2020 | **HSC Assessment Task 3**

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 65 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 43

Section I – 5 marks (pages 2 – 4)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 38 marks (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 57 minutes for this section

NAME:_______________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

Question	1 – 5	6	7	8	Total
Proof	/2	/2	/7	/10	**/21**
Further Integration	/3	/10	/5	/4	**/22**
	/5	**/12**	**/12**	**/14**	**/43**

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

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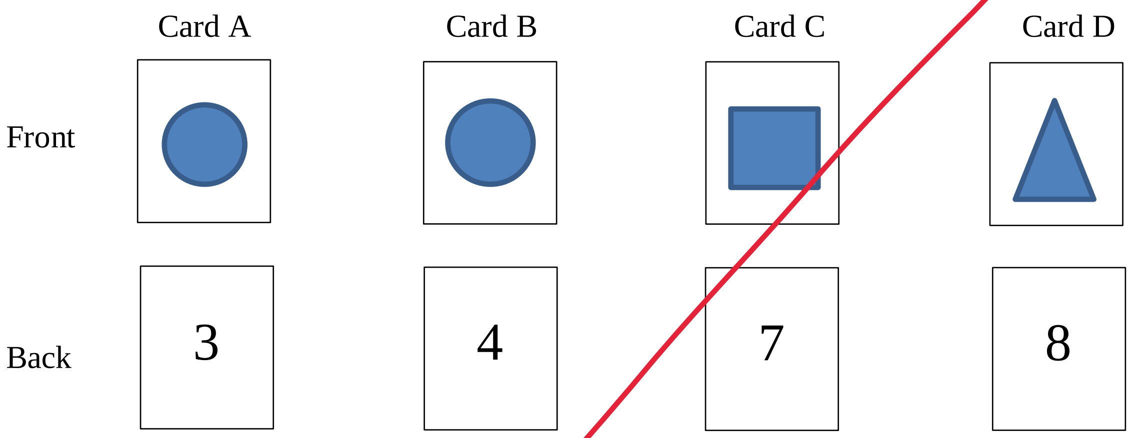
1. A set of four cards each with a shape printed on the front and a number printed on the back is shown below. Which one of the cards provides a counterexample to the following statement?

Every card that has an even number on its back has a circle on its front.



A. Card A

B. Card B

C. Card C

D. Card D

2. Consider the following statement about Fred:

Every day next week, Fred will do at least one maths problem.

If this statement is **not** true, which of the following is **certainly** true?

A. On no day next week will Fred do more than one Maths problem.

B. Every day next week, Fred will do no Maths problems.

C. On some day next week, Fred will do no Maths problems.

D. On no day next week will Fred do no Maths problems

3. Which of the following is equivalent to $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x}$ using the substitution $t=\tan\frac{x}{2}$?

A. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{2t^2+t+1}$

B. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{t^2+t+1}$

C. $\displaystyle\int_0^{1} \frac{dt}{2t^2+t+1}$

D. $\displaystyle\int_0^{1} \frac{dt}{t^2+t+1}$

4 Using a suitable substitution, which of the following is equivalent to $\displaystyle\int_0^{\frac{\pi}{3}} \sec^4 x\tan^3 x\,dx$?

A. $\displaystyle\int_0^{\sqrt{3}} \left(u^5-u^3\right)du$

B. $\displaystyle\int_{\sqrt{3}}^{0} \left(-u^3-u^5\right)du$

C. $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \left(u^3+u^5\right)du$

D $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \left(u^5-u^3\right)\,du$

5. Let $f(x)$ be an even function where $f(x) \neq 0$. Which of the following statements is not necessarily true?

A. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left(f(x) + f(-x) \right) dx$

B. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{2a} f(2a - x)\,dx$

C. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{2a} f(x - a)\,dx$

D. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{2a} f(a - x)\,dx$

End of Section I

Section II

38 marks
Attempt Questions 6-8
Allow about 57 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks**)** Use a SEPARATE writing booklet

(a) Find $\int 2x\sqrt{2x-3}\,dx$. **2**

(b) (i) Write $\frac{8}{(x+2)(x^2+4)}$ in the form $\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$. **2**

(ii) Hence evaluate $\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$. **3**

(c) Use the substitution $x=2\sin\theta$ to find $\int \frac{dx}{(4-x^2)^{\frac{3}{2}}}$. **3**

(d) If n^2 is an even integer, prove by contrapositive that n is even. **2**

Question 7 (12 marks**)** Use a SEPARATE writing booklet

(a) Find $\displaystyle\int \frac{dx}{\sqrt{13+4x-x^2}}$. **2**

(b) Find $\displaystyle\int x\sin 2x\,dx$. **3**

(c) (i) If $a,\, b \geq 0$, prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$. **1**

(ii) Hence prove that $e^a + e^b \geq 2e^{\frac{a+b}{2}}$ for all real *a*, *b*. **1**

(iii) Hence or otherwise, find the lower bound for the expression **2**

$e^{-2x} + e^{-x} + 1 + e^x + e^{2x}$ for all real *x*.

(d) Use mathematical induction to prove that $2^{n-1} \leq n!$ for all positive integers *n*. **3**

Question 8 (14 marks**)** Use a SEPARATE writing booklet

(a) (i) Use a proof by contradiction to prove that if $ab > 0$, then

3 $$\frac{a}{b} + \frac{b}{a} \geq 2 \qquad \text{where } a, b \in \mathbb{R}.$$

(ii) Prove that the converse is also true. **2**

(b) Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$.

(i) Show that $I_{n-1} - I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1-x}\, dx$. (No integration is needed.) **1**

(ii) Use integration by parts on the result of part (i) to show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$. **3**

(c) Consider $f(x) = \log x - x + 1$.

(i) Show that $f(x) \leq 0$ for all $x > 0$. **2**

(ii) Consider the set of *n* positive numbers $p_1, p_2, p_3, \dots p_n$. **1**

By using the result in part (i), deduce that

$$\sum_{r=1}^{n} \log(np_r) \leq np_1 + np_2 + np_3 \dots + np_n - n$$

(iii) If $p_1 + p_2 + p_3 + \dots\dots + p_n = 1$, deduce that $0 < n^n p_1\, p_2\, p_3 \dots\dots p_n \leq 1$ **2**

End of paper

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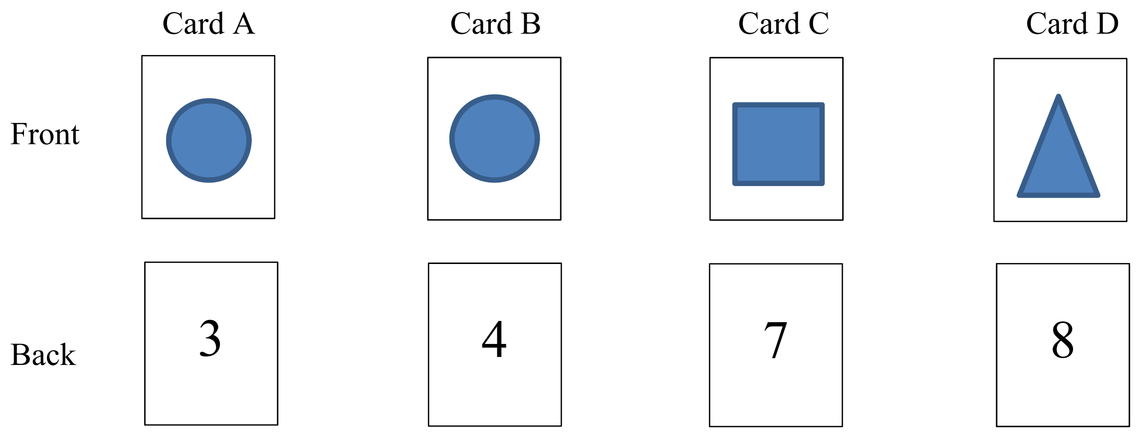
2020 HSC Task 3 Mathematics Ext. 2 - Solutions

1. A set of four cards each with a shape printed on the front and a number printed on the back is shown below. Which one of the cards provides a counterexample to the following statement?

Every card that has an even number on its back has a circle on its front.



A. Card A

B. Card B

C. Card C

D. Card D

The statement is a 'for all' statement, and can be expressed as a conditional statement:

For every card, **if** it has an even number on its back, **then** it has a circle on its front.

A counterexample to this statement is a card for which '**if** it has an even number on its back, **then** it has a circle on its front' is false. This is card D

2. Consider the following statement about Fred:

Every day next week, Fred will do at least one maths problem.

If this statement is **not** true, which of the following is **certainly** true?

A. On no day next week will Fred do more than one Maths problem.

B. Every day next week, Fred will do no Maths problems.

C. On some day next week, Fred will do no Maths problems.

D. On no day next week will Fred do no Maths problems

If a statement is false, then its negation is true.
The negation of 'for all x, P' is 'there exists x such that not P'.

3. Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x}$ using the substitution $t=\tan\frac{x}{2}$?

A. $\int_0^{\frac{\pi}{2}} \frac{dt}{2t^2+t+1}$

B. $\int_0^{\frac{\pi}{2}} \frac{dt}{t^2+t+1}$

C. $\int_0^{1} \frac{dt}{2t^2+t+1}$

D. $\int_0^{1} \frac{dt}{t^2+t+1}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x+\sin x} = \int_0^1 \frac{1}{3-\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{3(1+t^2)-1+t^2+2t}$$

$$= \int_0^1 \frac{2\,dt}{4t^2+2t+2}$$

4 Using a suitable substitution, which of the following is equivalent to $\int_0^{\frac{\pi}{3}} \sec^4 x \tan^3 x\,dx$?

A. $\int_0^{\sqrt{3}} \left(u^5-u^3\right)du$

B. $\int_{\sqrt{3}}^{0} \left(-u^3-u^5\right)du$

C. $\int_0^{\frac{1}{\sqrt{3}}} \left(u^3+u^5\right)du$

D $\int_0^{\frac{1}{\sqrt{3}}} \left(u^5-u^3\right)du$

$$\int_0^{\pi/3} \left(\sec^2 x\left(1+\tan^2 x\right)\tan^3 x\right)dx$$

$$= \int_0^{\pi/3} \left(\sec^2 x\left(\tan^3 x+\tan^5 x\right)\right)dx$$

$$= \int_0^{\sqrt{3}} \left(u^3+u^5\right)du$$

$u=\tan x$

$du=\sec^2 x\,dx$

$x=0,\ u=0$

$x=\frac{\pi}{3},\ u=\sqrt{3}$

5. Let $f(x)$ be an even function where $f(x)\neq 0$. Which of the following statements is not necessarily true?

A. $\int_{-a}^{a} f(x)\,dx = \int_0^a \left(f(x)+f(-x)\right)dx$

B. $\int_{-a}^{a} f(x)\,dx = \int_0^{2a} f(2a-x)\,dx$

C. $\int_{-a}^{a} f(x)\,dx = \int_0^{2a} f(x-a)\,dx$

D. $\int_{-a}^{a} f(x)\,dx = \int_0^{2a} f(a-x)\,dx$

$$\int_0^{2a} f(2a-x)\,dx = \int_0^{2a} f(x)\,dx$$

$$\int_0^{2a} f(a-x)\,dx = \int_0^{2a} f(x-a)\,dx$$

since f is an even function.

$\therefore$ D is the same as C.

Question 6

(a)	Find $\int 2x\sqrt{2x-3}\,dx$.	**2**

$$\int 2x\sqrt{2x-3}\,dx = \int\left((2x-3)\sqrt{2x-3} + 3\sqrt{2x-3}\right)dx$$

$$= \int\left((2x-3)^{3/2} + 3(2x-3)^{1/2}\right)dx$$

$$= \frac{\cancel{2}(2x-3)^{5/2}}{\cancel{2}\times 5} + \cancel{2}\times\cancel{3}\,\frac{(2x-3)^{3/2}}{\cancel{2}\times\cancel{3}} + C$$

$$= \frac{1}{5}(2x-3)^{5/2} + (2x-3)^{3/2} + C$$

(a) Generally well done. Most students used a substitution. A small number of students used integration by parts which was not an efficient method. Main errors were algebraic. Other errors were leaving the primitive in terms of the substituted variable or errors in integrating $(2x-3)^n$ and forgetting to divide by the 2.

(b)	(i)	Write $\dfrac{8}{(x+2)(x^2+4)}$ in the form $\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+4}$.	**2**
	(ii)	Hence evaluate $\displaystyle\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$.	**3**

(i) Let $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+4}$

$$8 = A(x^2+4)+(Bx+C)(x+2)$$

$(x=-2)$ $\quad 8=8A \quad \Rightarrow \quad A=1$

$(x=0)$ $\quad 8=4A+2C \quad \Rightarrow \quad 8=4+2C \quad \Rightarrow \quad C=2$

$(x=1)$ $\quad 8=5A+3B+3C \quad \Rightarrow \quad 8=5+3B+6 \quad \Rightarrow \quad B=-1$

$$\therefore \quad \frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2}+\frac{-x+2}{x^2+4}$$

(a) (i) Generally well done. Some students stopped after finding *A*, *B* and *C* and forgot to write the expression in the form required. Students are advised to read the question carefully.

(ii) $$\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)} = \int_0^2\left(\frac{1}{x+2}-\frac{x}{x^2+4}+\frac{2}{x^2+4}\right)dx$$

$$= \left[\ln(x+2)-\frac{1}{2}\ln(x^2+4)+\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \left(\ln 4-\frac{1}{2}\ln 8+\frac{\pi}{4}\right)-\left(\ln 2-\frac{1}{2}\ln 4+0\right)$$

$$= 2\ln 2-\frac{3}{2}\ln 2+\frac{\pi}{4}-\ln 2+\ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

(ii) Most students were able to integrate the expression correctly. Some errors included forgetting the $\frac{1}{a}$ in the $\tan^{-1}$ primitive and in simplifying the log expressions using log laws. A small number of students felt compelled to use integration by substitution for $\int \frac{x}{x^2+4}\,dx$ and are advised to practise identifying and integrating using the reverse chain rule to save time.

(c) Use the substitution $x = 2\sin\theta$ to find $\int \frac{dx}{\left(4-x^2\right)^{\frac{3}{2}}}$. **3**

$$\int \frac{dx}{\left(4-x^2\right)^{\frac{3}{2}}} = \int \frac{2\cos\theta\, d\theta}{\left(4-4\sin^2\theta\right)^{\frac{3}{2}}}$$

Let $x = 2\sin\theta$

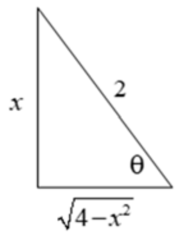
$dx = 2\cos\theta\, d\theta$

$$= \int \frac{2\cos\theta\, d\theta}{8\cos^3\theta}$$

$$= \frac{1}{4}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\tan\theta + c$$

$$= \frac{x}{4\sqrt{4-x^2}} + c$$



(c) Well done. Main errors were in forgetting to rewrite the primitive in terms of *x*.

(d) If n^2 is an even integer, prove by contrapositive that n is even. **2**

Contrapositive: If n is not even then n^2 is not even.

Let $n = 2m+1$ $\quad m \in \mathbb{Z}$

$n^2 = (2m+1)^2$

$= 4m^2 + 4m + 1$

$= 2(m^2 + 2m) + 1$

which is 1 more than a multiple of 2 and therefore odd.

Hence by the contrapositive, if n^2 is even then n is even.

(d) Very well done. Almost all students correctly stated the contrapositive, and proved it. Students are reminded that all proofs should conclude with the statement being proved. So in this case, a restatement of "Therefore, by the contrapositive, if n^2 is even, then n is even" was required.

Question 7

(a) Find $\int \frac{dx}{\sqrt{13+4x-x^2}}$. **2**

$$\int \frac{dx}{\sqrt{13+4-(x^2-4x+(-2)^2)}}$$

$$= \int \frac{dx}{\sqrt{17-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{\sqrt{17}}\right) + C$$

(b) Find $\int x\sin 2x\,dx$. **3**

$u = x \qquad v' = \sin 2x$

$u' = 1 \qquad v = -\frac{\cos 2x}{2}$

$$\int x\sin 2x\,dx = -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + C$$

(b) Very well done. Please ensure that you show enough work for 3 marks. It is not up to the marker to guess how you have determined the parts. Remember the instructions on the front of the paper..”show relevant reasoning and/or calculations.”

(c) (i) If $a, b \geq 0$, prove that $\frac{a+b}{2} \geq \sqrt{ab}$. **1**

(ii) Hence prove that $e^a + e^b \geq 2e^{\frac{a+b}{2}}$ for all real a, b. **1**

(iii) Hence or otherwise, find the lower bound for the expression **2**

$e^{-2x} + e^{-x} + 1 + e^x + e^{2x}$ for all real x.

i) $(\sqrt{a} - \sqrt{b})^2 \geq 0 \quad , \quad a, b \geq 0$

$a - 2\sqrt{ab} + b \geq 0$

$a + b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$

ii) Replacing a with e^a, and b with e^b in (i)

$$\frac{e^a+e^b}{2} \geq \sqrt{e^a . e^b}$$

$$= \left(e^{a+b}\right)^{1/2}$$

$$\therefore e^a+e^b \geq 2e^{\frac{a+b}{2}}$$

iii) $e^{-2x}+e^{-x}+1+e^{x}+e^{2x}$

$$= \left(e^{-2x}+e^{2x}\right)+\left(e^{x}+e^{-x}\right)+1$$

$$\geq 2e^0+2e^0+1$$

$$= 5.$$

$\therefore$ the lower bound is 5.

(c)(ii)	When using the result in part (i), it is better to write '**replace** a with e^a' rather than 'let $a=e^a$'. The latter can lead to confusion since there is an a on both sides of the equation. The main problems arose when students simply tried to use each side of the inequality in part (i) as the exponents of an inequality in part (ii). This did not lead to the required result.
(c)(iii)	Some students grouped terms incorrectly to get $e^{-2x}+e^{-x} \geq 2e^{-\frac{3x}{2}}$ and $e^{2x}+e^{x} \geq 2e^{\frac{3x}{2}}$. This did not really go anywhere but students were awarded 1mark for applying part (ii).

(d)	Use mathematical induction to prove that $2^{n-1} \leq n!$ for all positive integers n.	**3**

Prove true for $n = 1$

$LHS = 2^{1-1}$ $\quad$ $RHS = 1!$

$= 1$ $\quad$ $= 1$

$\therefore$ true for $n = 1$

Assume true for $n = k$

i.e. $2^{k-1} \leq k!$

Prove true for $n = k+1$

i.e. RTP $2^k \leq (k+1)!$

$LHS = 2^k$

$= 2 \cdot 2^{k-1}$

$\leq 2 \cdot k!$ (from assumption)

$\leq (k+1) \cdot k!$ $\quad$ since $k \geq 1$ $\quad k \in \mathbb{Z}^+$

$\quad (k+1) \geq 2$

$= (k+1)!$

$= RHS$

$\therefore LHS \leq RHS$ $\quad$ By mathematical induction the result is true for $n \geq 1$.

(d)	Generally well done. A surprising number of students tried to prove that RHS-LHS>=0 even though they were asked to prove LHS<=RHS. Occasionally this method led to errors with the inequality sign especially when trying to subtract a larger positive quantity. Make sure you link your final line of working back to what you were originally trying to prove. i.e LHS<=RHS. The third mark was awarded for a clear argument and explanation after the inductive step.

Question 8

(a)	(i)	Use a proof by contradiction to prove that if $ab > 0$, then $\dfrac{a}{b}+\dfrac{b}{a} \geq 2$ where $a, b \in \mathbb{R}$.	**3**
	(ii)	Prove that the converse is also true.	**2**

(i) Assume $ab > 0$ and $\dfrac{a}{b} + \dfrac{b}{a} < 2$

$$\frac{a^2+b^2}{ab} < 2$$

$a^2 + b^2 < 2ab \quad (ab > 0)$

$a^2 - 2ab + b^2 < 0$

$(a-b)^2 < 0$

But $(a-b)^2 \geq 0$ since squares are never negative.

This is a contradiction.

$\therefore$ our assumption is incorrect

$\therefore$ if $ab > 0$ then $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.

(a)(i) "If A then B" means "$A \Rightarrow B$". The correct negation of this statement is "A and not B".

So your assumption in this question should have been "Assume $ab > 0$ AND $\dfrac{a}{b}+\dfrac{b}{a} < 2$".

Very few students had the correct assumption. Many reversed the inequality of the wrong statement, which meant they were negating the converse.
A correct proof would have begun with the assumption and developed the contradiction, but many students didn't start with the assumption. Provided the proof was logically consistent, the marks were paid, but this was a case of leniency.

A number of students incorrectly believed that $\dfrac{a}{b}+\dfrac{b}{a} > 0$ contradicts $\dfrac{a}{b}+\dfrac{b}{a} < 2$.

ii) Converse: If $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ then $ab > 0$

Method 1: Direct Proof

Suppose $\dfrac{a}{b} + \dfrac{b}{a} \geq 2 \quad a, b \in \mathbb{R} \quad a, b \neq 0$

$$\frac{a^2+b^2}{ab} \geq 2$$

But $a^2 + b^2 > 0$

$\therefore ab > 0$ If numerator is positive, denominator is also positive.

$\therefore$ If $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$, $ab > 0$

Method 2: Contrapositive

Suppose if $ab \leq 0$, then $\frac{a}{b} + \frac{b}{a} < 2$

Ignore when $ab = 0$ since $\frac{a}{b} + \frac{b}{a}$ is undefined.

If $ab < 0$ then a & b are opposite in sign.

$\therefore$ $\frac{a}{b}$ & $\frac{b}{a}$ are negative

and $\frac{a}{b} + \frac{b}{a} < 2$

By contrapositive if $\frac{a}{b} + \frac{b}{a} \geq 2$, then $ab > 0$.

(ii) Many students did not understand what the converse is.

The converse is merely a reversal of the direction of implication. So if the original statement is "If A then B" then the converse is "If B then A". Negation only comes into play when you attempt to prove the converse by contradiction or by proving its contrapositive. Students who had the wrong converse could not be awarded marks.

Many students made the mistake of assuming the result by turning $\frac{a^2+b^2}{ab} \geq 2$ into $a^2+b^2 \geq 2ab$.

(b) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$.

(i) Show that $I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\,dx$. (No integration is needed.) **1**

(ii) Use integration by parts on the result of part (i) to show that $I_n = \frac{2n}{2n+1} I_{n-1}$. **3**

(i)

$$\begin{aligned} I_{n-1} - I_n &= \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}}\,dx - \int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx \\ &= \int_0^1 \frac{x^{n-1} - x^n}{\sqrt{1-x}}\,dx \\ &= \int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\,dx \\ &= \int_0^1 x^{n-1}\sqrt{1-x}\,dx \end{aligned}$$

(b) (i) Generally well done.

(ii)

$$I_{n-1} - I_n = \int_0^1 x^{n-1}\sqrt{1-x}\,dx$$

$$= \int_0^1 \sqrt{1-x}\cdot\frac{d}{dx}\left(\frac{x^n}{n}\right)dx$$

$$= \frac{1}{n}\left[x^n\sqrt{1-x}\right]_0^1 - \frac{1}{n}\int_0^1 x^n\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1)\,dx$$

$$= 0 + \frac{1}{2n}\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}}$$

$$I_{n-1} - I_n = \frac{1}{2n}I_n$$

$$2nI_{n-1} - 2nI_n = I_n$$

$$2nI_{n-1} = (2n+1)I_n$$

$$I_n = \frac{2n}{2n+1}I_{n-1}$$

(ii)	Many students chose u and v incorrectly and could not be awarded marks. The instruction was to perform IBP on $I_{n-1} - I_n$. Students who performed IBP on I_n could not receive full marks. A few students could not correctly differentiate $\sqrt{1-x}$, forgetting to multiply by -1, then fudged the sign later.

(c) Consider $f(x) = \log x - x + 1$.

(i) Show that $f(x) \le 0$ for all $x > 0$. **2**

(ii) Consider the set of n positive numbers p_1, p_2, p_3,... p_n. **1**

By using the result in part (i), deduce that

$$\sum_{r=1}^{n} \log(np_r) \le np_1 + np_2 + np_3 ... + np_n - n$$

(iii) If $p_1 + p_2 + p_3 + ... + p_n = 1$, deduce that $0 < n^n p_1\ p_2\ p_3 ..... p_n \le 1$ **2**

(i) $f(x) = \log x - x + 1$

$f'(x) = \frac{1}{x} - 1$

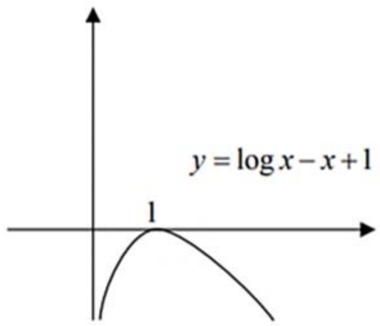
$= \frac{1-x}{x}$

$\therefore$ stationary point at $x = 1$

$f''(x) = -\frac{1}{x^2} < 0\ \forall x$

$\therefore$ minimum turning point at $(1, 0)$

Also domain $x > 0$ (and continuous for all x in the domain).



$\therefore\ f(x) \le 0\ \ \forall x > 0$

(c)(i) Poorly done, considering this was essentially a simple 2U optimisation question.

A majority of students argued incorrectly that, for $x > 0$, $\frac{1}{x} < 1$, or $\frac{1-x}{x} < 0$, which can be seen to be false by letting $x = \frac{1}{2}$.

Students who did this, or otherwise claimed that $f(x)$ is decreasing for $x > 0$, could not be awarded marks. Even if correct, this reasoning would require knowing $f(0)$, which is undefined.

Students who argued by drawing $y = \log x$ and $y = x - 1$ could not be awarded marks unless they showed that $y = x - 1$ is in fact a tangent to the log graph. Simply passing through the same point does not mean the curves are tangent at that point.

(ii) $\sum_{r=1}^{n}\log(np_r) \le \sum_{r=1}^{n}(np_r - 1)$ (from part i - $\log x \le x - 1 \quad \forall x > 0$)

$$= \sum_{r=1}^{n} np_r - \sum_{r=1}^{n} 1$$

$$\sum_{r=1}^{n}\log(np_r) \le \sum_{r=1}^{n} np_r - n$$

(ii) A number of students did not sufficiently explain where the $-n$ came from.

Others invented their own log laws, such as $n\sum \log x = \sum \log nx$.

A significant number of students did not treat n as a constant, believing its value should vary between terms in the sum.

(iii) Continuing from part ii:

$$\sum_{r=1}^{n}\log np_r \le n\sum_{r=1}^{n} p_r - n \quad \text{(since } n \text{ is a constant)}$$

$$= n \cdot 1 - n$$

$$\sum_{r=1}^{n}\log np_r \le 0$$

$$\log np_1 + \log np_2 + \ldots + \log np_n \le 0$$

$$\log(np_1 \cdot np_2 \cdot \ldots \cdot np_n) \le 0$$

$$\log(n^n \cdot p_1 p_2 \ldots p_n) \le 0$$

$$n^n \cdot p_1 p_2 \ldots p_n \le 1$$

Also, since $p_1,\ p_2,\ \ldots,\ p_n$ and n are all positive, then $n^n \cdot p_1 p_2 \ldots p_n > 0$

$\therefore\ 0 < n^n \cdot p_1 p_2 \ldots p_n \le 1$

(iii) A large number of students managed to perform the first step but could not take it further.

Converting the sum to a product must be developed, not just stated.

END OF SOLUTIONS