

$\therefore x = 1, y = -2$

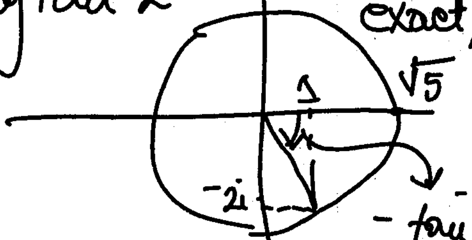
ii) $|z| = \sqrt{1^2+2^2}$

$= \sqrt{5}$ ✓

iii) $\operatorname{Arg} z = \arg\tan \frac{-2}{1} = \arg\tan(-2)$

$= -\arg\tan 2$ ✓

please use exact value



iv) $\bar{z} = \overline{(1-2i)}$

$= 1+2i$ ✓

CFE - carried forward error considered

b)

i) Find $\sqrt{8+6i}$

$\sqrt{8+6i} = x+iy \quad)^2$

$8+6i = x^2-y^2+2xyi$

① $x^2-y^2=8$

② $2xy = 6 \Rightarrow x = \frac{3}{y}$

② → ① $\Rightarrow \frac{9}{y^2} - y^2 = 8$

$9 - y^4 - 8y^2 = 0$

$y^4+8y^2-9=0$

$(y^2-1)(y^2+9)=0$

$y = \pm 1, \; y = \pm 3i \notin \mathbb{R}$

$\therefore y = \pm 1 \Rightarrow x = \pm 3$

$\therefore \pm(3+i)$ ✓✓

Alternately:

① $x^2-y^2=8$

② $x^2+y^2 = \sqrt{8^2+6^2}$
$= 10$

①+②:

$2x^2 = 18$

$x = \pm 3$

② − ①

$2y^2 = 2$

$y = \pm 1$

$\therefore$

$\sqrt{8+6i} = \pm(3+i)$

ii) Solve $z^2+(2+4i)z-11-2i=0$

$$\Delta = b^2-4ac$$
$$= (2+4i)^2 + 4\times 1\times(11+2i)$$
$$= 4+16i-16+44+8i$$
$$= 32+24i$$
$$= 4(8+6i)$$

$$z = \frac{-b\pm\sqrt{\Delta}}{2a}$$
$$= \frac{-(2+4i)\pm\sqrt{4(8+6i)}}{2\times 1}$$
$$= \frac{-2-4i\pm 2(3+i)}{2} \quad \text{from(i)}$$
$$= -1-2i\pm(3+i)$$
$$= 2-i \quad \text{or} \quad -4-3i$$

✓ ✓

c) z is complex number such as $z=k(\cos\theta+i\sin\theta)$, $k\in R$. Show $\arg(z+k)=\frac{\theta}{2}$

Solution 1:

$z=k(\cos\theta+i\sin\theta)$, $k\in R$

$$z+k = k(\cos\theta+i\sin\theta)+k$$
$$= k(\cos\theta+1)+ik\sin\theta$$

$$\arg(z+k) = \tan^{-1}\left(\frac{\cancel{k}\sin\theta}{\cancel{k}(\cos\theta+1)}\right)$$
$$= \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$
$$= \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-1+1}\right)$$
$$= \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$
$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$
$$= \frac{\theta}{2}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$\cos\theta = \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$$
$$= \cos^2\frac{\theta}{2}-1+\cos^2\frac{\theta}{2}$$
$$= 2\cos^2\frac{\theta}{2}-1$$

Solution 2:

$$\arg(z+k) = \arg\left(k\left(\cos\theta + i\sin\theta + 1\right)\right)$$

$$= \arg\left(\cos\theta + i\sin\theta + 1\right)$$

$$= \arg\left(e^{i\theta} + 1\right)$$

$$= \arg\left(e^{i\theta/2}\left(e^{i\theta/2} + e^{-i\frac{\theta}{2}}\right)\right)$$

$$= \arg\left(e^{i\theta/2} \times \underbrace{2\cos\tfrac{\theta}{2}}_{\text{real part}}\right)$$

$$= \arg e^{i\theta/2} + \underbrace{\arg\left(2\cos\theta/2\right)}_{=0}$$

$$= \frac{\theta}{2}$$

as for $z = e^{i\theta} \Rightarrow z^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$

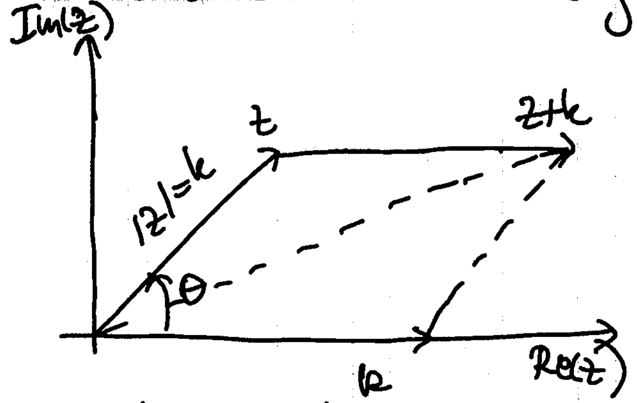
$\bar{z} = e^{-i\theta} \Rightarrow \bar{z}^n = e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$

$$e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$$

in our case $n = \frac{1}{2}$

$$\therefore e^{i\theta/2} + e^{-i\frac{\theta}{2}} = 2\cos\frac{\theta}{2}$$

Solution 3



$\arg z = \theta$

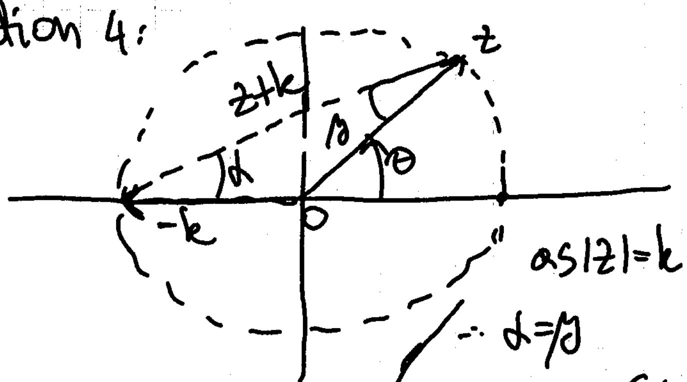
$|z| = k$

$k \in \mathbb{R}$

$z+k$ form a rhombus

$\therefore$ angle in a rhombus is bisected by its diagonals $\therefore \arg(z+k) = \frac{\theta}{2}$

Solution 4:



$|z| = k$

$\arg(z) = \theta$

as $|z| = k$ & $|-k| = k$

$\therefore \alpha = \beta$

$\alpha + \beta = \theta$ Exterior angle theorem

$\therefore 2\alpha = \theta$

$\therefore \alpha = \frac{\theta}{2}$, $\arg(z+k) = \frac{\theta}{2}$

or: from circle geometry angle at the center is double the angle at circumference with the same arc

Question 2

a)

i) For all integers a and b, if a or b is odd, then a+b is odd

ii) For all integers a and b, if a and b are even, then a+b is even

iii) There exists integers a and b such that a + b is even
and a or b is odd

iv) False. Counterexample: $a = 3$ and $b = 5$

$a+b = 8$ but neither a nor b are even.

b) $(\exists x \in \mathbb{Z})(x = 3 \wedge x^2 \neq 9)$

c) The contrapositive is proven instead. That is:

If n is even, then $4^n - 1$ is not prime.

Let $n = 2k$, $k \in \mathbb{Z}$, then n is even and

$$\begin{aligned} 4^n - 1 &= 4^{2k} - 1 \\ &= (4^k + 1)(4^k - 1) \end{aligned}$$

which is a product of two integers of which neither is 1 $(n > 0)$
and is thus not prime.

Hence, by the contrapositive, if $4^n - 1$ is prime, then n is odd.

d) By way of contradiction, assume there are integers x and y
such that $x^2 = 4y + 2$ (*)

Now $4y + 2 = 2(y+1)$ which is even, so x^2 is even.

Let $x = 2k$, $k \in \mathbb{Z}$

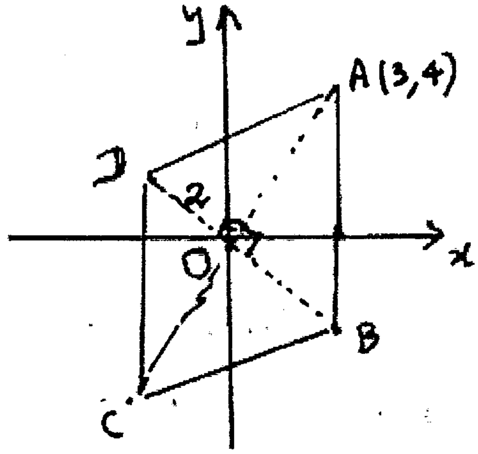
$$\begin{aligned} (2k)^2 &= 4y + 2 \\ 4k^2 &= 4y + 2 \\ 2k^2 &= 2y + 1 \end{aligned}$$

LHS is even and RHS is odd.
Even $\neq$ Odd

Thus we have a contradiction, so there must not be any
integers x and y such that $x^2 = 4y + 2$

$\underline{Q_3}$

(a)



(i) Diagonals bisect in a Rhombus

$\therefore \quad \vec{OC} = -\vec{OA}$
$= -(3+4i)$
$= \underline{\underline{-3-4i}}$

$\vec{OC} = \vec{OA}\, cis\,\pi$
$= -1(3+4i)$
$= \underline{\underline{-3-4i}}$

Diagonals bisect + correct Answer for 1 mark

No reason no mark

(ii) $|\vec{OD}| = 2,\ |\vec{OA}| = \sqrt{3^2+4^2} = 5$

$\Rightarrow \vec{OD} = \frac{2}{5}\vec{OA}\, cis\frac{\pi}{2}$ ✓ (diagonals $\perp$ar in a rhombus)

$= \frac{2}{5}(3+4i)i$

$= \underline{\underline{-\frac{8}{5} + \frac{6i}{5}}}$ ✓

(b) $2z^3 + pz^2 + qz + 16 = 0 \quad p, q \in \mathbb{R}$

roots α, β and γ

(i) $\alpha = 2 + 2\sqrt{3}i$

Since the coefficients are all real,

the complex conjugate of α is also a root.

let $\beta = \bar{\alpha} = \underline{2 - 2\sqrt{3}i}$ ✓ (correct answer)

(Reason: given the benefit of doubt)

(ii) Product of roots $\alpha\beta\gamma = -\frac{16}{2} = -8$ ✓

$\Rightarrow \alpha\bar{\alpha}\gamma = -8$

$\Rightarrow [2^2 + (2\sqrt{3})^2]\gamma = -8$

$\Rightarrow \gamma = -\frac{8}{16} = \underline{-\frac{1}{2}}$ ✓

(iii) sum of roots $\alpha + \beta + \gamma = -\frac{p}{2}$

$\alpha + \bar{\alpha} + \gamma = -\frac{p}{2}$

$2(2) + -\frac{1}{2} = -\frac{p}{2}$

$\Rightarrow \underline{p = -7}$ ✓

sum of product of 2 roots at a time $= \frac{q}{2}$

$\Rightarrow \alpha\bar{\alpha} + \bar{\alpha}\gamma + \gamma\alpha = \frac{q}{2}$

$16 + (4)(-\frac{1}{2}) = \frac{q}{2}$

$\Rightarrow \underline{q = 28}$ ✓ ← (or by factorising)

(iv) $\alpha = 2+2\sqrt{3}i$

$= 4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$= 4\,\text{cis}\left(\frac{\pi}{3}\right)$

$\alpha^n = 4^n\,\text{cis}\left(\frac{n\pi}{3}\right)$ (de Moivré's theorem.)

$\therefore \beta^n = (\bar{\alpha})^n = 4^n\,\text{cis}\left(-\frac{n\pi}{3}\right)$

(v) $\Longrightarrow \alpha^n + \beta^n + \gamma^n = 4^n\,\text{cis}\left(\frac{n\pi}{3}\right) + 4^n\,\text{cis}\left(-\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$

$= 4^n\left[2\cos\left(\frac{n\pi}{3}\right)\right] + \left(-\frac{1}{2}\right)^n$

$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$

Question 4

a) $z^5 = 1$

$z_1 = 1$

$z_2 = cis\,\frac{2\pi}{5} = \omega$

$z_3 = cis\,\frac{4\pi}{5} = \omega^2$

$z_4 = cis\,\frac{6\pi}{5} = cis\left(-\frac{4\pi}{5}\right) = \omega^3$

$z_5 = cis\,\frac{8\pi}{5} = cis\left(-\frac{2\pi}{5}\right) = \omega^4$

b) M1 $z^5 - 1 = (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ ①

$= (z-1)(z^4+z^3+z^2+z+1)$ ②

Equating ① & ②,

$$(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4) = z^4+z^3+z^2+z+1$$

Let $z=1$:

$$(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 1+1+1+1+1$$
$$= 5$$

M2

$(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$

$= (1-\omega-\omega^2+\omega^3)(1-\omega^3-\omega^4+\omega^7)$

$= 1-\omega^3-\omega^4+\omega^7-\omega+\omega^4+\omega^5-\omega^8-\omega^2+\omega^5+\omega^6$
$\quad -\omega^9+\omega^3-\omega^6-\omega^7+\omega^{10}$

$= 1-\omega-\omega^2+2-\omega^3-\omega^4+1$ since $\omega^5=1$

$= 4-(\omega-\omega^2-\omega^3-\omega^4)$

$= 4-(-1)$ since $\omega^5 = 1$ as ω is a root of z^5

$= 5$

$\omega^5 - 1 = 0$

$(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4) = 0$

$1+\omega+\omega^2+\omega^3+\omega^4 = 0$ since $\omega \neq 1$

$\omega+\omega^2+\omega^3+\omega^4 = -1$

c) $(1-\omega)(1-\omega^4) = (1-\text{cis}\frac{2\pi}{5})(1-\text{cis}(-\frac{2\pi}{5}))$ from (a)

$= 1-(\text{cis}\frac{2\pi}{5}+\text{cis}(-\frac{2\pi}{5})) + \text{cis}(\frac{2\pi}{5}+(-\frac{2\pi}{5}))$

$= 1-2\cos\frac{2\pi}{5} + \text{cis}\,0$ since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$= 1-2\cos\frac{2\pi}{5}+1$

$= 2-2\cos\frac{2\pi}{5}$

d) Similarly, $(1-\omega^2)(1-\omega^3) = 2-2\cos\frac{4\pi}{5}$

$(1-\omega)(1-\omega^4)(1-\omega^2)(1-\omega^3) = 5$ from (b)

$\therefore\ (2-2\cos\frac{2\pi}{5})(2-2\cos\frac{4\pi}{5}) = 5$

$(2-2(1-2\sin^2\frac{\pi}{5}))(2-2(1-2\sin^2\frac{2\pi}{5})) = 5$ since $\cos 2x = 1-2\sin^2 x$

$4\sin^2\frac{\pi}{5}\times 4\sin^2\frac{2\pi}{5} = 5$

$(\sin\frac{\pi}{5}\sin\frac{2\pi}{5})^2 = \frac{5}{16}$

$\therefore\ \sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$, $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5} > 0$

e) i) M1 $|\overrightarrow{P_0P_1}| = |\omega - 1|$

$= |\text{cis}\frac{2\pi}{5} - \text{cis}\,0|$

$= |\text{cis}\frac{\pi}{5}(\text{cis}\frac{\pi}{5} - \text{cis}(-\frac{\pi}{5}))|$

$= |\text{cis}\frac{\pi}{5}(2i\sin\frac{\pi}{5})|$

$= |\text{cis}\frac{\pi}{5}||2i\sin\frac{\pi}{5}|$

$= 1\times 2\sin\frac{\pi}{5}$

M2 Using the cosine rule:

$|\overrightarrow{P_0P_1}| = \sqrt{1^2+1^2-2(1)(1)\cos\frac{2\pi}{5}}$

$= \sqrt{2-2\cos\frac{2\pi}{5}}$

$= \sqrt{2-2(1-2\sin^2\frac{\pi}{5})}$ since $\cos 2x = 1-2\sin^2 x$

$= \sqrt{4\sin^2\frac{\pi}{5}}$

$= 2\sin\frac{\pi}{5}$

ii) Similarly, $|\overrightarrow{P_0P_2}| = 2\sin\left(\frac{4\pi}{5} \times \frac{1}{2}\right) = 2\sin\frac{2\pi}{5}$

$|\overrightarrow{P_0P_3}| = \left|2\sin\left(-\frac{4\pi}{5} \times \frac{1}{2}\right)\right| = 2\sin\frac{2\pi}{5}$

$|\overrightarrow{P_0P_4}| = \left|2\sin\left(-\frac{2\pi}{5} \times \frac{1}{2}\right)\right| = 2\sin\frac{\pi}{5}$

$|\overrightarrow{P_0P_1}| \times |\overrightarrow{P_0P_2}| \times |\overrightarrow{P_0P_3}| \times |\overrightarrow{P_0P_4}|$

$= \left(2\sin\frac{2\pi}{5} \times 2\sin\frac{\pi}{5}\right)^2$

$= 16 \times \left(\frac{\sqrt{5}}{4}\right)^2$

$= 5$