NORTH SYDNEY GIRLS HIGH SCHOOL

2021 | HSC Assessment Task 3

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 41

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 7 minutes for this section

Section II – **36 marks** (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 48 minutes for this section

NAME:____________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

Question	1 – 5	6	7	8	Total
Integration	/2	/5	/7	/5	**/19**
Proof	/3	/6	/6	/7	**/22**
Total	/5	/11	/13	/12	**/41**

Section I

5 marks
Attempt Questions 1-5
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

---

1 Which of the following is a primitive of $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$?

A. $2e^{\sqrt{x}}$

B. $\dfrac{1}{2}e^{\sqrt{x}}$

C. $\left(e^{\sqrt{x}}\right)^2$

D. $\dfrac{1}{2}\left(e^{\sqrt{x}}\right)^2$

2 Which of the following is the result of integrating $\int \sin^3 \theta d\theta$?

A. $\dfrac{\sin^4 \theta}{4} + C$

B. $\cos^3 \theta + C$

C. $-\cos\theta + \dfrac{\cos^3 \theta}{3} + C$

D. $\cos\theta - \dfrac{\cos^3 \theta}{3} + C$

3 Consider the statement "$\forall x \in D, P(x) \vee Q(x)$".

Which of the following is the negation of this statement?

A. $\exists x \in D, \neg P(x) \wedge \neg Q(x)$

B. $\forall x \in D, \neg P(x) \vee \neg Q(x)$

C. $\exists x \in D, \neg P(x) \vee \neg Q(x)$

D. $\forall x \in D, \ \neg P(x) \wedge \neg Q(x)$

4 Consider the statement below:

If z is even, then x and y are either both even or both odd.

Which of the following is equivalent to the above statement?

A. If x and y are either both even or both odd, then z is even.

B. If z is odd, then exactly one of x and y is even.

C. If x and y are either both even or both odd, then z is odd.

D. If exactly one of x and y is even, then z is odd.

5 Given that $0 < a < b < 1$ and $n \in \mathbb{Z}^+$, which of the following statements is true?

A. $b^n < a^n$

B. $a^{n+1} < a^n$

C. $\sqrt{a} < a$

D. $ab < a^2$

BLANK PAGE

Section II

36 marks
Attempt Questions 6-8
Allow about 48 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks**)** Use a SEPARATE writing booklet

(a) Use integration by parts to find $\int x^2 e^x dx$. **3**

(b) Use a suitable method to find $\int \frac{2+x}{\sqrt{4-x^2}} dx$. **2**

(c) (i) Prove that if a^3 is even, then a is even. **2**

(ii) Use a proof by contradiction to establish that $\sqrt[3]{2}$ is irrational. **2**

(d) If $x \le 1 \le y$ show that $x + y \ge 1 + xy$. **2**

Question 7 (13 marks) Use a SEPARATE writing booklet

(a) Use the substitution $u = \tan\dfrac{y}{2}$ to find $\displaystyle\int \frac{1}{1+\sin y}\,dy$. **3**

(b) By first decomposing the integrand into partial fractions, **4**

evaluate $\displaystyle\int_2^4 \frac{3}{x^2+x-2}\,dx$.

Show all working.

(c) A recurrence relation is defined by $u_{n+1} = \dfrac{3u_n - 1}{4u_n - 1}$ and $u_1 = 1$. **3**

Use Mathematical Induction to prove that $u_n = \dfrac{n}{2n-1}$ for $n \geq 1$.

(d) (i) Prove that $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$. **1**

(ii) Hence, or otherwise, prove that for positive real numbers p, q, r, s **2**

$$\frac{p}{q} + \frac{q}{r} + \frac{r}{s} + \frac{s}{p} \geq 4.$$

Question 8 (12 marks) Use a SEPARATE writing booklet

(a) Consider $I_n = \int_0^1 x^n \sqrt{1-x^2}\,dx \quad , \; n \geq 0$.

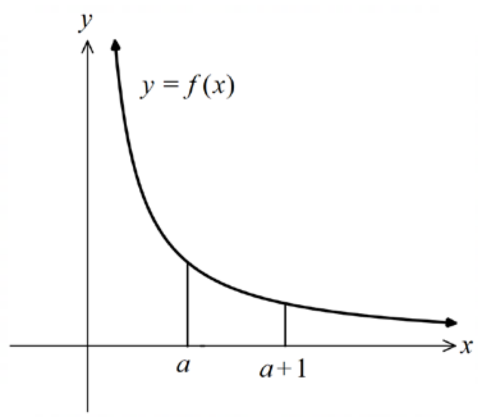
(i) Use integration by parts to show that $I_n = \dfrac{n-1}{n+2} I_{n-2}$. **3**

(ii) Hence, evaluate $\int_0^1 x^4 \sqrt{1-x^2}\,dx$. **2**

(b) Use the principle of Mathematical Induction to prove that **3**

$$(2n)! > 2^n (n!)^2 \text{ , for } n \geq 2$$

(c) A part of the graph of $y = \dfrac{1}{x}$ is shown below.



(i) By considering areas, show that for $a > 0$: **2**

$$\frac{1}{a+1} < \log_e\left(\frac{a+1}{a}\right) < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)$$

(ii) Hence deduce that $\dfrac{5}{6} < \ln 3 < \dfrac{7}{6}$. **2**

End of paper

BLANK PAGE

NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2021 | HSC Assessment Task 3

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 41

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 7 minutes for this section

Section II – 36 marks (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 48 minutes for this section

NAME:______________________________ TEACHER:___________________

STUDENT NUMBER:_________________________

Question	1 – 5	6	7	8	Total
Integration	/2	/5	/7	/5	**/19**
Proof	/3	/6	/6	/7	**/22**
Total	/5	/11	/13	/12	**/41**

Section I

5 marks
Attempt Questions 1-5
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

---

1 Which of the following is a primitive of $\frac{e^{\sqrt{x}}}{\sqrt{x}}$?

A. $2e^{\sqrt{x}}$

B. $\frac{1}{2}e^{\sqrt{x}}$

C. $\left(e^{\sqrt{x}}\right)^2$

D. $\frac{1}{2}\left(e^{\sqrt{x}}\right)^2$

$$\frac{e^{\sqrt{x}}}{\sqrt{x}} = 2\left[e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}\right]$$

2 Which of the following is the result of integrating $\int \sin^3 \theta d\theta$?

A. $\frac{\sin^4 \theta}{4} + C$

B. $\cos^3 \theta + C$

C. $-\cos\theta + \frac{\cos^3 \theta}{3} + C$

D. $\cos\theta - \frac{\cos^3 \theta}{3} + C$

$$\int \sin^3\theta d\theta = \int \sin\theta\,(1-\cos^2\theta)d\theta$$
$$= \int(\sin\theta - \sin\theta\cos^2\theta)d\theta$$
$$= -\cos\theta + \frac{\cos^3\theta}{3} + C$$

3 Consider the statement "$\forall x \in D, P(x) \vee Q(x)$".
Which of the following is the negation of this statement?

A. $\exists x \in D, \neg P(x) \wedge \neg Q(x)$

B. $\forall x \in D, \neg P(x) \vee \neg Q(x)$

C. $\exists x \in D, \neg P(x) \vee \neg Q(x)$

D. $\forall x \in D, \neg P(x) \wedge \neg Q(x)$

4 Consider the statement below:
If z is even, then x and y are either both even or both odd.

Which of the following is equivalent to the above statement?

A. If x and y are either both even or both odd, then z is even.

B. If z is odd, then exactly one of x and y is even.

C. If x and y are either both even or both odd, then z is odd.

D. If exactly one of x and y is even, then z is odd.

5 Given that $0 < a < b < 1$ and $n \in \mathbb{Z}^+$, which of the following statements is true?

A. $b^n < a^n$

B. $a^{n+1} < a^n$

C. $\sqrt{a} < a$

D. $ab < a^2$

Question 6

Monday, 31 May 2021 6:10 AM

a) $\int x^2 e^x \, dx$

$u = x^2$	$v' = e^x$
$u' = 2x$	$v = e^x$
$u = x$	$v' = e^x$
$u' = 1$	$v = e^x$

$= x^2 e^x - \int 2x e^x \, dx$ ✓

$= x^2 e^x - 2\int x e^x \, dx$

$= x^2 e^x - 2\left[x e^x - \int e^x \, dx\right]$

$= x^2 e^x - 2x e^x + 2e^x + C$ ✓

b) $\int \frac{2+x}{\sqrt{4-x^2}} \, dx = \int \frac{2}{\sqrt{4-x^2}} \, dx + \int \frac{x}{\sqrt{4-x^2}} \, dx$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int (-2x)(4-x^2)^{-\frac{1}{2}} \, dx$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{\cancel{2}} \times \cancel{2}(4-x^2)^{\frac{1}{2}}$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C$

✓ ✓

c) (i) RTP: If a^3 is even then a is even

Contrapositive: If a is odd then a^3 is odd.

If a is odd, then $a = 2k+1$ for some $k \in \mathbb{Z}$ ✓

$a^3 = (2k+1)^3$

$= 8k^3 + 12k^2 + 6k + 1$

$= 2(4k^3 + 6k^2 + 3k) + 1$

$= 2M + 1$ where $M = 4k^3 + 6k^2 + 3k \in \mathbb{Z}$

$\therefore a^3$ is odd. ✓

Thus, by the contrapositive, if a^3 is even, a is even.

(ii) RTP : $\sqrt[3]{2}$ is irrational

Assume for the sake of contradiction that $\sqrt[3]{2}$ is rational.

Then $\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$
HCF $(p, q) = 1$ ✓ $\frac{1}{2}$

Cube both sides

$$2 = \frac{p^3}{q^3}$$

$$2q^3 = p^3$$

$\therefore$ p^3 is even. Then by part (a) p is even ✓ $\frac{1}{2}$

So $p = 2k$ for some $k \in \mathbb{Z}$

$$\therefore\ 2q^3 = (2k)^3$$

$$1\cancel{2}q^3 = 4\cancel{8}k^3$$

$$q^3 = 2(2k^3)$$

Thus, q^3 is even. By (a) q is even ✓ $\frac{1}{2}$

So p is even & q is even, so 2 is a common factor

But HCF $(p, q) = 1$ by assumption.

$\therefore$ we have a contradiction. ✓ $\frac{1}{2}$

$\therefore$ $\sqrt[3]{2}$ must be irrational.

d) Consider $x + y - xy - 1 = x(1-y) - (1-y)$

$$= (x-1)(1-y) \quad ✓$$

$$\geq 0$$

as $x \leq 1$ so $x - 1 \leq 0$ and $1 \leq y$ so $1 - y \leq 0$

and multiplying two negatives yields a positive ✓

so $x + y - xy - 1 \geq 0$ or $x + y \geq 1 + xy$

Question 6

(a) Well done. Some students either didn't recognise that IBP was required a second time or make an incorrect choice of u and v for the second application. There were some careless errors in applying the -2 to the second term of the second application.

(b) Generally well done. A number of students thought that the second integral led to a log. Some got incorrect coefficients, either because they didn't apply the arcsin integral rule properly or because they couldn't divide $\frac{1}{2}$ by $\frac{1}{2}$.

(c) (i) Well done. There were some careless errors in expanding the cube. It was frustrating to see how many students chose not to use binomial theorem to expand a simple cube, preferring a mult-step multiplication. A couple of students proved the converse and could be awarded no marks.

(ii) Generally well done. Part (i) was there to help with this part. I expected you to reference part (i) to explain why $\left(p^3 \text{ even}\right) \Rightarrow \left(p \text{ even}\right)$. A number of students believed they already had a contradiction half way through the proof and stopped. A couple of students believed that $\sqrt[3]{2} = 2^{\frac{3}{2}}$.

(d) Very poorly done. This is the simplest type of inequality proof. Students who didn't factorise (ie. most students) generally made an assortment of logical errors and could be awarded little if anything. You can't multiply an inequality by x if you don't know whether x is positive or negative. Do not start with the result to be proved or a rearrangement of it such as $x+y-1-xy \geq 0$. You are assuming the result to be proved. Instead start with $x+y-1-xy$ and show by algebraic manipulation that this is positive. You **could** assume the negation of the result and do a proof by contradiction or contraposition.

Question 7

Friday, 28 May 2021 9:27 pm

a) $\displaystyle\int \frac{1}{1+\sin y}\,dy$

$u = \tan\frac{y}{2}$

$\frac{y}{2} = \tan^{-1}u$

$y = 2\tan^{-1}u$

$dy = \frac{2\,du}{1+u^2}$

$\sin y = \frac{2u}{1+u^2}$ ✓

$= \displaystyle\int \frac{1}{1+\frac{2u}{1+u^2}} \times \frac{2\,du}{1+u^2}$

$= \displaystyle\int \frac{\cancel{1+u^2}}{1+u^2+2u} \times \frac{2\,du}{\cancel{1+u^2}}$

$= 2\displaystyle\int \frac{du}{(u+1)^2}$

$= 2\displaystyle\int (u+1)^{-2}\,du$

$= 2\,\frac{(u+1)^{-1}}{-1} + C$ ✓

$= \frac{-2}{u+1} + C$

$= \frac{-2}{\tan\frac{y}{2}+1} + C$ ✓

b) Consider $\dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)}$

$$\frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Equating numerators: $3 = A(x-1) + B(x+2)$

Sub $x = 1$: $3 = B(3)$

$B = 1$

Sub $x = -2$: $3 = A(-3)$

$A = -1$

Thus $\dfrac{3}{x^2+x-2} = \dfrac{-1}{x+2} + \dfrac{1}{x-1}$

$$\int_2^4 \frac{3}{x^2+x-2}\,dx = \int_2^4 \left(\frac{-1}{x+2} + \frac{1}{x-1}\right) dx$$

$$= \Big[-\ln|x+2| + \ln|x-1|\Big]_2^4$$

$$= \Big[(-\ln 6 + \ln 3) - (-\ln 4 + \ln 1)\Big]$$

$$= \ln 3 - \ln 6 + \ln 4$$

$$= \ln\left(\frac{3 \times 4}{6}\right)$$

$$= \ln 2$$

c) Given $u_{n+1} = \dfrac{3u_n - 1}{4u_n - 1}$, to prove $u_n = \dfrac{n}{2n-1}$

Let $S(n)$: $u_n = \dfrac{n}{2n-1}$

Test $n=1$

$LHS = u_1 = 1$ $\quad$ $RHS = \dfrac{1}{2(1)-1} = \dfrac{1}{1} = 1$ ✓

$LHS = RHS$ $\quad \therefore S(1)$ is true.

Assume $S(k)$ is true ie. $u_k = \dfrac{k}{2k-1}$

Prove $S(k+1)$ is true

ie. RTP $u_{k+1} = \dfrac{k+1}{2(k+1)-1} = \dfrac{k+1}{2k+1}$

$LHS = u_{k+1} = \dfrac{3u_k - 1}{4u_k - 1}$ using given recurrence relation

$$= \frac{3\left(\frac{k}{2k-1}\right) - 1}{4\left(\frac{k}{2k-1}\right) - 1} \begin{matrix} \times(2k-1) \\ \times(2k-1) \end{matrix}$$ using assumption ✓

$$= \frac{3k - (2k-1)}{4k - (2k-1)}$$

$$= \frac{3k - 2k + 1}{4k - 2k + 1}$$

$$= \frac{k+1}{2k+1}$$ ✓

$= RHS$

$\therefore S(k+1)$ is true when $S(k)$ is true

$\therefore S(n)$ is true by Mathematical Induction d)(i) Consider $(\sqrt{a}-\sqrt{b})^2 \geq 0$ perfect square

$$a+b-2\sqrt{ab} \geq 0$$

$$a+b \geq 2\sqrt{ab}$$ ✓

(ii) $\frac{p}{q}+\frac{q}{r} \geq 2\sqrt{\frac{p}{\cancel{q}} \times \frac{\cancel{q}}{r}}$ using (i) with $a=\frac{p}{q}$, $b=\frac{q}{r}$

$\therefore$ $\frac{p}{q}+\frac{q}{r} \geq 2\sqrt{\frac{p}{r}}$ (1)

Similarly using (i) with $a=\frac{r}{s}$, $b=\frac{s}{p}$

$\frac{r}{s}+\frac{s}{p} \geq 2\sqrt{\frac{r}{p}}$ (2) ✓

(1) + (2) yields

$$\frac{p}{q}+\frac{q}{r}+\frac{r}{s}+\frac{s}{p} \geq 2\left(\sqrt{\frac{p}{r}}+\sqrt{\frac{r}{p}}\right)$$

$$\geq 2\left(2\sqrt{\sqrt{\frac{p}{r}} \times \sqrt{\frac{r}{p}}}\right)$$ using (i) with $a=\sqrt{\frac{p}{r}}$, $b=\sqrt{\frac{r}{p}}$

$$= 4\sqrt[4]{\frac{\cancel{p}}{\cancel{r}} \times \frac{\cancel{r}}{\cancel{p}}}$$

$$= 4.$$ ✓

$\therefore$ $\frac{p}{q}+\frac{q}{r}+\frac{r}{s}+\frac{s}{p} \geq 4$

Question 7

(a) Mostly well done. Some students had difficulty making the switch to *y* and *u* instead of *x* and *t*, but generally, corrected this later. Students who rearranged the substitution $u = \tan\frac{y}{2}$ to $y = 2\tan^{-1} u$ before differentiating were generally more likely to obtain the correct expression for dy. Students who differentiated directly to get an expression involving $\sec^2 \frac{y}{2}$ at times incorrectly used the chain rule or did not adequately show how they rearranged the result to get dy. Most students could then correctly integrate the resulting integral. A small number of students could not correctly simplify the integrand or forgot to rewrite the result in terms of the original variable *y*.

(b) Very well done. A small number of students made numerical errors and are advised to be more careful when dealing with routine questions such as this one. Some students are using less efficient methods when decomposing the fractions and are advised to study the suggested solutions for more efficient working.

(c) Generally well done. A small number of students incorrectly believed that this question required strong induction and verifies two base cases and assumed for two values before proving the subsequent value. This was not penalised. A recurrence relation specifies the relationship with previous terms and an initial or seed value. Thus, in this question u_1 is the initial case. Some students were confused about the initial value and tested u_0. This was not penalised. A small number of students did not know how to prove this result using induction. Either they were confused about what result they were proving and attempted to prove $u_{n+1} = \frac{3u_n - 1}{4u_n - 1}$ or simply evaluated a number of terms and did not commence the proof. The recurrence relation is given and can be used. The formula for u_n is the result required to be proved. This is a particular style of question which is listed in the syllabus and students need to recognise and learn how to set out the proof for Induction proofs of such results.

(d) (i) Generally well done. Some students are using less efficient ways to prove this very standard result and are advised to review the solutions.

(ii) Not well done. This part required successive applications of the result in (i) with new variables. Many students attempted to use the result in (i) but used it with $\frac{p}{q}$ and $\frac{q}{p}$. As $\frac{q}{p}$ is not a term in the required result, this does not lead anywhere. The most efficient way to deal with this question is modelled in the solutions. Some students manipulated the required result to combine the fractions in the LHS. Disappointingly, in many of these attempts algebraic errors in combining the fractions altered the required result. Those who dealt with the algebra correctly could often make progress towards the required result or successfully obtain the required result.

Question 8

Monday, [illegible] May [illegible] 6:10 AM

a) (i) $I_n = \int_0^1 x^n \sqrt{1-x^2}\,dx$

$$u = x^{n-1} \qquad v' = x\sqrt{1-x^2}$$
$$u' = (n-1)x^{n-2} \qquad v = -\frac{1}{\cancel{2}}(1-x^2)^{\frac{3}{2}} \times \frac{\cancel{2}}{3}$$
$$= -\frac{1}{3}(1-x^2)^{3/2}$$

$$= -\frac{1}{3}\left[x^{n-1}(1-x^2)^{3/2}\right]_0^1 - \int_0^1 (n-1)x^{n-2}\cdot\left(-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right)dx$$

$$= -\frac{1}{3}(0-0) + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$$

$$= \frac{n-1}{3}\left[\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx - \int_0^1 x^n\sqrt{1-x^2}\,dx\right]$$

$$I_n = \frac{n-1}{3} I_{n-2} - \frac{n-1}{3} I_n$$

$$I_n + \frac{n-1}{3} I_n = \frac{n-1}{3} I_{n-2}$$

$$\left(\frac{3+n-1}{3}\right) I_n = \frac{n-1}{3} I_{n-2}$$

$$\frac{n+2}{3} I_n = \frac{n-1}{3} I_{n-2}$$

$$I_n = \frac{n-1}{n+2} I_{n-2} \quad \text{as required.}$$

Monday, 31 May 2021 6:10 AM

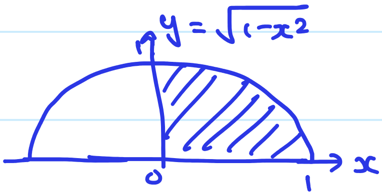
(ii) $\int_0^1 x^4 \sqrt{1-x^2}\, dx = I_4$

$I_4 = \frac{3}{6} I_2$ using (i)

$= \frac{1}{2} I_2$

$= \frac{1}{2} \times \frac{1}{4} I_0$ ✓

$= \frac{1}{8} \int_0^1 \sqrt{1-x^2}\, dx$



$= \frac{1}{8} \times \frac{\pi (1)^2}{4}$ area of quarter circle

$= \frac{\pi}{32}$ ✓

b) $S(n) : (2n)! > 2^n (n!)^2$ for $n \geq 2$

Test $S(2)$

$LHS = (2\times 2)!$ $\quad$ $RHS = 2^2 \times (2!)^2$

$= 4!$ $\quad$ $= 4 \times 2^2$

$= 24$ $\quad$ $= 16$

$LHS > RHS$ $\therefore$ $S(1)$ is true ✓

Assume $S(k)$ is true

Prove $S(k+1)$ is true

ie. RTP $(2(k+1))! > 2^{k+1}((k+1)!)^2$

$LHS = (2(k+1))!$

$= (2k+2)!$

$= (2k+2)(2k+1)(2k)!$

$> (2k+2)(2k+1) \quad 2^k.(k!)^2$ using assumption ✓

$= 2(k+1)(2k+1) \quad 2^k (k!)^2$

$> 2(k+1)(k+1) \quad 2^k (k!)^2$ as $2k+1 > k+1$

$= 2.2^k \; (k+1)k! \times (k+1)k!$

$= 2^{k+1}(k+1)! \; (k+1)!$

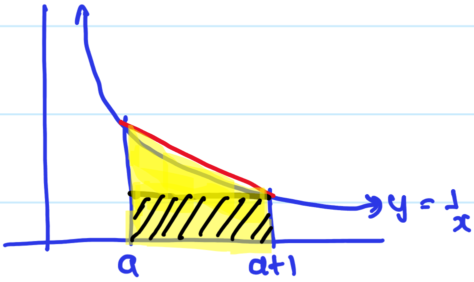
$= 2^{k+1}((k+1)!)^2$ ✓

$= RHS$

$LHS > RHS$ $\therefore$ $S(k+1)$ is true

$\therefore$ $S(n)$ is true by Mathematical Induction.

c)
(i)



Area lower rectangle < Exact area under curve < Area of trapezium

$$[(\cancel{a}+1)-\cancel{a}]\frac{1}{a+1} < \int_a^{a+1}\frac{1}{x}\,dx < \frac{1}{2}\left[\frac{1}{a}+\frac{1}{a+1}\right] \quad ✓$$

$$\frac{1}{a+1} < \left[\ln x\right]_a^{a+1} < \frac{1}{2}\left[\frac{1}{a}+\frac{1}{a+1}\right]$$

$$\frac{1}{a+1} < \ln(a+1) - \ln a < \frac{1}{2}\left(\frac{1}{a}+\frac{1}{a+1}\right]$$

NB: $a, a+1 > 0$

$$\frac{1}{a+1} < \ln\left(\frac{a+1}{a}\right) < \frac{1}{2}\left[\frac{1}{a}+\frac{1}{a+1}\right] \quad ✓$$

(ii) Using (i) with $a = 1$

$$\frac{1}{2} < \ln\left(\frac{2}{1}\right) < \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)$$

$$\frac{1}{2} < \ln 2 < \frac{3}{4} \qquad (1)$$

Using (i) with $a = 2$

$$\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)$$

$$\frac{1}{3} < \ln 3 - \ln 2 < \frac{5}{12} \qquad (2) \quad ✓$$

Adding (1) & (2) $\quad \frac{1}{2}+\frac{1}{3} < \cancel{\ln 2} + \ln 3 - \cancel{\ln 2} < \frac{3}{4}+\frac{5}{12}$

$$\frac{5}{6} < \ln 3 < \frac{\cancel{14}\ 7}{\cancel{12}\ 6}$$

$$\therefore \quad \frac{5}{6} < \ln 3 < \frac{7}{6} \qquad \text{as required} \quad ✓$$

Question 8

a) i) It was crucial to be able to see that an x was needed to be combined with the square root section so that it could be integrated. $\int_0^1 x^n \sqrt{1-x^2}\,dx = \int_0^1 \left(x^{n-1}\right)\left(x\sqrt{1-x^2}\right)dx$.

Almost all of the students who could see this were able to achieve 3 marks. However, there were a few errors with finding u' or v. Other errors from here were taking out constants or a few errors with negatives.

Students who did not choose correct expressions for u or v' could not proceed with this but were able to achieve 1 out of three if they applied the integration by parts correctly.

ii) Most students were able to get the first mark which was for developing the expression for I_4 in terms of I_0 , However very few got the second mark for correctly evaluating I_0. Most students decided to evaluate I_0 by using a trig substitution, whilst this is possible it is time consuming and many students made an error whilst attempting to evaluate it using this method. Only a few students could see that I_0 is the area of a quarter circle with radius 1.

There were a number of students who did not attempt to use the result from part i) to evaluate I_4 and instead attempted to evaluate it directly.

b) The vast majority of student achieved full marks for this question. Most errors occurred when students tried to prove by showing that $LHS - RHS > 0$ or made simple algebraic errors.

c) i) The students who attempted this question needed to establish that the exact area under the curve between a and a+1 was sandwiched between the lower rectangle and the trapezium. The best way to do this was by using a diagram.
Many students stated that the $\frac{1}{2}\left(\frac{1}{a}+\frac{1}{a+1}\right)$ was the area of the "upper rectangle".

ii) No student achieved both marks for this question. Many attempted to find $\ln 3$ by letting $\frac{a+1}{a}=3$ which gave $a=\frac{1}{2}$. This does yield an inequality, but unfortunately it is not the one required in the question. This was given 1 mark out of 2.