NORTH SYDNEY GIRLS HIGH SCHOOL

2022 | **HSC Assessment Task 1**

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 58 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Section II, show relevant mathematical reasoning and/or calculations

Total marks – 39

Section I – 6 marks (pages 2 – 4)

- Attempt Questions 1 – 6
- Allow about 8 minutes for this section

Section II – 33 marks (pages 5 – 7)

- Attempt Questions 7 – 10
- Allow about 50 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER

Question	1 – 6	7	8	9	10	Total
Complex Numbers	/6	/9	/7	/9	/8	/39

Section I

6 marks
Attempt Questions 1-6
Allow about 8 minutes for this section

Use the multiple choice answer sheet for Questions 1-6.

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1 What is $-1+i\sqrt{3}$ in exponential form?

A. $2e^{i\frac{\pi}{3}}$

B. $2e^{i\frac{2\pi}{3}}$

C. $2e^{i\frac{4\pi}{3}}$

D. $2e^{i\frac{5\pi}{3}}$

2 The equation $x^2-2ix+k=0$ has a root of $-1+2i$. What is the value of k?

A. $-1+2i$

B. $-1-2i$

C. 1

D. 5

3 If $z=\omega$ is a non-real solution of the equation $z^3=1$, which of the following expressions is equal to $\left(1+\omega-\omega^2\right)^7$?

A. 128ω

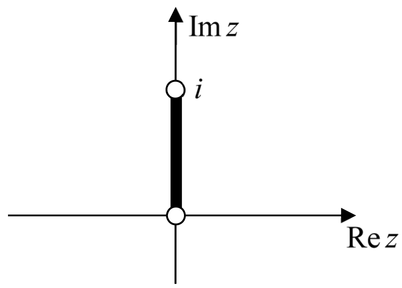
B. -128ω

C. $128\omega^2$

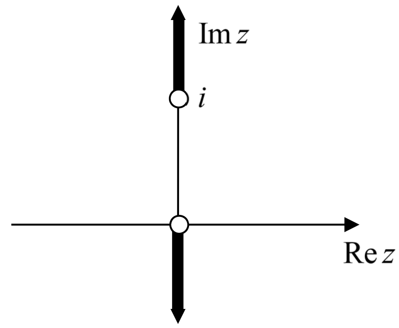
D. $-128\omega^2$

4 Which of the following loci is represented by the equation $\arg z = \arg i$?

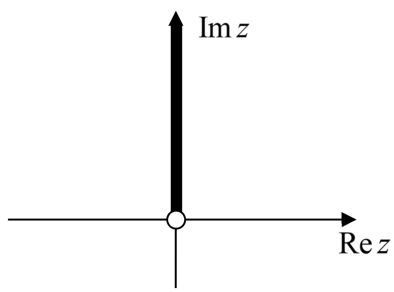
A.



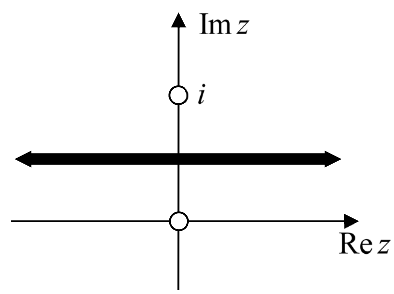
B.



C.



D.



5 Given $z_1 = 3\operatorname{cis}\theta$ and $z_2 = 7\operatorname{cis}\phi$, which of the following could be $|z_1 + z_2|$?

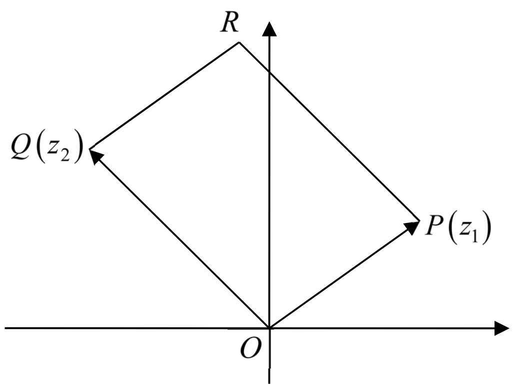
A. 1

B. 3

C. 7

D. 11

6 In the following diagram, $OPRQ$ is a parallelogram, and the vectors OP and OQ represent the complex numbers z_1 and z_2 respectively.



NOT TO SCALE

If $\dfrac{z_2}{z_1} = \sqrt{3}\,i$, what is the value of $\left|\dfrac{z_1 + z_2}{z_2 - z_1}\right|$?

A. 1

B. $\sqrt{3}$

C. $\dfrac{1}{\sqrt{3}}$

D. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

Section II

33 marks
Attempt Questions 7-10
Allow about 50 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 7-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 7 (9 marks)

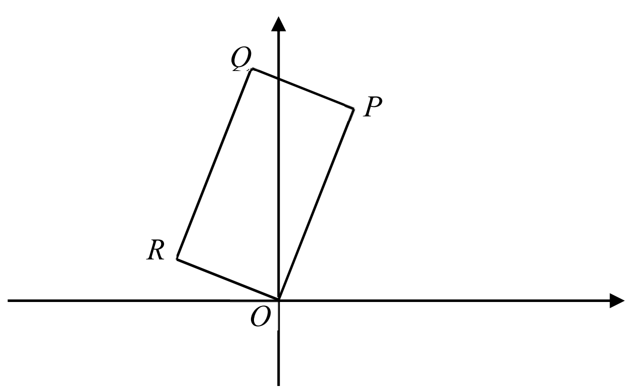
(a) Given $z = 2 - i$ and $w = 3 - 2i$, find in the form $a + ib$:

(i) $z + \overline{w}$ **1**

(ii) $\dfrac{z}{w}$ **2**

(b) Solve the equation $16i\,z^2 + (8 + 8i)z - 10 + 5i = 0$, expressing answers in the form $a + ib$. **4**

(c) $OPQR$ is a rectangle and $OP = 3OR$. **2**



If $\overrightarrow{OQ}$ represents the complex number $-1 + 6i$, determine what complex number is represented by the vector $\overrightarrow{OR}$.

Question 8 (7 marks) Start a new booklet.

(a) Solve the equation $z^5 = 4 + 4i$. Give your answers in exponential form. **3**

(b) The polynomial $P(x) = x^4 - 8x^3 + 24x^2 - 32x + 20$ has $1 + i$ as a zero.

(i) Express $P(x)$ as a product of two real quadratic factors. **3**

(ii) Explain briefly why $P(x) > 0$ for all real values of x. **1**

Question 9 (9 marks) Start a new booklet.

(a) (i) On the same Argand diagram, sketch the locus of all complex numbers z which satisfy: **2**

$$|z-3| = 1 \qquad \text{or} \qquad |z| = |z-2i|$$

(ii) Hence write down the complex number(s) which satisfy: **1**

$$|z-3| = 1 \qquad \text{and} \qquad |z| = |z-2i|$$

(b) (i) Given that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for positive values of n, deduce that it is also true for negative values of n. **2**

(ii) Hence show that if $Z = \cos\theta + i\sin\theta$ then $Z^n + Z^{-n} = 2\cos n\theta$. **1**

(iii) By considering $\left(Z + Z^{-1}\right)^4$, show that $\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$. **3**

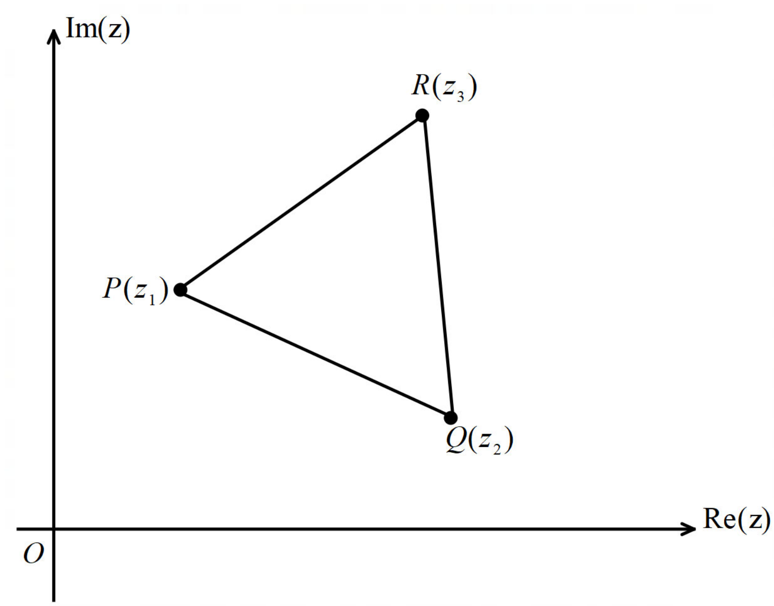
Question 10 (8 marks) Start a new booklet.

(a) You are given that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. (DO NOT PROVE THIS) **3**

Without using any other trigonometric identities, use this result to show that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

(b) z_1, z_2 and z_3 are three complex numbers which are represented by the points P, Q, R respectively in the complex plane, where ΔPQR is equilateral.



(i) Explain why $\dfrac{z_3 - z_1}{z_2 - z_1} = \text{cis}\left(\dfrac{\pi}{3}\right)$. **1**

(ii) Hence or otherwise, show that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$. **2**

(iii) If z_1, z_2 and z_3 are the three roots of the cubic equation $z^3 + az^2 + bz + c = 0$, find a relationship between a and b. **2**

End of paper

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2022 Ext 2 Ass 1 Solutions

Section I

Summary: 1. B 2. A 3. D
4. C 5. C 6. A

① $|-1+i\sqrt{3}| = 2$, $\arg(-1+i\sqrt{3}) = \frac{2\pi}{3}$

$\therefore -1+i\sqrt{3} = \underline{2e^{i\cdot\frac{2\pi}{3}}}$

② $(-1+2i)^2 - 2i(-1+2i) + k = 0$

$1 - 4i - 4 + 2i + 4 + k = 0$

$\underline{k = -1+2i}$

③ $(1+\omega-\omega^2)^7 = (1+\omega+\omega^2 - 2\omega^2)^7$

$= (0 - 2\omega^2)^7$

$= -128\omega^{14}$

$= -128(\omega^3)^4 \cdot \omega^2$

$= -128(1)^4\omega^2$

$= \underline{-128\omega^2}$

⑤ $\big||z_1| - |z_2|\big| < |z_1 + z_2| < |z_1| + |z_2|$

$|3-7| < |z_1+z_2| < 3+7$

$4 < |z_1+z_2| < 10$

$\therefore |z_1+z_2| = 7$

④ $\arg z = \arg i$

$= \frac{\pi}{2}$

⑥ $\frac{z_2}{z_1} = \sqrt{3}\,i \Rightarrow |z_2| = \sqrt{3}\cdot|z_1|$

and $OQ \perp OP \Rightarrow OPRQ$ is a rectangle

$\therefore |z_1+z_2| = |z_1 - z_2|$ (diagonals of rectangle equal)

$\therefore \left|\frac{z_1+z_2}{z_1-z_2}\right| = 1$

Question 7

(a) (i) $z + \bar{w} = (2-i) + (3+2i)$

$$= \boxed{5+i}$$

1

(ii) $$\frac{z}{w} = \frac{2-i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{6+4i-3i+2}{9+4}$$

$$= \boxed{\frac{8}{13} + \frac{1}{13}i}$$

2

(b)

$$16iz^2 + (8+8i)z - 10 + 5i = 0$$

$$z = \frac{-(8+8i) \pm \sqrt{(8+8i)^2 - 64i(-10+5i)}}{32i}$$

$$= \frac{-8-8i \pm \sqrt{320+768i}}{32i}$$

$$= \frac{-1-i \pm \sqrt{5+12i}}{4i}$$

let $\sqrt{5+12i} = a+ib$, a, b real

$$5+12i = (a^2-b^2) + 2abi$$

equating real & imag. parts:

$a^2 - b^2 = 5$ ①

$2ab = 12 \Rightarrow b = \frac{6}{a}$ ②

sub ② in ①: $a^2 - \frac{36}{a^2} = 5$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2-9)(a^2+4) = 0$$

$$a = \pm 3$$

$$b = \pm 2$$

$$\therefore \sqrt{5+12i} = \pm(3+2i)$$

$$Z = \frac{-1-i \pm (3+2i)}{4i}$$

$$= \frac{2+i}{4i} \times \frac{-i}{-i} \; , \; \frac{-4-3i}{4i} \times \frac{-i}{-i}$$

$$= \frac{1-2i}{4} \; , \; \frac{-3+4i}{4}$$

$$= \boxed{\frac{1}{4} - \frac{1}{2}i \, , \; -\frac{3}{4} + i}$$

OR

$$16iz^2 + (8+8i)z - 10 + 5i = 0$$

$$\times -i: \quad 16z^2 + (8-8i)z + 5 + 10i = 0$$

$$16z^2 + (8-8i)z - 2i = -5 - 12i$$

$$16z^2 + (8-8i)z + (1-i)^2 = i^2(3+2i)^2$$

$$[4z + (1-i)]^2 = i^2(3+2i)^2$$

$$4z + (1-i) = \pm i(3+2i)$$

$$4z + (1-i) = \pm(-2+3i)$$

$$4z = -3+4i, \; 1-2i$$

$$\boxed{z = -\frac{3}{4} + i, \; \frac{1}{4} - \frac{1}{2}i}$$

4

(c) $\vec{OQ} = \vec{OP} + \vec{OR}$

$$-1+6i = -3i \cdot \vec{OR} + \vec{OR}$$

$$= (1-3i) \cdot \vec{OR}$$

$$\vec{OR} = \frac{-1+6i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \boxed{\frac{-19}{10} + \frac{3}{10}i}$$

2

Question 8

(a) $z^5 = 4 + 4i$

$= 4\sqrt{2}\, \text{cis}\left(\frac{\pi}{4} + 2k\pi\right), \ k \in \mathbb{Z}$

$= (\sqrt{2})^5\, \text{cis}\, \frac{8k+1}{4}\pi$

$z = \sqrt{2}\, \text{cis}\, \frac{8k+1}{20}\pi$

$k = -2, -1, 0, 1, 2 : \quad z = \sqrt{2}\, \text{cis}\left(-\frac{3\pi}{4}\right), \sqrt{2}\, \text{cis}\left(\frac{-7\pi}{20}\right), \sqrt{2}\, \text{cis}\frac{\pi}{20}, \sqrt{2}\, \text{cis}\frac{9\pi}{20}, \sqrt{2}\, \text{cis}\frac{17\pi}{20}$

$\boxed{z = \sqrt{2}e^{-i\frac{3\pi}{4}}, \sqrt{2}e^{-i\frac{7\pi}{20}}, \sqrt{2}e^{i\frac{\pi}{20}}, \sqrt{2}e^{i\frac{9\pi}{20}}, \sqrt{2}e^{i\frac{17\pi}{20}}}$ 3

(b)(i) $P(x) = x^4 - 8x^3 + 24x^2 - 32x + 20 = 0$

$P(1+i) = 0$ (given)

$\therefore P(1-i) = 0$ (non-real roots of real polynomials occur in conjugate pairs)

let other roots be α, β

sum: $(1+i) + (1-i) + \alpha + \beta = 8 \Rightarrow 2 + \alpha + \beta = 8 \Rightarrow \alpha + \beta = 6$

product: $(1+i)(1-i)\alpha\beta = 20 \Rightarrow 2\alpha\beta = 20 \Rightarrow \alpha\beta = 10$

$\therefore \boxed{P(x) = (x^2 - 2x + 2)(x^2 - 6x + 20)}$ 3

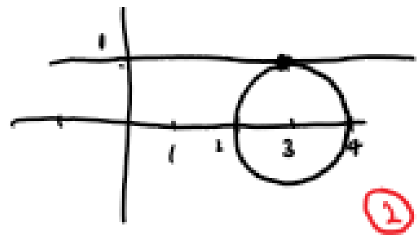
(ii) $x^2 - 2x + 2 = (x-1)^2 + 1 > 0 \quad \forall x \in \mathbb{R}$

$x^2 - 6x + 20 = (x-3)^2 + 11 > 0 \quad \forall x \in \mathbb{R}$

$\therefore P(x) > 0 \quad \forall x \in \mathbb{R}$ 1

Question 9

(a) (i)



2

(ii) $3+i$ 1

(b) (i) Let $m=-n$ where $n>0$, so $m<0$

$$(\cos\theta+i\sin\theta)^m = (\cos\theta+i\sin\theta)^{-n}$$
$$= \frac{1}{(\cos\theta+i\sin\theta)^n}$$
$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$
$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$
$$= \frac{\cos(-n\theta)+i\sin(-n\theta)}{1}$$
$$= \cos m\theta + i\sin m\theta$$

2

(ii) $$z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos(-n\theta)+i\sin(-n\theta))$$
$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$
$$= 2\cos n\theta$$

1

(iii) $$z^4 + z^{-4} = (z+z^{-1})^4 - 4(z^2+z^{-2}) - 6$$
$$= (z+z^{-1})^4 - 4\left[(z+z^{-1})^2 - 2\right] - 6$$
$$= (z+z^{-1})^4 - 4(z+z^{-1})^2 + 2$$
$$2\cos 4\theta = (2\cos\theta)^4 - 4(2\cos\theta)^2 + 2$$
$$= 16\cos^4\theta - 16\cos^2\theta + 2$$
$$8\cos^4\theta = \cos 4\theta + 8\cos^2\theta - 1$$
$$= \cos 4\theta + 8\times\tfrac{1}{2}(1+\cos 2\theta) - 1$$
$$= \cos 4\theta + 4\cos 2\theta + 3$$
$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

3

$$(z^1 + z^{-1})^4 = (2\cos\theta)^4$$
$$= 16\cos^4\theta \qquad (1)$$

Also

$$(z^1 + z^{-1})^4 = z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}$$
$$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$$
$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$
$$= 2\cos 4\theta + 8\cos 2\theta + 6. \qquad (2)$$

Equating expressions (1) & (2)

$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$
$$= 2[\cos 4\theta + 4\cos 2\theta + 3]$$
$$\cos^4\theta = \frac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$$

Question 10

(a) $\sin\theta = \dfrac{e^{-i\theta}-e^{-i\theta}}{2i}$

$$\sin^3 x = \frac{(e^{ix}-e^{-ix})^3}{8i^3}$$

$$= \frac{e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}}{-8i}$$

$$= -\frac{1}{4}\cdot\frac{e^{3ix}-e^{-3ix}}{2i} + \frac{3}{4}\cdot\frac{e^{ix}-e^{-ix}}{2i}$$

$$= -\frac{1}{4}\cdot\sin 3x + \frac{3}{4}\sin x$$

$$= \frac{3\sin x - \sin 3x}{4}$$ 3

(i) $z_3 - z_1$ & $z_2 - z_1$ have same length

so $\left|\dfrac{z_3-z_1}{z_2-z_1}\right| = 1$

and you rotate $z_2 - z_1$ $60°$ anti-clockwise to get $z_3 - z_1$

so $\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right) = \dfrac{\pi}{3}$ 1

(b) (ii) $\dfrac{z_3-z_1}{z_2-z_1} = \operatorname{cis}\left(\pm\dfrac{\pi}{3}\right)$

$$= \frac{1\pm\sqrt{3}i}{2}$$

$$2(z_3-z_1) = (1\pm\sqrt{3}i)(z_2-z_1)$$

$$= (z_2-z_1) \pm \sqrt{3}i(z_2-z_1)$$

$$2z_3 - z_1 - z_2 = \pm\sqrt{3}i(z_2-z_1)$$

$$(2z_3-z_1-z_2)^2 = -3(z_2-z_1)^2$$

$$4z_3^2 + z_1^2 + z_2^2 - 4z_1z_3 - 4z_2z_3 + 2z_1z_2 = -3z_2^2 - 3z_1^2 + 6z_1z_2$$

$$4z_3^2 + 4z_1^2 + 4z_2^2 = 4z_1z_2 + 4z_1z_3 + 4z_2z_3$$

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$$ 2

OR

$$z_3 - z_1 = \text{cis}\frac{\pi}{3}(z_2 - z_1) \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \frac{\pi}{3}$$

similarly

$$z_3 - z_2 = \text{cis}\frac{\pi}{3}(z_3 - z_1) \Rightarrow \frac{z_3 - z_2}{z_3 - z_1} = \frac{\pi}{3}$$

$$\therefore \frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3 - z_2}{z_3 - z_1}$$

$$(z_3 - z_1)^2 = (z_3 - z_2)(z_2 - z_1)$$

$$z_3^2 - 2z_3z_1 + z_1^2 = z_3z_2 - z_2^2 - z_3z_1 + z_2z_1$$

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

(ii) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$

$$(z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_1z_3 + z_2z_3) = z_1z_2 + z_1z_3 + z_2z_3$$

$$(z_1 + z_2 + z_3)^2 = 3(z_1z_2 + z_1z_3 + z_2z_3)$$

$$(\text{sum of roots})^2 = 3(\text{sum of roots in pairs})$$

$$(-a)^2 = 3b$$

$$a^2 = 3b \qquad 3$$