NORTH SYDNEY GIRLS HIGH SCHOOL

2022 | **HSC Assessment Task 3**

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 5 – 7, show relevant mathematical reasoning and/or calculations

Total marks – 41

Section I – 4 marks (pages 2 – 3)

- Attempt Questions 1 – 4
- Allow about 7 minutes for this section

Section II – 37 marks (pages 4 – 6)

- Attempt Questions 5 – 7
- Allow about 53 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

Question	1 – 4	5	6	7	Total
Integration	/3	/7	/9	/5	**/24**
Proof	/1	/7	/3	/6	**/17**
Total	/4	/14	/12	/11	**/41**

Section I

4 marks
Attempt Questions 1-4
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-4.

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1 If x, y and z are any real numbers and $x > y,$ which of the following statements must be true?

A. $x^2 > y^2$

B. $\frac{1}{x} < \frac{1}{y}$

C. $x + z > y + z$

D. $xz > yz$

2 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta + 2}$ using the substitution $t = \tan\frac{\theta}{2}$?

A. $\int_0^{\frac{\pi}{2}} \frac{2dt}{t^2 + 3}$

B. $\int_0^{1} \frac{2dt}{t^2 + 3}$

C. $\int_0^{\frac{\pi}{2}} \frac{2dt}{t^2 - 3}$

D. $\int_0^{1} \frac{2dt}{t^2 - 3}$

3 Which of the following integrals has the same value as $\int_0^{\sqrt{2}} \frac{x^2\,dx}{\sqrt{4-x^2}}$?

A. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta\ d\theta$

B. $\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\cos^2\theta\ d\theta$

C. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cot\theta\cos\theta\ d\theta$

D. $\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\cot\theta\cos\theta\ d\theta$

4 Without evaluating the integrals, which of the following is greater than zero?

A. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^{-1} x}{1+x^2}\,dx$

B. $\int_{-1}^{1} \left(e^{-|x|}-1\right)dx$

C. $\int_{-2}^{2} \tan^{-1}\left(x^2\right)dx$

D. $\int_{-\pi}^{\pi} x^2 \sin x\,dx$

End of Section I

Section II

37 marks
Attempt Questions 5-7
Allow about 53 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (14 marks**)** Use a SEPARATE writing booklet

(a) Find $\int \cos^3 x \ \sin^5 x \, dx$. **2**

(b) Find $\int x \tan^{-1} x \, dx$**.** **3**

(c) Find $\int \dfrac{dx}{\sqrt{5-8x-4x^2}}$. **2**

(d) (i) Given that $0 \le t \le 1$, prove that $\dfrac{1}{2} \le \dfrac{1}{1+t^2} \le 1$. **2**

(ii) Hence deduce that for $0 \le x \le 1$, $\dfrac{x}{2} \le \tan^{-1} x \le x$. **1**

(e) (i) Prove that $|x|+|y| \ge |x+y|$ for any real numbers x and y. **2**

(ii) Prove, by using mathematical induction, the more general triangle inequality: **2**

$|x_1|+|x_2|+\ldots+|x_n| \ge |x_1+x_2+\ldots+x_n|$ for all positive integers n.

Question 6 (12 marks) Use a SEPARATE writing booklet

(a) (i) Write $\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}$ in the form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$. **2**

(ii) Hence find $\displaystyle\int \frac{3x^2+3x+2}{(x-1)(x^2+1)}\,dx$. **3**

(b) (i) Prove that for all non-negative real numbers a and b, $\sqrt{ab} \le \dfrac{a+b}{2}$. **1**

(ii) Given $a+b+c=1$, where c is also a non-negative real number, prove using the result in (i), or otherwise, that **2**

$$(1-a)(1-b)(1-c) \ge 8abc.$$

(c) (i) Use a suitable substitution to show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x+\cos^3 x}\,dx$. **3**

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Question 7 (11 marks) Use a SEPARATE writing booklet

(a) Given that $y = \dfrac{1}{1+x}$, prove by mathematical induction that $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$ for all positive integers n. **3**

(b) Let $I_n = \displaystyle\int_1^2 \frac{(x-1)^n}{x^n}\,dx$ for all positive integers n.

(i) Show that $I_{n+1} = \dfrac{n+1}{n} I_n - \dfrac{1}{n \times 2^n}$. **3**

(ii) Hence, use the reduction formula in part (i) to find I_3. **2**

(c) (i) Given that k is a positive integer, prove that $\dfrac{1}{(k+1)^2} < \dfrac{1}{k(k+1)}$. **1**

(ii) It is given that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ for $k \in \mathbb{N}$. **2**

If $x_1, x_2, \ldots, x_n$ are positive integers, not necessarily consecutive, such that $1 < x_1 < x_2 < \ldots < x_n$, prove that

$$\frac{1}{x_1^{\,2}} + \frac{1}{x_2^{\,2}} + \frac{1}{x_3^{\,2}} + \ldots + \frac{1}{x_n^{\,2}} < 1$$

End of paper

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NORTH SYDNEY GIRLS HIGH SCHOOL

2022	HSC Assessment Task 3

Mathematics Extension 2 SOLUTIONS

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Section I

4 marks
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Use the multiple choice answer sheet for Questions 1-4.

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1 If x, y and z are any real numbers and $x > y$, which of the following statements must be true?

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2 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta + 2}$ using the substitution $t = \tan\frac{\theta}{2}$?

A. $\int_0^{\frac{\pi}{2}} \frac{2dt}{t^2+3}$

B. $\int_0^{1} \frac{2dt}{t^2+3}$

C. $\int_0^{\frac{\pi}{2}} \frac{2dt}{t^2-3}$

D. $\int_0^{1} \frac{2dt}{t^2-3}$

$\theta = 2\tan^{-1} t$

$\frac{d\theta}{dt} = \frac{2}{1+t^2}$

when $\theta = 0,\ t = 0$

$\theta = \frac{\pi}{2},\ t = 1$

$$\int_0^1 \frac{2\,dt}{\left(\frac{1-t^2}{1+t^2}\right)+2} \times \frac{1}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{1-t^2+2+2t^2}$$

$$= \int_0^1 \frac{2\,dt}{t^2+3}$$

3 Which of the following integrals has the same value as $\displaystyle\int_0^{\sqrt{2}} \frac{x^2\,dx}{\sqrt{4-x^2}}$?

A. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta\,d\theta$

B. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\cos^2\theta\,d\theta$

C. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cot\theta\cos\theta\,d\theta$

D. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\cot\theta\cos\theta\,d\theta$

Let $x = 2\cos\theta$

$dx = -2\sin\theta\,d\theta$

when $x = 0,\ \theta = \frac{\pi}{2}$

$x = \sqrt{2},\ \theta = \frac{\pi}{4}$

$$\int_{\frac{\pi}{2}}^{\pi/4} \frac{4\cos^2\theta \times (-2\sin\theta)\,d\theta}{\sqrt{4-4\cos^2\theta}}$$

$$= \int_{\frac{\pi}{4}}^{\pi/2} \frac{4\cos^2\theta \times 2\cancel{\sin}\theta\,d\theta}{2\cancel{\sin}\theta}$$

4 Without evaluating the integrals, which of the following is greater than zero?

A. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^{-1}x}{1+x^2}\,dx$

B. $\displaystyle\int_{-1}^{1} \left(e^{-|x|}-1\right)dx$

C. $\displaystyle\int_{-2}^{2} \tan^{-1}\left(x^2\right)dx$

D. $\displaystyle\int_{-\pi}^{\pi} x^2\sin x\,dx$

Since $\frac{\sin^{-1}x}{1+x^2}$ and $x^2\sin x$ are all odd functions then the integrals for these will all be zero.

$f(x) = e^{-|x|}-1$ is less than or equal to zero.

And since $\tan^{-1}\left(x^2\right)$ is an even function and always positive then $\displaystyle\int_{-2}^{2} \tan^{-1}\left(x^2\right)dx > 0$

End of Section I

Question 5 (14 **marks)** Use a SEPARATE writing booklet

(a)	Find $\int \cos^3 x \ \sin^5 x \, dx$.	**2**

$$\int \cos^3 x \ \sin^5 x \, dx = \int \cos x \cos^2 x \sin^5 x \, dx$$
$$= \int \cos x \ \left(1-\sin^2 x\right)\sin^5 x \, dx$$
$$= \int \cos x \ \sin^5 x \, dx - \int \cos x \ \sin^7 x \, dx$$
$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

Well done. Some students used an alternate method beginning with $\int \cos^3 x \ \left(1-\sin^2 x\right)^2 \sin x \, dx$ and then rewriting integrand in terms of $\cos x$ .They were also generally successful although this was more time consuming.

(b)	Find $\int x \tan^{-1} x \ dx$.	**3**

$$\int x \tan^{-1} x \ dx = \int \tfrac{d}{dx}\left(\tfrac{1}{2}x^2\right)\tan^{-1} x \ dx$$
$$= \tfrac{1}{2}x^2 \tan^{-1} x - \int \left(\tfrac{1}{2}x^2\right)\left(\frac{1}{1+x^2}\right) dx \qquad \text{(by parts)}$$
$$= \tfrac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} \, dx$$
$$= \tfrac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} \, dx$$
$$= \tfrac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} \, dx$$
$$= \tfrac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

Generally well done. Some careless mistakes when applying integration by parts method and some students incorrectly assigned u and v^1. Use the mnemonic LIATE (log, inverse trig, algebra, trig, exp.) as an aid on choosing u.

(c)	Find $\int \frac{dx}{\sqrt{5-8x-4x^2}}$.	**2**

$$\int \frac{dx}{\sqrt{5-8x-4x^2}} = \int \frac{dx}{\sqrt{9-4\left(x^2+2x+1\right)}}$$
$$= \int \frac{dx}{\sqrt{9-4\left(x+1\right)^2}}$$
$$= \int \frac{dx}{2\sqrt{\left(\frac{3}{2}\right)^2-\left(x+1\right)^2}}$$
$$= \frac{1}{2}\sin^{-1}\frac{2\left(x+1\right)}{3} + C$$

A few too many careless errors. Don't rush the more straightforward questions as these early marks are valuable. Every error was penalised 1 mark.

(d)	(i)	Given that $0 \le t \le 1$, prove that $\frac{1}{2} \le \frac{1}{1+t^2} \le 1$.	**2**
	(ii)	Hence deduce that for $0 \le x \le 1$, $\frac{x}{2} \le \tan^{-1} x \le x$.	**1**

(i)

$$0 \le t \le 1$$

$$0 \le t^2 \le 1$$

$$1 \le t^2+1 \le 2$$

$$\frac{1}{2} \le \frac{1}{t^2+1} \le 1$$

(ii)

$$\int_0^x \frac{1}{2}\,dt \le \int_0^x \frac{1}{t^2+1}\,dt \le \int_0^x 1.dt$$

$$\left[\frac{t}{2}\right]_0^x \le \left[\tan^{-1}(t)\right]_0^x \le \left[t\right]_0^x$$

$$\frac{x}{2} - \frac{0}{2} \le \tan^{-1}(x) - \tan^{-1}(0) \le x - 0$$

$$\therefore \quad \frac{x}{2} \le \tan^{-1}(x) \le x$$

(i) The setting out can be improved by many here. Many students tackled this as two separate inequalities and their method was time consuming. The best approach here is to start with a known result and build up. Working from the result was penalised unless each line was justified because in effect you are proving the converse instead.

(ii) Many students incorrectly used indefinite integrals here. The property you are relying on is: If $f(x) \le g(x)$ over an interval $a \le x \le b$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

This was only penalised ½ mark.

(e)	(i)	Prove that $\lvert x\rvert+\lvert y\rvert \ge \lvert x+y\rvert$ for any real numbers x and y.	2
	(ii)	Prove, by using mathematical induction, the more general triangle inequality: $\lvert x_1\rvert+\lvert x_2\rvert+\ldots+\lvert x_n\rvert \ge \lvert x_1+x_2+\ldots+x_n\rvert$ for all positive integers n.	2

(i) Consider

$$(|x|+|y|)^2 = |x|^2 + 2|x||y| + |y|^2$$
$$= x^2 + y^2 + 2|xy|$$
$$\geq x^2 + y^2 + 2xy$$
$$= (x+y)^2$$
$$= |x+y|^2$$

$\therefore (|x|+|y|)^2 \geq |x+y|^2$

since $|x|+|y| \geq 0$ and $|x+y| \geq 0$

then $|x|+|y| \geq |x+y|$

(ii) Prove true for $n=1$

$|x_1| \geq |x_1|$

$\therefore$ true for $n=1$

Assume true for $n=k$ $k \in \mathbb{Z}^+$

$|x_1|+|x_2|+\ldots+|x_k| \geq |x_1+x_2+\ldots+x_k|$

Prove true for $n=k+1$

R.T.P. $|x_1|+|x_2|+\ldots+|x_k|+|x_{k+1}| \geq |x_1+x_2+\ldots+x_k+x_{k+1}|$

LHS $= |x_1+|x_2|+\ldots+|x_k|+|x_{k+1}|$

$\geq |x_1+x_2+\ldots+x_k|+|x_{k+1}|$ from assumption

$\geq |x_1+x_2+\ldots+x_k+x_{k+1}|$ from part (i) since this is the sum of two integers.

$\therefore$ Hence by mathematical induction is is true for $n \geq 1$

(i) **Not well done. Some students tried proofs involving complex numbers or vectors but only some were successful. They needed to explain why these proofs were relevant to real numbers as described in the question.**

(ii) **Induction proof was generally well done. Some students had difficulty with the initial case using part (i) rather than proving for n=1. Students needed to explain how they got from second last line to last line otherwise it was not clear that they understood how to arrive at the desired result.**

Question 6 (12 marks) Use a SEPARATE writing booklet

(a)	(i)	Write $\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}$ in the form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$.	**2**
	(ii)	Hence find $\displaystyle\int \frac{3x^2+3x+2}{(x-1)(x^2+1)}\,dx$.	**3**

(i) $$\frac{3x^2+3x+2}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$=\frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$3x^2+3x+2=A(x^2+1)+(Bx+C)(x-1)$$

Let $x=1$	Let $x=0$	Let $x=-1$
$3+3+2=A(1+1)+0$ $8=2A$ $A=4$	$2=A(0+1)+(0+C)(0-1)$ $2=A-C$ $2=4-C$ $C=2$	$3-3+2=A(1+1)+(-B+C)(-2)$ $2=2A+2B-2C$ $2=8+2B-4$ $-2=2B$ $B=-1$

$$\frac{3x^2+3x+2}{(x-1)(x^2+1)}=\frac{4}{x-1}+\frac{-x+2}{x^2+1}$$

(ii) $$\int\frac{3x^2+3x+2}{(x-1)(x^2+1)}dx=\int\left(\frac{4}{x-1}+\frac{-x+2}{x^2+1}\right)dx$$

$$=\int\frac{4}{x-1}dx+\int\frac{-x+2}{x^2+1}dx$$

$$=\int\frac{4}{x-1}dx-\int\frac{x}{x^2+1}dx+\int\frac{2}{x^2+1}dx$$

$$=4\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{2x}{x^2+1}dx+2\int\frac{1}{x^2+1}dx$$

$$=4\ln(x-1)-\frac{1}{2}\ln(x^2+1)+2\tan^{-1}x+C$$

(a)	Very well done, except for a handful of careless errors.

(b)	(i)	Prove that for all non-negative real numbers a and b, $\sqrt{ab} \le \frac{a+b}{2}$.	**1**
	(ii)	Given $a+b+c=1$, where c is also a non-negative real number, prove using the result in (i), or otherwise, that $$(1-a)(1-b)(1-c) \ge 8abc.$$	**2**

(i) Consider

$$(\sqrt{a}-\sqrt{b})^2 \geq 0 \qquad a, b \geq 0$$

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{a}\sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

(ii) $(1-a)(1-b)(1-c)$

$$= (a+b+c-a)(a+b+c-b)(a+b+c-c)$$

$$= (b+c)(a+c)(a+b)$$

$$\geq 2\sqrt{bc} \times 2\sqrt{ac} \times 2\sqrt{ab} \quad \text{from (i)}$$

$$= 8\sqrt{a^2b^2c^2}$$

$$= 8abc \qquad a, b, c \geqslant 0$$

(b)	(i)	Well done. Some students expanded $\left(\sqrt{a}+\sqrt{b}\right)^2$, requiring them to complete a different square to get the result.
	(ii)	Students who expanded out the LHS were generally not successful. Please show every step, including the intermediate step between $2\sqrt{bc} \times 2\sqrt{ac} \times 2\sqrt{ab}$ and $8abc$.

(c)	(i)	Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.	**1**
	(ii)	Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$.	**3**

(i) $\int_0^a f(a-x)\,dx$

Let $u = a - x$

$-du = dx$

when $x = 0,\ u = a$

$x = a,\ u = 0$

$= \int_a^0 f(u)\ -du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$ (dummy variable)

(ii) $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{\sin^3\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right) + \cos^3\left(\frac{\pi}{2}-x\right)}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^3 x + \sin^3 x - \sin^3 x}{\cos^3 x + \sin^3 x}\,dx$

$= \int_0^{\frac{\pi}{2}} 1\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x + \sin^3 x}\,dx$

$2I = \int_0^{\pi/2} dx$

$2I = [x]_0^{\pi/2} = \frac{\pi}{2}$ $\quad \therefore I = \frac{\pi}{4}$

(c)	(i)	This is a stock question. Please learn it. A number of students proved the result for a specific function instead of the general case. The notation involving "~" is not a recognised one. Please don't use it. Some students essentially proved LHS = LHS.
	(ii)	Most students could successfully apply the result of part (i) to get an alternate integral. For those who didn't split the integral as in the solutions, you should probably have said $2I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x \ dx}{\sin^3 x + \cos^3 x} + \int_0^{\frac{\pi}{2}} \frac{\cos^3 x \ dx}{\sin^3 x + \cos^3 x}$ before combining into one integral. Failure to do so was not penalised, but could have been.

Question 7 (11 marks) Use a SEPARATE writing booklet

(a) Given that $y = \dfrac{1}{1+x}$, prove by mathematical induction that $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$ for all positive integers n. **3**

Consider $n = 1$.

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} \text{ and } \frac{(-1)^1 1!}{(1+x)^{1+1}} = -\frac{1}{1+x^2} = \frac{dy}{dx}$$

True when $n = 1$.

Suppose true for $n = k$.

So $\dfrac{d^k y}{dx^k} = \dfrac{(-1)^k k!}{(1+x)^{k+1}}$.

Required to show it is true for $n = k + 1$.

That is, $\dfrac{d^{k+1} y}{dx^{k+1}} = \dfrac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$.

$$\begin{aligned}
\text{LHS} &= \frac{d^{k+1} y}{dx^{k+1}} \\
&= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right) \\
&= \frac{d}{dx}\left(\frac{(-1)^k k!}{(1+x)^{k+1}}\right) \\
&= (-1)^k k!(-(k+1)(1+x)^{-(k+2)}) \\
&= \frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}} \\
&= \text{RHS}
\end{aligned}$$

If true for $n = k$, then true for $n = k + 1$.

Hence, by mathematical induction, true for $n \geq 1$.

Q7 a)

Generally well done. It's a proof, you need to show all steps. Some people were unfamiliar with how $\frac{d}{dx}$ notation works and will need to refresh. Ideally you should define *k*.

(b) Let $I_n = \int_1^2 \frac{(x-1)^n}{x^n}\,dx$ for all positive integers n.

(i) Show that $I_{n+1} = \frac{n+1}{n} I_n - \frac{1}{n \times 2^n}$. **3**

(ii) Hence, use the reduction formula in part (i) to find I_3. **2**

(i)
$$I_n = \int_1^2 \frac{(x-1)^n}{x^n}dx$$
$$= \int_1^2 \frac{d}{dx}\left(\frac{(x-1)^{n+1}}{n+1}\right)\left(\frac{1}{x^n}\right)dx$$
$$= \int_1^2 \frac{d}{dx}\left(\frac{(x-1)^{n+1}}{n+1}\right)\left(x^{-n}\right)dx$$
$$= \left[\frac{(x-1)^{n+1}}{n+1}\frac{1}{x^n}\right]_1^2 - \int_1^2 \frac{(x-1)^{n+1}}{n+1}\left(-nx^{-n-1}\right)dx$$
$$= \left[\frac{(x-1)^{n+1}}{n+1}\frac{1}{x^n}\right]_1^2 + \frac{n}{n+1}\int_1^2 \frac{(x-1)^{n+1}}{x^{n+1}}dx$$
$$= \left[\left(\frac{(2-1)^{n+1}}{n+1}\frac{1}{2^n}\right) - (0)\right] + \frac{n}{n+1}I_{n+1}$$
$$I_n = \frac{1}{(n+1)2^n} + \frac{n}{n+1}I_{n+1}$$
$$(n+1)I_n = \frac{1}{2^n} + nI_{n+1}$$
$$\left(\frac{n+1}{n}\right)I_n = \frac{1}{n2^n} + I_{n+1}$$
$$I_{n+1} = \left(\frac{n+1}{n}\right)I_n - \frac{1}{n2^n}$$

b(i)Generally well done. Many people started by integrating I_n however it was more efficient to integrate from I_{n+1} here. Also many choices for the parts to integrate by. They were all accepted as long as you did so correctly. Make sure you put your equation into the form given in the question before subbing in. To get full marks, you had to show evidence that you had evaluated, rather than jumping to the answer given in the question.

(ii)
$$I_1 = \int_1^2 \left(1-\frac{1}{x}\right)^n dx$$
$$= [x - \ln x]_1^2$$
$$= 1 - \ln 2$$
$$I_2 = (2)(1-\ln 2) - \frac{1}{2}$$
$$= \frac{3}{2} - 2\ln 2$$

$$I_3 = \left(\frac{2+1}{2}\right)I_2 - \frac{1}{2\times 2^2}$$
$$= \frac{3}{2}\left(\frac{3}{2} - 2\ln 2\right) - \frac{1}{8}$$
$$= \frac{17}{8} - 3\ln 2$$

b(ii)Generally well done. You could have started from I_3 or I_1. Most common mistake was not keeping track of terms or unsuccessfully adding fractions. Common mistake was using the formula incorrectly: $I_3 = I_{n+1}$ then $n = 2$.

(c) (i) Given that k is a positive integer, prove that $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$. **1**

(ii) It is given that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k \in \mathbb{N}$. **2**

If $x_1, x_2, ..., x_n$ are positive integers, not necessarily consecutive, such that $1 < x_1 < x_2 < ... < x_n$, prove that $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + ... + \frac{1}{x_n^2} < 1$

(i) $k < k+1$

$k(k+1) < (k+1)^2$

$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$

ii) Since $x_1, x_2 \dots x_n$ are not necessarily consecutive

$\frac{1}{x_1^2} \leq \frac{1}{2^2}$

$\frac{1}{x_2^2} \leq \frac{1}{3^2}$

$\vdots$

$\frac{1}{x_n^2} \leq \frac{1}{(n+1)^2}$

$\therefore \frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_n^2} \leq \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(n+1)^2}$

$$< \frac{1}{1\times 2} + \frac{1}{2\times 3} + \dots + \frac{1}{n(n+1)}$$

from part i)

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$< 1 \quad \text{(n is a positive integer)}$$

$$\therefore \frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_n^2} < 1$$

c (i)
Two main methods used, LHS – RHS or starting simple and building up. You should be concluding your proof. Working from the result was not accepted without justifying each line.
c (ii)
Not well done. To get both marks you had to correctly apply the given results as well as deal with the fact that the numbers were not necessarily sequential: don't ignore information given in the question. A common mistake was subbing in x_1 for k, instead of $k+1$. Proof by induction was not relevant in this question.