
Student Number

Name ______________________

North Sydney Girls High School

2023 **HSC ASSESSMENT TASK 1**

Mathematics Extension 2

General Instructions

- Reading time – 2 minutes
- Working time – 60 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- A double-sided A4 page of handwritten notes may be used

Total Marks: 38

Section I – 5 marks (pages 2–4)

- Attempt Questions 1–5
- Allow about 8 minutes for this section

Section II – 33 marks (pages 5–8)

- Attempt Questions 6–8
- Allow about 52 minutes for this section

	Multiple Choice	Question 6(a)	Question 6(b-e)	Question 7	Question 8	Total
Complex Numbers	/5	/2	/11	/11	/9	/38

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

---

1 Given $z^2 = 4\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$, which of the following are possible values of z?

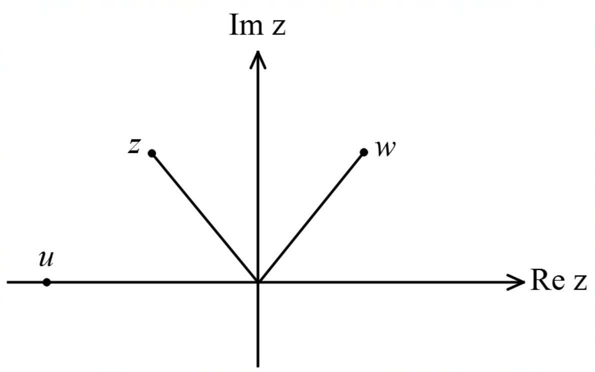
A. $\sqrt{3}+i,\ -\sqrt{3}-i$

B. $\sqrt{3}-i,\ \sqrt{3}+i$

C. $1-\sqrt{3}i,\ -1+\sqrt{3}i$

D. $1-\sqrt{3}i,\ 1+\sqrt{3}i$

2 The Argand diagram below shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant, and u lies on the negative real axis.



Which of the following statements could be true?

A. $u = zw$ and $u = z + w$

B. $u = zw$ and $u = z - w$

C. $z = uw$ and $u = z + w$

D. $z = uw$ and $u = z - w$

3 The polynomial $P(z)$ has real coefficients, and $z = 3 - i$ is a root of $P(z)$.

Which quadratic polynomial must be a factor of $P(z)$?

A. $z^2 - 6z + 10$

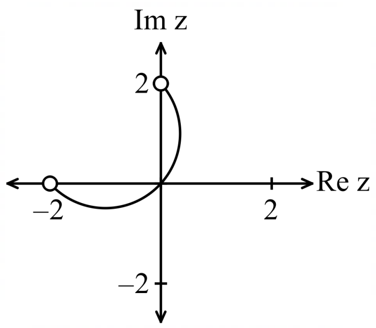
B. $z^2 + 6z + 10$

C. $z^2 - 6z + 8$

D. $z^2 + 6z + 8$

4 Which diagram best represents the solutions to the equation $\arg\left(\dfrac{z-2}{z+2i}\right) = \dfrac{\pi}{2}$?

A.



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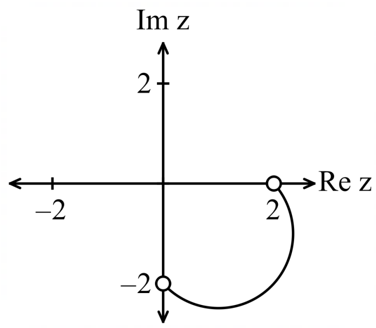

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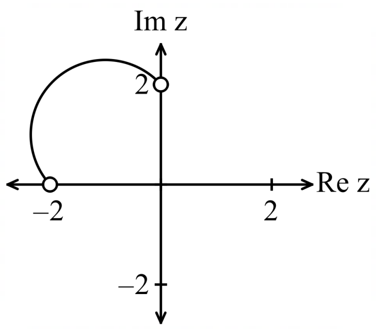
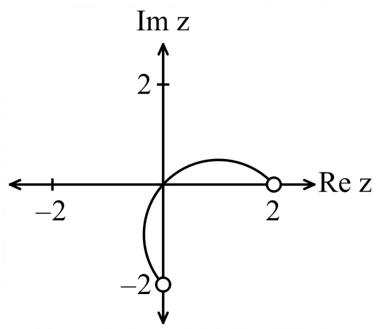


D.



5 Consider the complex numbers $2z$, $-iz$ and $2z - iz$, where $z \neq 0$. These three numbers are plotted on the Argand diagram and together with the origin, O, they form the vertices of a quadrilateral.

What is the area of the quadrilateral?

A. $2|z|^2$

B. $2|z|$

C. $|z^2|$

D. $|z| + 2|z|$

Section II

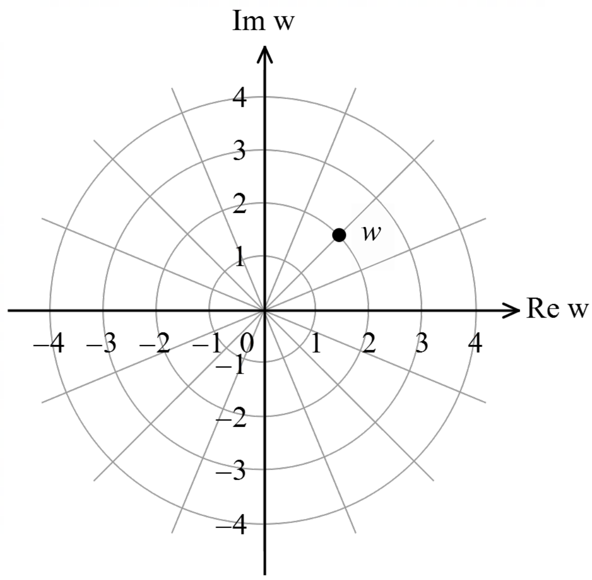
33 marks
Attempt Questions 6–8
Allow about 52 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 6–8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks) Use the Question 6 Writing Booklet.

(a) Consider the complex number $w = 2\operatorname{cis}\dfrac{\pi}{4}$, as shown on the diagram below.



Answer part (i) and part (ii) on the diagram provided on the back of the Multiple Choice answer sheet.

(i) Indicate the location of $A = w^2$. **1**

(ii) Indicate the location of $B = -\dfrac{1}{2}iw$. **1**

Question 6 continues on the next page

Question 6 (continued)

(b) Given that $z = 2 - i$ and $w = -2 - i$, find in the form $a + ib$:

(i) $\overline{z + w}$ **1**

(ii) $\dfrac{z}{w}$ **2**

(c) Given $z = 3\sqrt{3} - 3i$:

(i) Express z in exponential form. **2**

(ii) Find z^5. Express your answer in exponential form. **1**

(d) By expanding $(\cos\theta + i\sin\theta)^3$, find an expression for $\sin 3\theta$ in terms of powers of $\sin\theta$. **2**

(e) (i) Indicate the region in the Argand diagram which satisfies: **2**

$$|z - (1+i)| \leq 1 \text{ and } \frac{\pi}{4} \leq \arg(z - (1+i)) \leq \frac{\pi}{2}$$

(ii) Find the complex number z, in the region specified in (i), corresponding to the maximum value of $|z|$. Express your answer in modulus-argument form. **1**

End of Question 6

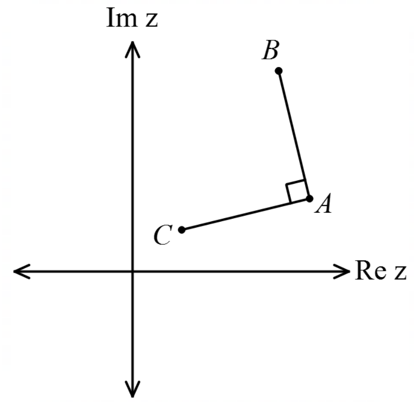
Question 7 (11 marks) Use the Question 7 Writing Booklet.

(a) Solve $z^2 - (2-i)z - (1+7i) = 0$, giving your simplified answers in the form $a+ib$. **4**

(b) Simplify fully, leaving your answer in modulus-argument form: **2**

$$(\cos 2\theta - i\sin 2\theta)^5 (\cos 3\theta + i\sin 3\theta)^2$$

(c) On the Argand diagram below, the point A represents the complex number a, and the point B represents the complex number b. The point B is rotated about the point A through a right angle in the anticlockwise direction to take up the point C, representing the complex number c. **2**



Find c in terms of a and b.

(d) Let $P(x) = x^4 - 10x^3 + 38x^2 - 64x + 40$. **3**

Given that $x = 2$ is a root of $P'(x) = 0$, fully factorise $P(x)$ over $\mathbb{C}$.

Question 8 (9 marks) Use the Question 8 Writing Booklet.

(a) Given that $(2+i)^3 = 2+11i$ find the other two cube roots of $2+11i$, in Cartesian form. **2**

(b) Solve $z^4 = (z-1)^4$. Express your answers in Cartesian form. **3**

(c) (i) By using the method of completing the square, or otherwise, solve $z^2 - 2z\cos\theta + 1 = 0$. Give simplified solutions in terms of θ. **2**

(ii) Let α and β be the two solutions found in part (i). P and Q are the points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, where n is a positive integer. **2**

Show that PQ is of constant length.

End of paper

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$\boxed{\text{C}}$

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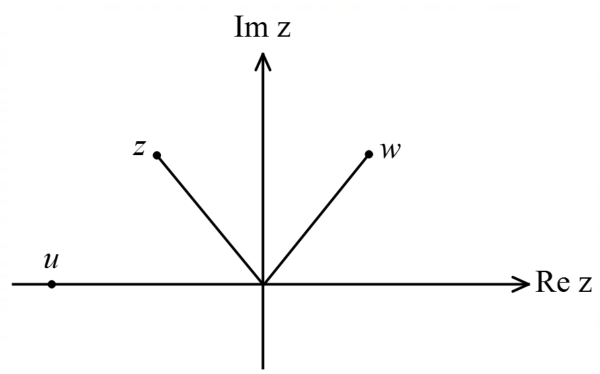

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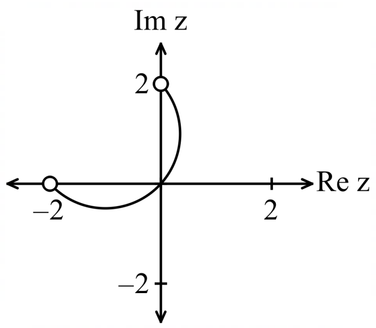

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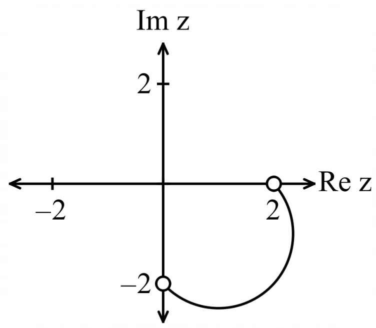

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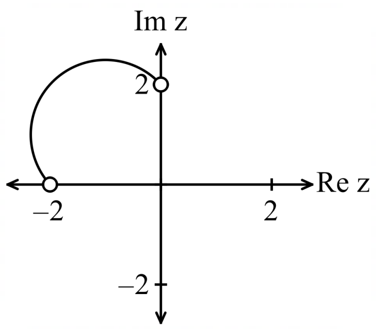

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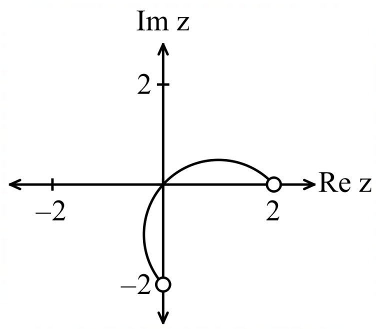

$\boxed{\text{D}}$

5 Consider the complex numbers $2z$, $-iz$ and $2z-iz$, where $z \neq 0$. These three numbers are plotted on the Argand diagram and together with the origin, O, they form the vertices of a quadrilateral.

What is the area of the quadrilateral?

A. $2|z|^2$

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C. $|z^2|$

D. $|z|+2|z|$

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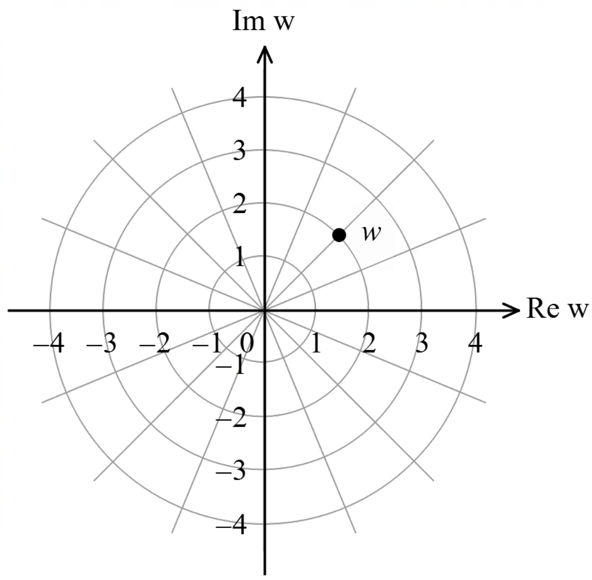

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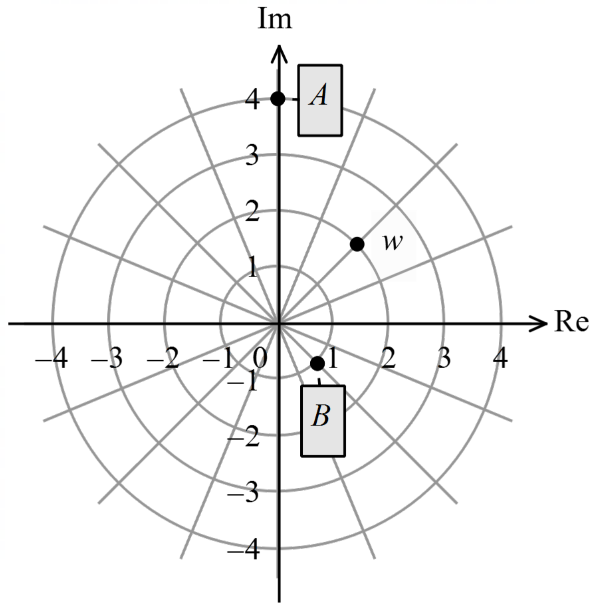


Question 6 continues on the next page

Question 6 (continued)

(b) Given that $z = 2 - i$ and $w = -2 - i$, find in the form $a + ib$:

(i) $\overline{z+w}$ **1**

(ii) $\dfrac{z}{w}$ **2**

(i) $2i$

(ii) $$\frac{z}{w} = \frac{2-i}{-2-i} \times \frac{-2+i}{-2+i}$$
$$= -\frac{-4+2i+2i+1}{4+1}$$
$$= -\frac{3}{5} + \frac{4}{5}i$$

(c) Given $z = 3\sqrt{3} - 3i$:

(i) Express z in exponential form. **2**

(ii) Find z^5. Express your answer in exponential form. **1**

(i) $$|z| = \sqrt{(3\sqrt{3})^2 + 3^2} = 6$$
$$\theta = \tan^{-1}\frac{-3}{3\sqrt{3}} = -\frac{\pi}{6}$$
$$\therefore z = 6e^{-i\frac{\pi}{6}}$$

(ii) $$z^5 = 6^5 e^{-i\frac{5\pi}{6}} = 7776e^{-i\frac{5\pi}{6}}$$

(d) By expanding $(\cos\theta + i\sin\theta)^3$, find an expression for $\sin 3\theta$ in terms of powers of $\sin\theta$. **2**

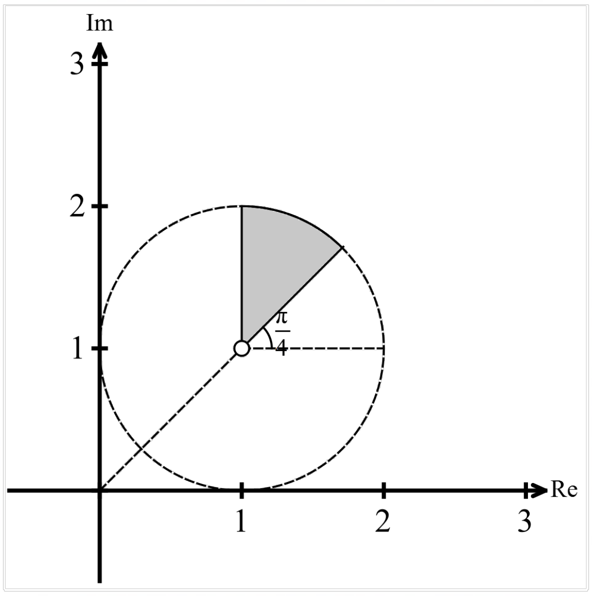
$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^3\theta - i\sin^3\theta$$

equating imaginary parts

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$\sin 3\theta = 3\left(1-\sin^2\theta\right)\sin\theta - \sin^3\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

(e) (i) Indicate the region in the Argand diagram which satisfies: **2**

$$|z-(1+i)| \le 1 \text{ and } \frac{\pi}{4} \le \arg(z-(1+i)) \le \frac{\pi}{2}$$



(ii) Find the complex number z, in the region specified in (i), corresponding to the maximum value of $|z|$. Express your answer in modulus-argument form. **1**

The maximum modulus is on the line through centre and the origin, opposite the origin.

Distance to centre, plus distance to edge of circle:

$$= \sqrt{2}\operatorname{cis}\frac{\pi}{4} + \operatorname{cis}\frac{\pi}{4}$$
$$= \left(1+\sqrt{2}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Question 7 (11 marks) Use the Question 7 Writing Booklet.

(a) Solve $z^2-(2-i)z-(1+7i)=0$, giving your simplified answers in the form $a+ib$. **4**

$$\text{Consider: } \Delta = (-(2-i))^2 - 4\times 1\times(-(1+7i))$$
$$= 7+24i$$

let $(x+iy)^2 = 7+24i$

expanding and equating real and imaginary parts:

$x^2-y^2=7$

$2xy=24$

$xy=12$

by inspection, $x=\pm 4,\ y=\pm 3$

so square roots are $4+3i$, $-4-3i$.

Therefore $z=\dfrac{2-i\pm(4+3i)}{2}$

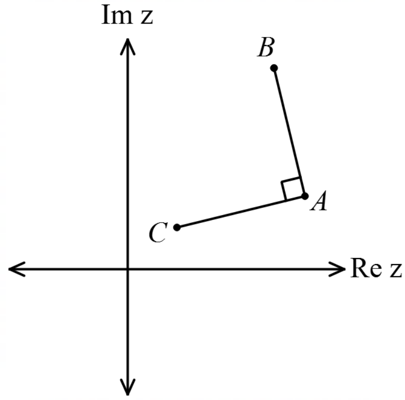
$z=3+i,\quad z=-1-2i$

(b) Simplify fully, leaving your answer in modulus-argument form: **2**

$$(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^2$$

$$\begin{aligned}(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^2 &= [\cos(-2\theta)+i\sin(-2\theta)]^5[\cos 3\theta + i\sin 3\theta]^2\\ &= [\cos(-10\theta)+i\sin(-10\theta)][\cos 6\theta + i\sin 6\theta]\\ &\qquad \text{by de Moivre's Theorem}\\ &= \cos(-4\theta)+i\sin(-4\theta)\end{aligned}$$

(c) On the Argand diagram below, the point A represents the complex number a, and the point B represents the complex number b. The point B is rotated about the point A through a right angle in the anticlockwise direction to take up the point C, representing the complex number c. **2**



Find c in terms of a and b.

$$
\begin{aligned}
&\text{Let } O \text{ be the origin.}\\
&\vec{AB} \equiv b - a\\
&\therefore \vec{AC} \equiv i(b-a)\\
&\vec{OC} = \vec{OA} + \vec{AC}\\
&\qquad \equiv a + i(b-a)\\
&\therefore c = a + i(b-a)
\end{aligned}
$$

(d) Let $P(x) = x^4 - 10x^3 + 38x^2 - 64x + 40$. **3**

Given that $x = 2$ is a root of $P'(x) = 0$, fully factorise $P(x)$ over $\mathbb{C}$.

$P(2) = 0$, therefore 2 is a double root.

$\therefore x^4 - 10x^3 + 38x^2 - 64x + 40 = (x-2)^2(ax^2 + bx + c)$

Equating coefficients of x^4 : $a = 1$.

Equating coefficients of x^0 : $c = 10$.

Equating coefficients of x^3 : $-10 = b - 4a$.

$\therefore b = -6$

$x^2 - 6x + 10$ is a factor.

$(x-3)^2 + 1 = (x-3+i)(x-3-i)$ are factors.

$\therefore P(x) = (x-2)^2(x-3+i)(x-3-i)$ over $\mathbb{C}$

Question 8 (9 marks) Use the Question 8 Writing Booklet.

(a) Given that $(2+i)^3 = 2+11i$ find the other two cube roots of $2+11i$, in Cartesian form. **2**

Let the other two cube roots be α and β.

The cube roots are evenly spaced around the circle with radius $|2+i| = \sqrt{5}$, at an angle of $\pm\dfrac{2\pi}{3}$

$$\begin{aligned}
\text{Let } \alpha &= (2+i)\operatorname{cis}\frac{2\pi}{3} \\
&= (2+i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\
&= -\frac{\sqrt{3}}{2}-1+\left(\sqrt{3}-\frac{1}{2}\right)i
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, let } \beta &= (2+i)\operatorname{cis}\left(-\frac{2\pi}{3}\right) \\
&= (2+i)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \\
&= \frac{\sqrt{3}}{2}-1+\left(-\sqrt{3}-\frac{1}{2}\right)i
\end{aligned}$$

(b) Solve $z^4 = (z-1)^4$. Express your answers in Cartesian form. **3**

$$z^4 = (z-1)^4$$

$$z^4 - (z-1)^4 = 0$$

$$[z^2-(z-1)^2][z^2+(z-1)^2] = 0$$

$$[z-(z-1)][z+(z-1)][z^2+z^2-2z+1] = 0$$

$$1[2z-1][2z^2-2z+1] = 0$$

$$z = \frac{1}{2},\ z = \frac{2 \pm \sqrt{4-8}}{4}$$

$$z = \frac{1}{2},\ \frac{1}{2} \pm \frac{1}{2}i$$

Alternatively:

$$z^4 = (z-1)^4$$

$$\frac{z^4}{(z-1)^4} = 1$$

$$\left(\frac{z}{(z-1)}\right)^4 = 1$$

$$\frac{z}{(z-1)} = \pm 1,\ \pm i$$

$\frac{z}{(z-1)} = 1$	$\frac{z}{(z-1)} = -1$	$\frac{z}{(z-1)} = +i$	$\frac{z}{(z-1)} = -i$
$z = z-1$	$z = 1-z$	$z = i(z-1)$	as coefficients are real
no solutions	$2z = 1$	$z(1-i) = -i$	$\frac{1}{2}+\frac{1}{2}i$
	$z = \frac{1}{2}$	$z = \frac{i}{1-i} \times \frac{1+i}{1+i}$	is also a solution
		$= \frac{-1+i}{-2}$	
		$= \frac{1}{2} - \frac{1}{2}i$	

$$\therefore z = \frac{1}{2},\ \frac{1}{2} \pm \frac{1}{2}i$$

(c) (i) By using the method of completing the square, or otherwise, solve $z^2 - 2z\cos\theta + 1 = 0$. Give simplified solutions in terms of θ. **2**

$$z^2 - 2z\cos\theta + 1 = 0$$

$$z^2 - 2z\cos\theta + \cos^2\theta = -1 + \cos^2\theta$$

$$(z - \cos\theta)^2 = -\sin^2\theta$$

$$z - \cos\theta = \pm i\sin\theta$$

$$z = \cos\theta \pm i\sin\theta$$

(ii) Let α and β be the two solutions found in part (i). P and Q are the points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, where n is a positive integer. **2**

Show that PQ is of constant length.

$$\alpha = \cos\theta + i\sin\theta$$

$\alpha^n = \cos(n\theta) + i\sin(n\theta)$ by de Moivre's Theorem

$$\beta = \cos\theta - i\sin\theta$$

$$\beta = \cos(-\theta) - i\sin(-\theta)$$

$\beta^n = \cos(-n\theta) + i\sin(-n\theta)$ by de Moivre's Theorem

$$\beta^n = \cos(n\theta) - i\sin(n\theta)$$

$$\alpha^n + \beta^n = 2\cos(n\theta)$$

$$\alpha^n - \beta^n = 2i\sin(n\theta)$$

$$\begin{aligned} PQ &= |(\alpha^n + \beta^n) - (\alpha^n - \beta^n)| \\ &= |2\cos(n\theta) - 2i\sin(n\theta)| \\ &= 2\sqrt{\cos^2(n\theta) - 2i\sin^2(n\theta)} \\ &= 2\sqrt{1} \end{aligned}$$

$$PQ = 2$$

End of paper

Question 6

(b) Generally, well done.

(ii) When realising the denominator, some people are skipping a step and directly writing the results of the multiplication, which is leading to errors.
Also, students had to write the answer correctly in the required form.

(c) Well done.

(d) Generally well done. Again, doing two steps at once can lead to errors, such as expanding, converting powers of i and collecting like terms all in one step. Other errors were not converting powers of cos to the required powers of sin.

(e) (i) Generally well done. Complete answers indicated the necessary angles, or that the angled line passed through the origin. Otherwise, some students did not identify the correct region.

(ii) Not done as well. Here, you had to answer the question which asked for the point, not the maximum modulus. Common errors included incorrectly finding the modulus, or writing the answer incorrectly. You were required to write out the $r(\cos\theta + i\sin\theta)$, not leave it in cis form.

The final step required factorising this quadratic factor – most common approach was to solve the associated quadratic equation using the formula. Other approaches could be completing the square (shown in solutions) or recognising that the roots will be conjugates (ok to do this at this stage) and hence that $2\operatorname{Re}(\alpha) = -6$ and $|\alpha|^2 = 10$ which aids in finding α and $\overline{\alpha}$.

A small number of students factorised $P'(x)$ instead of $P(x)$ - possibly not reading the question carefully.

Question 8

(a) Poorly done. Students who knew to multiply by $cis\left(\pm\frac{2\pi}{3}\right)$ generally multiplied before converting to cartesian form, making the final conversion near impossible.

Many students attempted a purely algebraic approach and made no significant progress.

(b) Poorly done. Factorising the difference of two squares should have led to a simple algebraic solution.

(c) (i) Many students could not correctly identify $-2cos\theta$ as the coefficient of z, and included the z in the constant term. Otherwise, generally well done.

(ii) Many students did not appear to understand what 'constant length' meant. The values of α and β depend only on θ, so 'constant' means independent of the choice of θ.

Students made two common errors when finding the modulus:

(1) Believing that $|z + w| = \sqrt{z^2 + w^2}$. This calculation can be performed only on the real and imaginary parts of a complex number, not on the sum of two complex numbers.

(2) Including i in the calculation of the modulus.