NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2020

HSC
Assessment
Task 1

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 63 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Section II, show relevant mathematical reasoning and/or calculations

Total marks – 38

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 33 marks (separate booklets)

- Attempt Questions 6 – 14
- Allow about 55 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER

Question	1 – 5	A	B	C	Total
Complex Numbers	/5	/12	/10	/11	/38

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

---

1 What is the value of $i + i^2 + i^3 + ... + i^{2019}$?

A. i

B. $-i$

C. 1

D. -1

2 Multiplying a non-zero complex number by $\dfrac{1-i}{1+i}$ results in a rotation about the origin. What is the rotation?

A. Clockwise by $\dfrac{\pi}{2}$ radians

B. Clockwise by $\dfrac{\pi}{4}$ radians

C. Anticlockwise by $\dfrac{\pi}{2}$ radians

D. Anticlockwise by $\dfrac{\pi}{4}$ radians

3 You are given that $3+i$ is a zero of the polynomial $P(z) = z^3 + az^2 + bz + 10$, where a and b are real. Which of the following is the correct factorisation for $P(z)$?

A. $P(z) = (z-1)(z^2 + 6z - 10)$

B. $P(z) = (z-1)(z^2 - 6z - 10)$

C. $P(z) = (z+1)(z^2 + 6z + 10)$

D. $P(z) = (z+1)(z^2 - 6z + 10)$

4 If z is any complex number, which of the following is equal to $z\overline{z} - |z|^2$?

A. 1

B. 0

C. $(z+\overline{z})(z-\overline{z})$

D. $(z+iz)(z-iz)$

5 The relation $(z+2)(\overline{z}+2) = 4$, when graphed on an Argand diagram, would be a:

A. circle of radius 4 with centre at $(-2,0)$

B. circle of radius 2 with centre at $(-2,0)$

C. circle of radius 4 with centre at $(2,0)$

D. circle of radius 2 with centre at $(2,0)$

See Instructions for Section II on back page

Mathematics Extension 2
Section II

33 marks

Attempt Questions 6 – 14

Allow about 55 minutes for this section

Instructions

- Section II is divided into three parts, A, B and C.
  Each part is in a different booklet.

- Answer the questions in the space provided in the booklets.
  Sufficient space is provided for typical responses.

- Your responses should include relevant mathematical reasoning and/or calculations.

- Extra writing space is provided at the back of this booklet. If you use this space, clearly indicate which question you are answering. Insert this sheet into the appropriate booklet.

Section II Answer Booklet

Part A: 12 marks

STUDENT NUMBER

Attempt Questions 6 – 9

Question 6 (2 marks)

Given $z = 3 + i$ and $w = 1 - 2i$, express $\dfrac{5z}{w}$ in the form $a + ib$.

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Question 7 (3 marks)

(i) Write $\sqrt{3}+i$ in polar (modulus-argument) form. **2**

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(ii) Find the smallest integer n such that $\left(\sqrt{3}+i\right)^n$ is a real number. **1**

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Question 8 (3 marks)

If $z = 1 - i$, find the values of u such that $u^2 = 1 + 3\overline{z}$.

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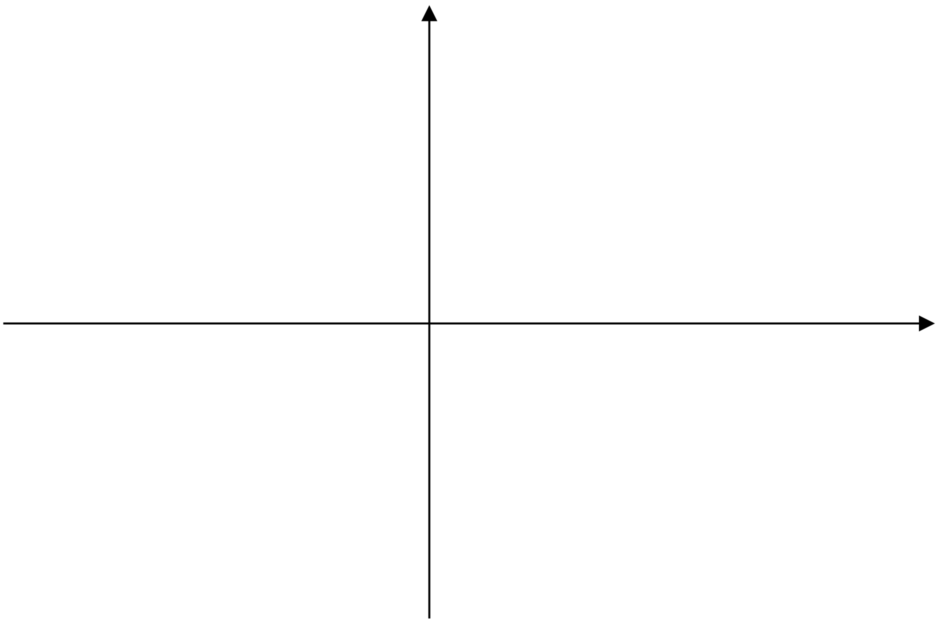
Question 9 (4 marks)

(i) Sketch the locus in the Argand plane defined by: **2**

$$|z-3| = 3 \quad \text{and} \quad \operatorname{Arg} z > 0$$



(ii) z is a complex number which lies on this locus. Let $\operatorname{Arg} z = \theta$. **2**

Find $\arg\left(z^2 - 9z + 18\right)$ in terms of θ.

End of Part A

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Section II Answer Booklet

STUDENT NUMBER

Part B: 10 marks

Attempt Questions 10 – 12

Question 10 (3 marks)

The polynomial $P(x)=x^4+3x^3-x^2-13x-10$ has a zero at $x=-2-i$.

(i) Explain why $x=-2+i$ is also a zero. **1**

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(ii) Fully factorise $P(x)$ over the real numbers. **2**

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Question 11 (2 marks)

ω is a non-real root of the equation $z^3 = 1$.

Find the value of $(1+\omega)^3$.

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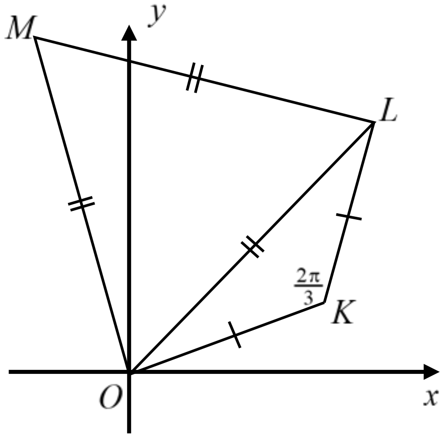
Question 12 (5 marks)

On the Argand diagram below, the complex numbers z and w are represented by the points K and M respectively. It is given that ΔOKL is isosceles with $\angle OKL = \frac{2\pi}{3}$, and ΔOLM is equilateral.



(i) Show that $\angle MOK = \frac{\pi}{2}$. **1**

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(ii) Show that $\left|\overrightarrow{OL}\right| = \sqrt{3}\,|z|$. **2**

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Question 11 continues on next page

(iii) Hence show that $3z^2 + w^2 = 0$. **2**

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End of Part B

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Section II Answer Booklet

Part C: 11 marks

STUDENT NUMBER

Attempt Questions 13 – 14

Question 13 (5 marks)

(i) Find the five 5th roots of $32i$. **2**

Leave your answers in modulus argument form.

It is acceptable to use cis notation.

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Question 13 continues on next page

(ii) It can be shown that $z^5 - 32i = (z-2i)\left(z^4 + 2iz^3 - 4z^2 - 8iz + 16\right)$.
DO NOT PROVE THIS.

Use this result and part (i) to find the value of $\sin\frac{3\pi}{10} - \sin\frac{\pi}{10}$. **3**

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Question 14 (6 marks)

(i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **1**

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(ii) Let $\omega = z + \dfrac{1}{z}$. Show that: **2**

$$\omega^3 + \omega^2 - 2\omega - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

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Question 14 continues on next page

(iii) Hence or otherwise find all solutions in the interval $[0, 2\pi]$ of: **3**

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0.$$

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End of examination

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Section II extra writing space
If you use this space, clearly indicate which question you are answering.
At the end of the examination, place this sheet inside the appropriate booklet.

(Start writing on other side)

North Sydney Girls High School

Student Number

Name:____________________________ Teacher: _________

Multiple Choice – Preliminary Mathematics Advanced

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

A B D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A D

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer

correct

A D

Start Here

1. A B C D

2. A B C D

3. A B C D

4. A B C D

5. A B C D

MULTIPLE CHOICE

1 What is the value of $i + i^2 + i^3 + \ldots + i^{2019}$?

A. i

B. $-i$

C. 1

D. -1

$i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$

Same for $i^5 + i^6 + i^7 + i^8, \ldots, i^{2013} + i^{2014} + i^{2015} + i^{2016}$

$\therefore$ answer is $i^{2017} + i^{2018} + i^{2019} = i + i^2 + i^3$

$= -1$

2 Multiplying a non-zero complex number by $\dfrac{1-i}{1+i}$ results in a rotation about the origin.

What is the rotation?

A. Clockwise by $\dfrac{\pi}{2}$ radians

B. Clockwise by $\dfrac{\pi}{4}$ radians

C. Anticlockwise by $\dfrac{\pi}{2}$ radians

D. Anticlockwise by $\dfrac{\pi}{4}$ radians

$\arg \dfrac{1-i}{1+i} = \arg(1-i) - \arg(1+i)$

$= -\dfrac{\pi}{4} - \dfrac{\pi}{4}$

$= -\dfrac{\pi}{2}$

3 You are given that $3+i$ is a zero of the polynomial $P(z) = z^3 + az^2 + bz + 10$, where a and b are real. Which of the following is the correct factorisation for $P(z)$?

A. $P(z) = (z-1)(z^2 + 6z - 10)$

B. $P(z) = (z-1)(z^2 - 6z - 10)$

C. $P(z) = (z+1)(z^2 + 6z + 10)$

D. $P(z) = (z+1)(z^2 - 6z + 10)$

roots: $3+i, 3-i, \alpha$

Product: $(3+i)(3-i)\alpha = -10$

$10\alpha = -10$

$\alpha = -1$

Product of complex roots: $(3+i)(3-i) = 10$

4 If z is any complex number, which of the following is equal to $z\bar{z}-|z|^2$?

A. 1

B. 0

C. $(z+\bar{z})(z-\bar{z})$

D. $(z+iz)(z-iz)$

$z\bar{z} = |z|^2$

5 The relation $(z+2)(\bar{z}+2)=4$, when graphed on an Argand diagram, would be a:

A. circle of radius 4 with centre at $(-2,0)$

B. circle of radius 2 with centre at $(-2,0)$

C. circle of radius 4 with centre at $(2,0)$

D. circle of radius 2 with centre at $(2,0)$

$z\bar{z} + 2(z+\bar{z}) + 4 = 4$

$x^2+y^2 + 4x = 0$

$(x+2)^2 + y^2 = 4$

Question 6 (2 marks)

Given $z = 3 + i$ and $w = 1 - 2i$, express $\frac{5z}{w}$ in the form $a + ib$.

$$\frac{5z}{w} = \frac{5(3+i)}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{5(3+6i+i-2)}{1+4}$$

$$= 1 + 7i$$

Question 7 (3 marks)

(i) Write $\sqrt{3}+i$ in polar (modulus-argument) form. **2**

$$|\sqrt{3}+i|^2 = (\sqrt{3})^2 + i^2$$
$$= 4$$
$$|\sqrt{3}+i| = 2$$

$$\tan\left[\arg(\sqrt{3}+i)\right] = \frac{1}{\sqrt{3}}$$
$$\arg(\sqrt{3}+i) = \frac{\pi}{6} \quad \text{(1st quadrant)}$$

$$\therefore \sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

(ii) Find the smallest positive integer n such that $\left(\sqrt{3}+i\right)^n$ is a real number. **1**

Need: $\arg(\sqrt{3}+i)^n = k\pi$ (k an integer)

$$n \cdot \arg(\sqrt{3}+i) = k\pi$$

$$n \cdot \frac{\pi}{6} = k\pi$$

Smallest +ve integer n : 6

Question 8 (3 marks)

If $z = 1 - i$, find the values of u such that $u^2 = 1 + 3\bar{z}$.

$u^2 = 1 + 3(1+i)$

$= 4 + 3i$

Let $u = a + ib$: $(a+ib)^2 = 4 + 3i$

$(a^2 - b^2) + 2abi = 4 + 3i$

Equating real & imaginary parts:

$a^2 - b^2 = 4$ - ① $2ab = 3$ - ②

$b = \frac{3}{2a}$

$a^2 - \left(\frac{3}{2a}\right)^2 = 4$

$4a^4 - 16a^2 - 9 = 0$

$(2a^2 + 1)(2a^2 - 9) = 0$

$a^2 = \frac{9}{2}$ (a is real)

$a = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

$2 \cdot \pm\frac{3\sqrt{2}}{2} \cdot b = 3$

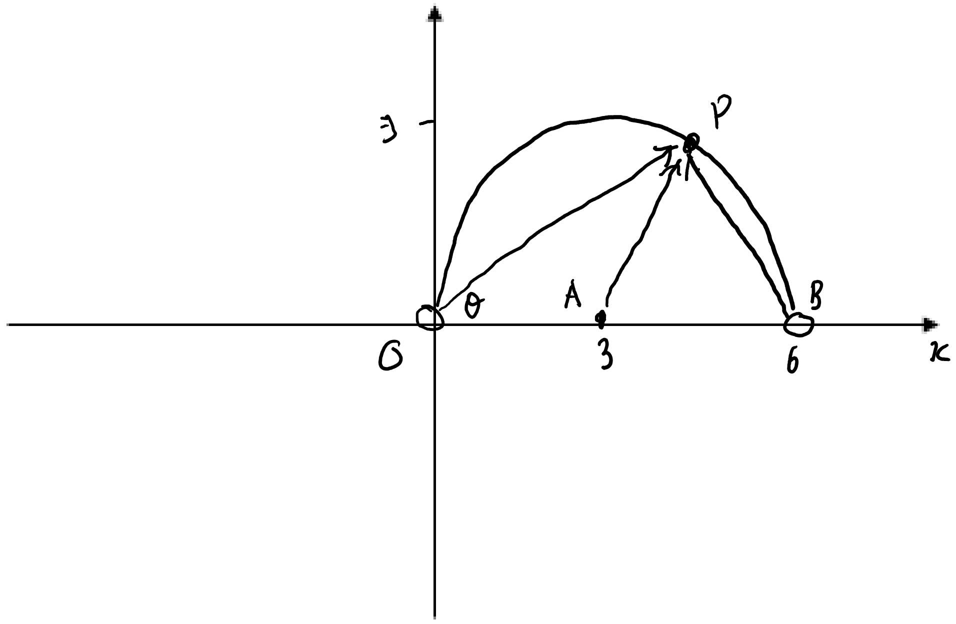
$b = \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\therefore u = \pm\frac{\sqrt{2}}{2}(3 + i)$

Question 9 (4 marks)

(i) Sketch the locus in the Argand plane defined by: **2**

$$|z-3| = 3 \quad \text{and} \quad \operatorname{Arg} z > 0$$



(ii) z is a complex number which lies on this locus. Let $\operatorname{Arg} z = \theta$. **2**

Find $\arg\left(z^2 - 9z + 18\right)$ in terms of θ.

$\operatorname{Arg}(z^2 - 9z + 18) = \operatorname{Arg}(z-6)(z-3)$

$= \operatorname{Arg}(z-6) + \operatorname{Arg}(z-3)$

$= x\hat{B}P + B\hat{A}P$

$= (\pi - A\hat{B}P) + (A\hat{O}P + A\hat{P}O)$

(angles on straight line + exterior angle of triangle)

$= \pi - (\frac{\pi}{2} - \theta) + (\theta + \theta)$

(angle sum of right-angled Δ + isosceles Δ)

$= 3\theta + \frac{\pi}{2}$

Part B: 10 marks

STUDENT NUMBER

Attempt Questions 10 – 12

Question 10 (3 marks)

The polynomial $P(x) = x^4 + 3x^3 - x^2 - 13x - 10$ has a zero at $x = -2 - i$.

(i) Explain why $x = -2 + i$ is also a zero. **1**

Non-real roots of real polynomials occur in conjugate pairs.

(ii) Fully factorise $P(x)$ over the real numbers. **2**

Let roots be $-2-i, -2+i, \alpha, \beta$

Sum of roots : $\alpha + \beta - 4 = -3$

$\alpha + \beta = 1$

Product of roots : $(-2-i)(-2+i) \cdot \alpha\beta = -10$

$5\alpha\beta = -10$

$\alpha\beta = -2$

$\therefore$ quadratic with roots α, β is $x^2 - x - 2 = 0$

ie. $(x-2)(x+1) = 0$

Consider only non-real roots : $-2-i$, $-2+i$

Sum of roots : -4

Product of roots : 5

$\therefore$ quadratic with roots $-2 \pm i$ is $x^2 + 4x + 5 = 0$

$\therefore$ $P(x) = (x-2)(x+1)(x^2+4x+5)$ over $\mathbb{R}$

Question 11 (2 marks)

ω is a non-real root of the equation $z^3 = 1$.

Find the value of $(1+\omega)^3$.

As ω is a root, $\omega^3 = 1$

$\omega^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$

$\omega = 1$ or $\omega^2 + \omega + 1 = 0$

but ω is non-real, so $\omega^2 + \omega + 1 = 0$

$1 + \omega = -\omega^2$

$(1+\omega)^3 = (-\omega^2)^3$

$= -\omega^6$

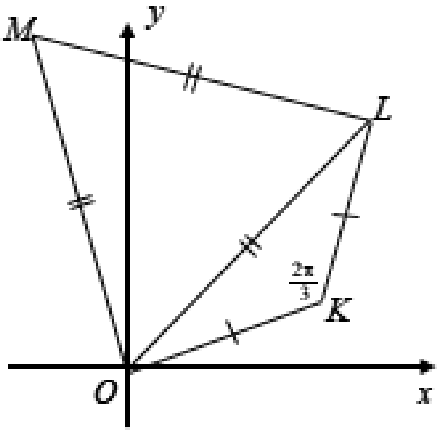
$= -(\omega^3)^2$

$= -1^2$

$= -1$

Question 12 (5 marks)

On the Argand diagram below, the complex numbers z and w are represented by the points K and M respectively. It is given that ΔOKL is isosceles with $\angle OKL = \frac{2\pi}{3}$, and ΔOLM is equilateral.



(i) Show that $\angle MOK = \frac{\pi}{2}$. **1**

$M\hat{O}L = \frac{\pi}{3}$ (angle in equilateral triangle)

$L\hat{O}K + O\hat{L}K + O\hat{K}L = \pi$ (angle sum of triangle)

$2L\hat{O}K + \frac{2\pi}{3} = \pi$ (Δ OKL isosceles)

$2L\hat{O}K = \frac{\pi}{3}$

$L\hat{O}K = \frac{\pi}{6}$

$M\hat{O}K = \frac{\pi}{3} + \frac{\pi}{6}$

$= \frac{\pi}{2}$

(ii) Show that $\left|\overrightarrow{OL}\right| = \sqrt{3}\,|z|$. **2**

$OL^2 = OK^2 + LK^2 - 2 \cdot OK \cdot LK \cdot \cos O\hat{K}L$ (cosine rule)

$= OK^2 + OK^2 - 2 \cdot OK \cdot OK \cdot \cos \frac{2\pi}{3}$ ($OK = LK$, isosceles)

$= 2OK^2 - 2OK^2 \cdot \left(-\frac{1}{2}\right)$

$= 2OK^2 + OK^2$

$= 3OK^2$

$OL = \sqrt{3}\, OK$

ie. $\left|\overrightarrow{OL}\right| = \sqrt{3} \cdot |z|$

(iii) Hence show that $3z^2 + w^2 = 0$. **2**

$w = \sqrt{3}\, i z$ $\left(\begin{array}{l} |w| = OM = OL = \sqrt{3}\, OK = \sqrt{3}\,|z| \\ w \text{ rotated } \frac{\pi}{2} \text{ anti-clockwise from } z \end{array} \right)$

$w^2 = 3 i^2 z^2$

$= -3z^2$

$3z^2 + w^2 = 0$

Part C: 11 marks

STUDENT NUMBER

Attempt Questions 13 – 14

Question 13 (5 marks)

(i) Find the five 5^{th} roots of $32i$. **2**

Leave your answers in modulus argument form.

It is acceptable to use cis notation.

Let $z^5 = 32i$

$|32i| = 32$ $\quad\quad$ $\arg(32i) = \frac{\pi}{2} + 2k\pi$ (k integer)

$\therefore |z| = 32^{1/5} = 2$ $\quad\quad$ $= \frac{4k+1}{2}\pi$

$\therefore \arg z = \frac{4k+1}{10}\pi$

$\therefore z = 2\operatorname{cis}\frac{4k+1}{10}\pi$

Let $k = 0, \pm 1, \pm 2$: $z = 2\operatorname{cis}\left(\frac{-7\pi}{10}\right), 2\operatorname{cis}\left(\frac{-3\pi}{10}\right), 2\operatorname{cis}\frac{\pi}{10}, 2\operatorname{cis}\frac{\pi}{2}, 2\operatorname{cis}\frac{9\pi}{10}$

(ii) It can be shown that $z^5 - 32i = (z-2i)(z^4 + 2iz^3 - 4z^2 - 8iz + 16)$.
DO NOT PROVE THIS.

Use this result and part (i) to find the value of $\sin\frac{3\pi}{10} - \sin\frac{\pi}{10}$. 3

$2\operatorname{cis}\frac{\pi}{2} = 2i$, which is the root of $z - 2i = 0$

$\therefore$ roots of $z^4 + 2iz^3 - 4z^2 - 8iz + 16 = 0$
are $2\operatorname{cis}\left(\frac{-7\pi}{10}\right)$, $2\operatorname{cis}\left(-\frac{3\pi}{10}\right)$, $2\operatorname{cis}\frac{\pi}{10}$, $2\operatorname{cis}\frac{9\pi}{10}$

Sum of roots: $2\operatorname{cis}\left(\frac{-7\pi}{10}\right) + 2\operatorname{cis}\left(-\frac{3\pi}{10}\right) + 2\operatorname{cis}\frac{\pi}{10} + 2\operatorname{cis}\frac{9\pi}{10} = -2i$
Imaginary parts: $2\sin\left(\frac{-7\pi}{10}\right) + 2\sin\left(-\frac{3\pi}{10}\right) + 2\sin\frac{\pi}{10} + 2\sin\frac{9\pi}{10} = -2$
Divide by 2 and using $\sin(-x) = -\sin x$:
$$-\sin\frac{7\pi}{10} - \sin\frac{3\pi}{10} + \sin\frac{\pi}{10} + \sin\frac{9\pi}{10} = -1$$
Using $\sin(\pi - x) = \sin x$:
$$-\sin\frac{3\pi}{10} - \sin\frac{3\pi}{10} + \sin\frac{\pi}{10} + \sin\frac{\pi}{10} = -1$$
$$2\sin\frac{\pi}{10} - 2\sin\frac{3\pi}{10} = -1$$
$$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$$

Question 14 (6 marks)

(i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **1**

$z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ (DMT)

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) Let $\omega = z + \frac{1}{z}$. Show that: **2**

$$\omega^3 + \omega^2 - 2\omega - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$w^3 = \left(z + \frac{1}{z}\right)^3$

$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

$= \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$

$w^2 = \left(z + \frac{1}{z}\right)^2$

$= \left(z^2 + \frac{1}{z^2}\right) + 2$

$\therefore LHS = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + 2$
$- 2\left(z + \frac{1}{z}\right) - 2$

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$

(iii) Hence or otherwise find all solutions in the interval $[0, 2\pi]$ of: **3**

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0 .$$

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) + \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) = 0$ (part (i))

$\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right) = 0$

$w^3 + w^2 - 2w - 2 = 0$ (part ii)

$w^2(w+1) - 2(w+1) = 0$

$(w^2 - 2)(w+1) = 0$

$w = \pm\sqrt{2}, \ -1$

$z + \frac{1}{z} = \pm\sqrt{2}, \ -1$

$2\cos\theta = \pm\sqrt{2}, \ -1$

$\cos\theta = \pm\frac{\sqrt{2}}{2}, \ -\frac{1}{2}$

$\theta = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$