NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2021 | **HSC Assessment Task 1**

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 40

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 7 minutes for this section

Section II – 35 marks (pages 4 – 6)

- Attempt Questions 6 – 8
- Allow about 48 minutes for this section

NAME:_______________________________ TEACHER:___________________

STUDENT NUMBER:__________________________

Question	1 – 5	6	7	8	Total
Complex Numbers	/5	/13	/11	/11	**/40**

Section I

5 marks
Attempt Questions 1-5
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

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1 Given that $i^n = p$ and $i^2 = -1$, what is the value of i^{2n-3} ?

A. $p^2 + i$

B. $p^2 - i$

C. ip^2

D. $-ip^2$

2 One root of the equation $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ is $x = 2 - 3i$. Which of the following is another root?

A. $1 - 2i$

B. $-1 - 2i$

C. $-2 - i$

D. $-2 + i$

3 Which of the following relations has a graph that does not pass through $1 + i$ in the complex plane?

A. $z\overline{z} = 2$

B. $z + \overline{z} = 2$

C. $\arg(z - 2i) = \arg(z - 2)$

D. $|z + i| = |z + 1|$

4 Multiplying a non-zero complex number by $\dfrac{1-i\sqrt{3}}{\sqrt{3}+i}$ results in a rotation.

What is the rotation?

A. Rotation anti-clockwise by $\dfrac{\pi}{2}$

B. Rotation clockwise by $\dfrac{\pi}{2}$

C. Rotation anti-clockwise by $\dfrac{\pi}{6}$

D. Rotation clockwise by $\dfrac{\pi}{6}$

5 Given that $x^4+x^3+x^2+x+1=0$, what is the value of $x^{2020}+\dfrac{1}{x^{2020}}$?

A. 0

B. 1

C. $\dfrac{2\pi}{5}$

D. 2

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Section II

35 marks
Attempt Questions 6-8
Allow about 48 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks**)** Use a SEPARATE writing booklet

(a) Given that $z = -1 + i$ and $w = 2 - i$:

(i) Find $z - iw$. **2**

(ii) Find a and b such that $\frac{z}{w} = a + ib$. **2**

(b) Consider the complex number $z = 1 - i\sqrt{3}$.

(i) Write $z = 1 - i\sqrt{3}$ in exponential form. **2**

(ii) Find $\text{Arg}\left(z^5\right)$ **1**

(iii) Hence, write $\left(1 - i\sqrt{3}\right)^5$ in Cartesian or $x + iy$ form. **2**

(c) Solve $z^2 - (1 - 3i)z + 4 + i = 0$. **4**

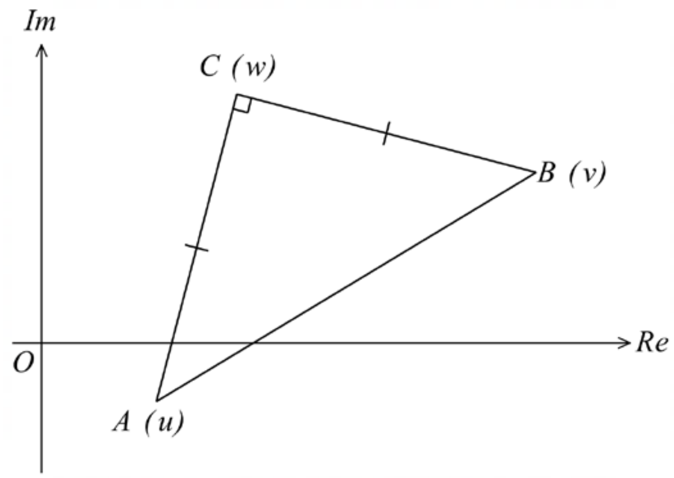
Question 7 (11 marks) Use a SEPARATE writing booklet

(a) (i) Find the fifth roots of i showing all working. **2**
Answer using principal arguments.

(ii) Plot these roots on an Argand Diagram. **1**

(iii) If ω is the root of smallest positive argument, find the value of **1**
$\omega^{-3}+\omega^{-7}+\omega+\omega^{9}$.

(b) The points A, B, C represent the complex numbers u, v and w respectively, where ABC is an isosceles right triangle as shown.



(i) Express $\overrightarrow{CA}$ in terms of u and w. **1**

(ii) By expressing $\overrightarrow{CB}$ in terms of u and w, or otherwise, find **2**
the complex number v in terms of u and w.

(c) Given $z=e^{-i\frac{\pi}{4}}$ and $w=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$:

(i) Draw a diagram and use a geometrical argument to explain why **2**
$\arg(z+w)=\dfrac{5\pi}{24}$.

(ii) Hence, find the exact value of $\tan\left(\frac{5\pi}{24}\right)$. **2**

Question 8 (11 marks) Use a SEPARATE writing booklet

(a) (i) Sketch the region in the Argand plane defined by: **2**

$$|z-1+i|<1 \text{ and } |\arg(z+i)| \le \frac{\pi}{4}$$

(ii) Find the minimum value of $|z|$. **1**

(b) By considering $e^{i\theta}+e^{-i\theta}$ show that: **2**

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$

(c) (i) Use De Moivre's theorem to show that: **2**

$$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$$

(ii) Hence, find the roots of the equation $32x^5-40x^3+10x+1=0$. **2**

(iii) Hence, using the results of (b) or otherwise, show that **2**

$$\left(1-2\cos\frac{2\pi}{5}\right)\left(1+2\cos\frac{\pi}{5}\right)=1$$

End of paper

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SUGGESTED
SOLUTIONS

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A. $p^2 + i$

B. $p^2 - i$

C. ip^2

D. $-ip^2$

$i^{2n-3} = i^{2n} \times i^{-3}$

$= \frac{(i^n)^2}{i^3} = \frac{p^2}{-i} \times \frac{i}{i}$

$= \frac{ip^2}{1}$

$\therefore$ C

2 One root of the equation $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ is $x = 2 - 3i$. Which of the following is another root?

A. $1 - 2i$

B. $-1 - 2i$

C. $-2 - i$

D. $-2 + i$

$\Sigma\alpha: (2-3i)+(2+3i)$
$+ (\alpha+i\beta)+(\alpha-i\beta) = 6$

$4 + 2\alpha = 6$
$\alpha = 1$

$\therefore$ A

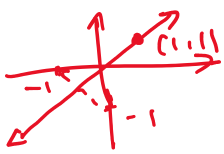
3 Which of the following relations has a graph that does not pass through $1+i$ in the complex plane?

A. $z\bar{z} = 2$ circle radius $\sqrt{2}$

B. $z + \bar{z} = 2$ $2\,\mathrm{Re}\,z = 2$ $\therefore \mathrm{Re}\,z = 1$

C. $\arg(z - 2i) = \arg(z - 2)$ does not pass thro' (1,1)

D. $|z + i| = |z + 1|$



4 Multiplying a non-zero complex number by $\dfrac{1-i\sqrt{3}}{\sqrt{3}+i}$ results in a rotation.

What is the rotation?

A. Rotation anti-clockwise by $\dfrac{\pi}{2}$

B. Rotation clockwise by $\dfrac{\pi}{2}$

C. Rotation anti-clockwise by $\dfrac{\pi}{6}$

D. Rotation clockwise by $\dfrac{\pi}{6}$

$$\frac{1-i\sqrt{3}}{\sqrt{3}+1} = \frac{2\operatorname{cis}\left(\frac{-\pi}{3}\right)}{2\operatorname{cis}\frac{\pi}{6}}$$
$$= \operatorname{cis}\left(\frac{-\pi}{3}-\frac{\pi}{6}\right)$$
$$= \operatorname{cis}\left(\frac{-\pi}{2}\right)$$
$$= -i$$

$\therefore$ B

5 Given that $x^4+x^3+x^2+x+1=0$, what is the value of $x^{2020}+\dfrac{1}{x^{2020}}$?

A. 0

B. 1

C. $\dfrac{2\pi}{5}$

D. 2

Roots of $x^4+x^3+x^2+x+1=0$
are among roots of $x^5=1$ excl $x=1$
ie. $x=\operatorname{cis}\frac{2k\pi}{5}$ $k=1,2,3,4$

$$x^{2020} = \left(\operatorname{cis}\frac{2k\pi}{5}\right)^{2020} = \operatorname{cis}\left(\frac{4040k\pi}{5}\right)$$
$$= \operatorname{cis}(808k\pi) = 1$$

$\therefore x^{2020}+\dfrac{1}{x^{2020}} = 1+\dfrac{1}{1} = 2$ $\quad\therefore$ D

Section II

35 marks
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Question 6 (13 marks**)** Use a SEPARATE writing booklet

(a) Given that $z = -1 + i$ and $w = 2 - i$:

(i) Find $z - iw$. **2**

(ii) Find a and b such that $\dfrac{z}{w} = a + ib$. **2**

(b) Consider the complex number $z = 1 - i\sqrt{3}$.

(i) Write $z = 1 - i\sqrt{3}$ in exponential form. **2**

(ii) Find $\operatorname{Arg}\left(z^5\right)$ **1**

(iii) Hence, write $\left(1 - i\sqrt{3}\right)^5$ in Cartesian or $x + iy$ form. **2**

(c) Solve $z^2 - (1 - 3i)z + 4 + i = 0$. **4**

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(a) $z = -1+i \qquad w = 2-i$

(i) $z - iw = (-1+i) - i(2-i)$

$= -1+i-2i+i^2$

$= -2-i$

(ii) $\frac{z}{w} = \frac{-1+i}{2-i} \times \frac{2+i}{2+i}$

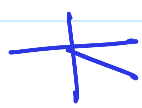
$= \frac{-2-i+2i+i^2}{4+1}$

$= \frac{-3+i}{5}$

$= -\frac{3}{5} + \frac{1}{5}i \qquad$ Thus, $a = -\frac{3}{5} \quad b = \frac{1}{5}$

(b) $z = 1 - i\sqrt{3}$

(i) $|z| = \sqrt{1+3} = 2$



$\tan(\arg z) = -\sqrt{3}$

$\arg z = -\frac{\pi}{3}$

$\therefore \quad z = 2e^{-i\frac{\pi}{3}}$

(ii) $\arg(z^5) = 5\arg z$

$= 5 \times -\frac{\pi}{3} = -\frac{5\pi}{3}$

$\text{Arg}(z^5) = -\frac{5\pi}{3} + 2\pi$

$= \frac{\pi}{3}$

(iii) $(1-i\sqrt{3})^5 = 32e^{i\frac{\pi}{3}}$

$= 32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 32\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 16 + 16\sqrt{3}i$

(c) $z^2 - (1-3i)z + 4 + i = 0$

$z^2 - (1-3i)z + (4+i) = 0$

$\Delta = [-(1-3i)]^2 - 4(1)(4+i)$

$= 1 - 9 - 6i - 16 - 4i$

$= -24 - 10i$

let $(x+iy)^2 = -24 - 10i$

$x^2 - y^2 + 2ixy = -24 - 10i$

Equating real & imaginary parts

$x^2 - y^2 = -24$ (1)

$2xy = -10$

$xy = -5$ (2)

$x^2 - y^2 = -24$ (1)

$xy = -5$ (2)

By inspection $x = 1$, $y = -5$ is a soln

as $1^2 - (-5)^2 = -24$ and $1 \times -5 = -5$

$\therefore$ $1 - 5i$ is a solution

$-1 + 5i$ is the other solution

$$\therefore z = \frac{(1-3i) \pm (1-5i)}{2}$$

$$= \frac{1-3i+1-5i}{2}, \frac{1-3i-1+5i}{2}$$

$$= 1 - 4i, i$$

Question 6

(a) (i) Well done, except for a few careless errors.

(ii) The question asks for values of a and b. You need to explicitly state those values.

(b) (i) Well done, other than a few who could not correctly evaluate the argument.
Please do not write $e^{i-\frac{\pi}{3}}$. If you wish to write it this way you need brackets: $e^{i\left(-\frac{\pi}{3}\right)}$.

(ii) The capital A means you need to give the principal value of the argument.

Any other value of the argument was awarded zero.

A few students added π instead of 2π to convert to the principal value.

(iii) Well done.

(c) This is a standard question type. Please revise all standard questions.

Generally done well by students who could get a start.

Be careful when solving by inspection, A number of students had the values of x and y reversed.

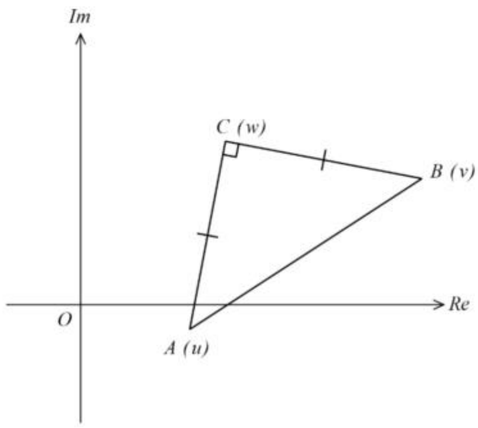
Question 7 (11 marks) Use a SEPARATE writing booklet

(a) (i) Find the fifth roots of i showing all working. **2**
Answer using principal arguments.

(ii) Plot these roots on an Argand Diagram. **1**

(iii) If ω is the root of smallest positive argument, find the value of **1**
$\omega^{-3}+\omega^{-7}+\omega+\omega^{9}$.

(b) The points A, B, C represent the complex numbers u, v and w respectively, where ABC is an isosceles right triangle as shown.



(i) Express $\overrightarrow{CA}$ in terms of u and w. **1**

(ii) By expressing $\overrightarrow{CB}$ in terms of u and w, or otherwise, find **2**
the complex number v in terms of u and w.

(c) Given $z=e^{-i\frac{\pi}{4}}$ and $w=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$:

(i) Draw a diagram and use a geometrical argument to explain why **2**
$\arg(z+w)=\dfrac{5\pi}{24}$.

(ii) Hence, find the exact value of $\tan\left(\frac{5\pi}{24}\right)$. **2**

(a) Let $z^5 = i$

$$z^5 = \text{cis}\left(\frac{\pi}{2} + 2k\pi\right) \qquad k \in \mathbb{Z}$$

$$= \text{cis}\,(4k+1)\frac{\pi}{2}$$

$$z = \left[\text{cis}\,(4k+1)\frac{\pi}{2}\right]^{\frac{1}{5}}$$

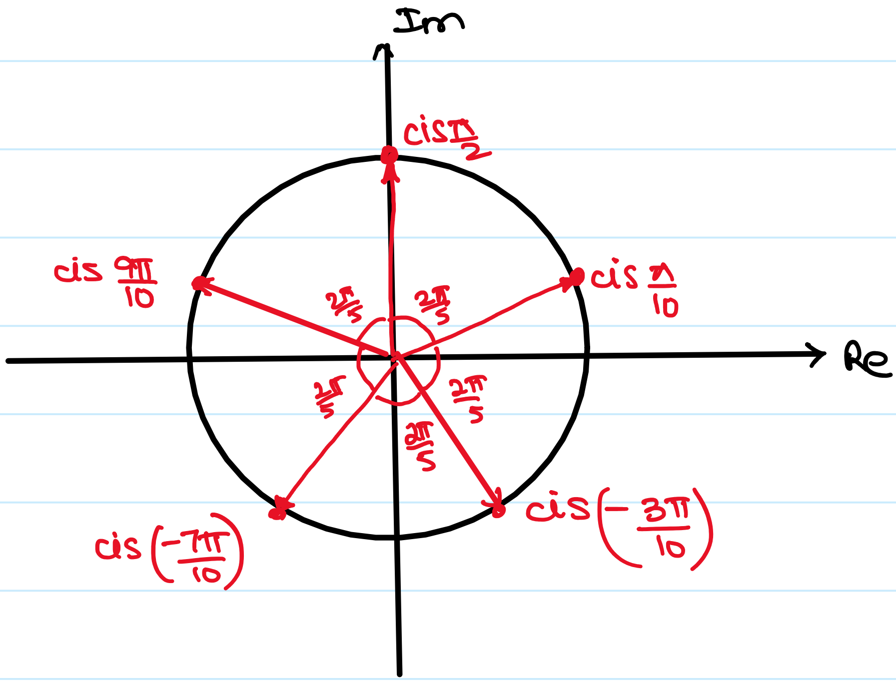
$$= \text{cis}\,(4k+1)\frac{\pi}{10} \qquad \text{using De Moivre's Thrm}$$

$$z = \text{cis}\,\frac{\pi}{10},\ \text{cis}\,\frac{5\pi}{10},\ \text{cis}\,\frac{9\pi}{10},\ \text{cis}\,\frac{13\pi}{10},\ \text{cis}\,\frac{17\pi}{10}$$

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$

$$= \text{cis}\,\frac{\pi}{10},\ \text{cis}\,\frac{\pi}{2},\ \text{cis}\,\frac{9\pi}{10},\ \text{cis}\left(-\frac{7\pi}{10}\right),\ \text{cis}\left(-\frac{3\pi}{10}\right)$$

(b)



c) Consider $z^5 - i = 0$ Using sum of roots

$$\text{cis}\,\frac{\pi}{10} + \text{cis}\,\frac{\pi}{2} + \text{cis}\,\frac{9\pi}{10} + \text{cis}\,\frac{-7\pi}{10} + \text{cis}\left(\frac{-3\pi}{10}\right) = 0$$

$$\omega + i + \omega^9 + \omega^{-7} + \omega^{-3} = 0$$

$$\therefore \quad \omega^{-7} + \omega^{-3} + \omega + \omega^9 = -i$$

(b)

(i) $\vec{CA} = \vec{OA} - \vec{OC}$

$\equiv u - \omega$

(ii) $\vec{CB}$ is obtained by rotating $\vec{CA}$ clockwise by $\frac{\pi}{2}$

as $\angle ACB = \frac{\pi}{2}$ and $|\vec{CB}| = |\vec{CA}|$

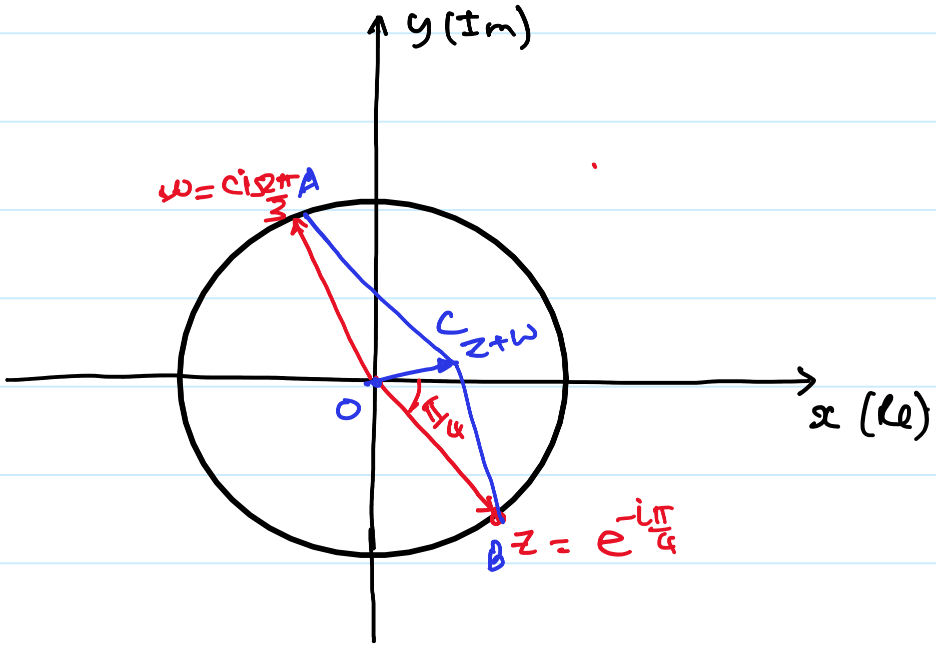
$\therefore \vec{CB} \equiv -i(u-\omega)$

Now, $\vec{OB} = \vec{OC} + \vec{CB}$

$\equiv \omega - i(u-\omega)$

$= \omega(1+i) - iu$

(c) $z = e^{-i\frac{\pi}{4}}$ $\qquad \omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$



Let $\vec{OA}$ represent the complex number ω & $\vec{OB}$ represent the complex number z

Complete the parallelogram of vectors OACB

$\vec{OC} = \vec{OA} + \vec{OB}$ (parallelogram law of addition)

Thus $\vec{OC}$ represents the complex number $z+\omega$.

Now $|\vec{OA}| = |\vec{OB}| = 1$ as $|z| = |\omega| = 1$

Thus, OACB is a rhombus. As the diagonals of a rhombus bisect the vertex angles

$\angle COA = \angle COB = \frac{1}{2}\angle AOB$

$\angle AOx = \frac{2\pi}{3}$ and $\angle BOx = \frac{\pi}{4}$

$\therefore \angle AOB = \angle AOx + \angle BOx$

$= \frac{2\pi}{3} + \frac{\pi}{4}$

$= \frac{11\pi}{12}$

Then $\angle COx = \angle COB - \angle BOx$

$= \frac{1}{2} \times \frac{11\pi}{12} - \frac{\pi}{4}$

$= \frac{11\pi}{24} - \frac{\pi}{4}$

$= \frac{5\pi}{24}$

(c) Now $z = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

So, $z + \omega = \left(+\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + i\left(-\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\right)$

$$z+w = \frac{(\sqrt{2}-1)}{2} + i\left(\frac{-\sqrt{2}+\sqrt{3}}{2}\right)$$

$$\tan(\arg(z+w)) = \tan\frac{5\pi}{24}$$

$$= \frac{-\sqrt{2}+\sqrt{3}}{2} \div \frac{\sqrt{2}-1}{2}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$$

Question 7

(a) (i) Standard question and generally well done. Answer was required in principal argument form.

(ii) Generally well done. Students are encouraged to show the roots on a unit circle rather than as points in space and also communicate that the roots are equally spaced by marking the angular distance between them as $\frac{2\pi}{5}$.

(iii) Not well done. However, students who saw the connection with the sum of roots had little difficulty.

(b) (i) Very well done. A small number of students wrote down the expression for $\overrightarrow{AC}$ instead.

(ii) Students who knew that $\overrightarrow{CB}$ is obtained by rotating $\overrightarrow{CA}$ anticlockwise by $\frac{\pi}{2}$ generally got the correct result. Students who did not provide any justification incurred a small penalty although they may have the correct expression for $\overrightarrow{OB}$. Some students stopped after finding $\overrightarrow{CB}$ and didn't realise that what was asked for was $\overrightarrow{OB}$ and could only be awarded 1 mark.

(c) (i) Not well done. Many students stated that $\arg(z+w)=\frac{1}{2}\left[\arg(z)+\arg(w)\right]$. This is not a quotable result and does not constitute a geometrical argument. Some students incorrectly stated that the angles of a parallelogram are bisected by the diagonal. Properties of special quadrilaterals often feature in these types of complex number problems are worth reviewing. No credit was awarded for students who drew a diagram and then stopped.

(ii) The key to this part was writing $z+w$ algebraically and obtaining an expression for $\tan(\arg(z+w))$. Students who made a start by writing z and w in Cartesian form were awarded some partial credit.

Question 8 (11 marks) Use a SEPARATE writing booklet

(a) (i) Sketch the region in the Argand plane defined by: **2**

$$|z-1+i|<1 \text{ and } |\arg(z+i)| \leq \frac{\pi}{4}$$

(ii) Find the minimum value of $|z|$. **1**

(b) By considering $e^{i\theta}+e^{-i\theta}$ show that: **2**

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$

(c) (i) Use De Moivre's theorem to show that: **2**

$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

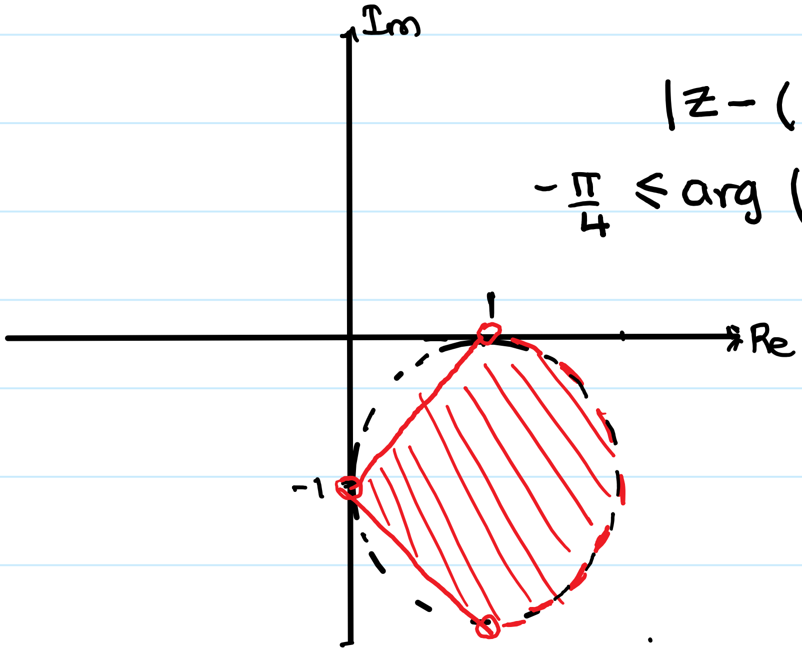
(ii) Hence, find the roots of the equation $32x^5-40x^3+10x+1=0$. **2**

(iii) Hence, using the results of (b) or otherwise, show that **2**

$$\left(1-2\cos\frac{2\pi}{5}\right)\left(1+2\cos\frac{\pi}{5}\right)=1$$

End of paper

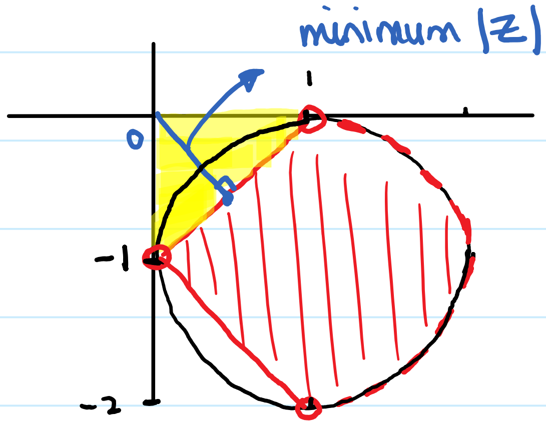
(a)



$|z-(1-i)|=1$

$-\frac{\pi}{4} \leq \arg(z-(-i)) \leq \frac{\pi}{4}$

b)



|z| is the distance of z from the origin. This is minimum when it is the perpendicular distance.

Calculating the area of the yellow triangle in two different ways

$$\cancel{\frac{1}{2}} \times 1 \times 1 = \cancel{\frac{1}{2}} \times \sqrt{2} \times |z|_{min}$$

$$|z|_{min} = \frac{1}{\sqrt{2}}$$

(b) Consider $e^{i\theta} = \cos\theta + i\sin\theta$ ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$ But cos is even sin is odd

$e^{-i\theta} = \cos\theta - i\sin\theta$ ②

① + ② $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad (*)$$

LHS = $\cos A \cos B$

$= \left(\frac{e^{iA}+e^{-iA}}{2}\right) \times \left(\frac{e^{iB}+e^{-iB}}{2}\right)$ using (*).

$= \frac{1}{4}\left(e^{iA}.e^{iB} + e^{iA}.e^{-iB} + e^{-iA}.e^{iB} + e^{-iA}.e^{-iB}\right)$

$= \frac{1}{4}\left(e^{i(A+B)} + e^{-i(A+B)} + e^{i(A-B)} + e^{-i(A-B)}\right)$

$= \frac{1}{2}\left[\frac{e^{i(A+B)}+e^{-i(A+B)}}{2} + \frac{e^{i(A-B)}+e^{-i(A-B)}}{2}\right]$

$= \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ using (*)

= RHS

(c) Consider $(\cos\theta + i\sin\theta)^5$ Let $c = \cos\theta$ $s = \sin\theta$

(i) $= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5c i^4 s^4 + i^5 s^5$

$= (c^5 - 10c^3 s^2 + 5cs^4) + i(5c^4 s - 10c^2 s^3 + s^5)$

Now $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre's thm)

Equating real parts of the two expressions

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1+\cos^4\theta - 2\cos^2\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

or $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

(ii) Let $x = \cos\alpha$

$\cos 5\alpha = 16x^5 - 20x^3 + 5x$

$2\cos 5\alpha = 32x^5 - 40x^3 + 10x$

$32x^5 - 40x^3 + 10x - 2\cos 5\alpha = 0$

Let $2\cos 5\alpha = -1$

$32x^5 - 40x^3 + 10x + 1 = 0$

Solutions of $32x^5 - 40x^3 + 10x + 1 = 0$ correspond to the solutions of $2\cos 5\alpha = -1$ ie. $\cos 5\alpha = \frac{-1}{2}$

$5\alpha = \frac{2\pi}{3} + 2k\pi, \quad \frac{4\pi}{3} + 2k\pi \qquad k \in \mathbb{Z}$

$\alpha = (6k+2)\frac{\pi}{15}, \quad (6k+4)\frac{\pi}{15}$

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$$\alpha = (6k+2)\frac{\pi}{15}, \ (6k+4)\frac{\pi}{15}$$

$$\alpha = \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{8\pi}{15}, \frac{10\pi}{15}, \frac{14\pi}{15}$$ (5 distinct solutions)

$$x = \cos\frac{2\pi}{15}, \cos\frac{4\pi}{15}, \cos\frac{8\pi}{15}, \overset{-\frac{1}{2}}{\cancel{\cos\frac{2\pi}{3}}}, \cos\frac{14\pi}{15}$$

(iii) Using product of roots for the equation in (ii)

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\left(-\frac{1}{2}\right)\cos\frac{14\pi}{15} = -\frac{1}{32}$$

$$\left(\cos\frac{2\pi}{15}\cos\frac{8\pi}{15}\right)\left(\cos\frac{4\pi}{15}\cos\frac{14\pi}{15}\right) \times \cancel{-\frac{1}{2}} = \cancel{-\frac{1}{32}} \ \frac{1}{16}$$

$$\cancel{\frac{1}{2}}\left[\cos\frac{10\pi}{15} + \cos\frac{6\pi}{15}\right] \times \cancel{\frac{1}{2}}\left[\cos\frac{18\pi}{15} + \cos\frac{10\pi}{15}\right] = \cancel{\frac{1}{16}} \ \frac{1}{4}$$

$$\left(\cos\frac{2\pi}{3} + \cos\frac{2\pi}{5}\right)\left(\cos\frac{6\pi}{5} + \cos\frac{2\pi}{3}\right) = \frac{1}{4}$$

$$\left(-\frac{1}{2} + \cos\frac{2\pi}{5}\right)\left(-\cos\frac{\pi}{5} - \frac{1}{2}\right) = \frac{1}{4}$$

$$\cancel{-\frac{1}{2}}\left(1 - 2\cos\frac{2\pi}{5}\right) \times \cancel{-\frac{1}{2}}\left(1 + 2\cos\frac{\pi}{5}\right) = \cancel{\frac{1}{4}} \ 1$$

$$\therefore \left(1 - 2\cos\frac{2\pi}{5}\right)\left(1 + 2\cos\frac{\pi}{5}\right) = 1$$ as required.

Question 8

(a) (i) A majority of students did not deal with the absolute value. If $\left|\arg(z+i)\right| \le \frac{\pi}{4}$ then $-\frac{\pi}{4} \le \arg(z+i) \le \frac{\pi}{4}$.

A number of students did not have the correct centre for the circle, or the correct starting point for the ray(s).

A few students did not dash the circle or keep the ray solid.

There should be open circles where the ray and circle intersect, as these points are not in the locus. Failure to do this was not penalised, but there did need to be an open circle at the start of the ray.

(ii) Poorly done. The main issue was with interpretation - the point on the circle which is closest to the origin is NOT in the locus.

(b) The result $\cos\theta = \frac{e^{ix}+e^{-ix}}{2}$ could not just be stated - it had to be derived.

Students who used the expansion results for $\cos(A \pm B)$ generally received no further marks, as they were not using exponential form as specified in the question.