NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2023 | HSC Assessment Task 3

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 5 – 7, show relevant mathematical reasoning and/or calculations

Total marks – 38

Section I – 4 marks (pages 2 – 3)

- Attempt Questions 1 – 4
- Allow about 6 minutes for this section

Section II – 34 marks (pages 4 – 6)

- Attempt Questions 5 – 7
- Allow about 54 minutes for this section

NAME:_______________________________ TEACHER:___________________

STUDENT NUMBER:__________________________

Question	1 – 4	5	6	7	Total
Integration	/2	/6	/8	/5	**/21**
Proof	/2	/5	/3	/7	**/17**
Total	/4	/11	/11	/12	**/38**

Section I

4 marks
Attempt Questions 1-4
Allow about 6 minutes for this section

Use the multiple choice answer sheet for Questions 1-4.

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1 If x and y are real numbers such that $xy < 0$, which of the following must be true?

A. $\frac{x}{y} < 0$

B. $x + y < 0$

C. $x^2 > y^2$

D. $x^2 - y^2 < 0$

2 Which of the following statements is true $\forall a, b \in \mathbb{R}$?

A. $|a+b| < |a| + |b|$

B. $\frac{a}{b} + \frac{b}{a} \geq 2$

C. $a^2 + b^2 \geq 2ab$

D. $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$

3 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta + \sin\theta}$ using the substitution $t = \tan\frac{\theta}{2}$?

A. $\int_0^1 \frac{2dt}{(t+1)^2}$

B. $\int_0^1 \frac{dt}{(t+1)^2}$

C. $\int_0^1 \frac{dt}{2-(t-1)^2}$

D. $\int_0^1 \frac{2\,dt}{2-(t-1)^2}$

4 Of the following expressions, which one need NOT contain a term involving a logarithm in its anti-derivative?

A. $\frac{x+3}{x^2+6x+13}$

B. $\frac{x^2-1}{x^3-2x^2-x+2}$

C. $\frac{x^2-1}{x+2}$

D. $\frac{x^3+x+1}{1+x^2}$

End of Section I

Section II

34 marks
Attempt Questions 5-7
Allow about 54 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 5-7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (11 marks**)** Use a SEPARATE writing booklet

(a) Use integration by parts to evaluate $\int xe^{-x}dx$. **2**

(b) Find $\int \frac{x}{\sqrt{6-x}}\,dx$. **2**

(c) Find $\int \cos^2 x \sin^2 x\,dx$. **2**

(d) Prove that $16a^4 + b^2 \geq 8a^2b \ \forall\ a, b \in \mathbb{R}$. **2**

(e) Use Mathematical Induction to prove that $4^n < (n+2)!$ for all positive integers n. **3**

Question 6 (11 marks) Use a SEPARATE writing booklet

(a) Find $\displaystyle\int \frac{1}{\sqrt{8+4x-4x^2}}\,dx$. **2**

(b) (i) Express $\dfrac{x}{(x^2+9)(x-1)}$ in the form of $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. **2**

(ii) Hence evaluate $\displaystyle\int \frac{x}{(x^2+9)(x-1)}\,dx$. **2**

(c) You are given that $\dfrac{1}{1+\sin 2x}=\dfrac{\sec^2 x}{(1+\tan x)^2}$. (Do NOT prove this.) **2**

Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{1+\sin 2x}\,dx$.

(d) A sequence is given by the following recurrence relation, **3**

$$T_1 = 1,\ T_n = \frac{3T_{n-1}-1}{4T_{n-1}-1} \text{ for } n \geq 2.$$

Use Mathematical Induction to prove that the general formula for the sequence is $T_n = \dfrac{n}{2n-1}$.

Question 7 (12 marks) Use a SEPARATE writing booklet

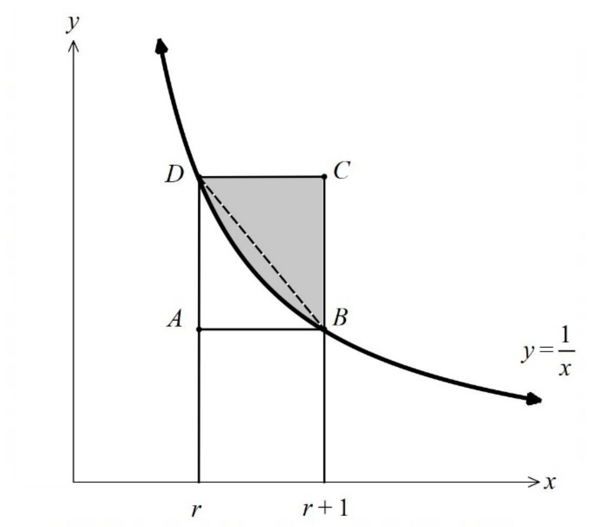
(a) Consider $I_n = \int_1^e x^{-3} (\ln x)^n \, dx$, $n \geq 0$.

(i) Prove that for $n \geq 1$, $I_n = \dfrac{-1}{2e^2} + \dfrac{n}{2} I_{n-1}$. **3**

(ii) Hence, evaluate $\int_1^e x^{-3} (\ln x)^2 \, dx$. **2**

(b) Prove that $t \ln t - (1+t)\ln(1+t) < 0$ for all $t > 0$. **2**

(c) Consider the graph of $y = \dfrac{1}{x}$ shown below.



(i) The area of the shaded region is denoted by A_r. **2**

Show that $\dfrac{1}{2}\left(\dfrac{1}{r} - \dfrac{1}{r+1}\right) \leq A_r \leq \dfrac{1}{r} - \dfrac{1}{r+1}$.

(ii) Let the sum S_n be defined as $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$. **3**

Using part (i), show that $\dfrac{1}{2}\left(1 + \dfrac{1}{n}\right) \leq S_n - \ln n \leq 1$.

End of paper

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NORTH
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SCHOOL

2023	HSC Assessment Task 3

Mathematics Extension 2 (Solutions)

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 5 – 7, show relevant mathematical reasoning and/or calculations

Total marks – 38

Section I – 4 marks (pages 2 – 3)

- Attempt Questions 1 – 4
- Allow about 6 minutes for this section

Section II – **34 marks** (pages 4 – 6)

- Attempt Questions 5 – 7
- Allow about 54 minutes for this section

NAME:________________________________ **TEACHER:**____________________

STUDENT NUMBER:___________________________

Question	1 – 4	5	6	7	Total
Integration	/2	/6	/8	/5	**/21**
Proof	/2	/5	/3	/7	**/17**
Total	/4	/11	/11	/12	**/38**

Section I

4 marks
Attempt Questions 1-4
Allow about 6 minutes for this section

Use the multiple choice answer sheet for Questions 1-4.

---

1 If x and y are real numbers such that $xy < 0$, which of the following must be true?

A. $\frac{x}{y} < 0$

B. $x + y < 0$

C. $x^2 > y^2$

D. $x^2 - y^2 < 0$

A test using some values e.g., $x = -1,\ y = 2$ will show that B and C are not true. Testing with $x = 2,\ y = -1$ will show that D is not true.

2 Which of the following statements is true $\forall a, b \in \mathbb{R}$?

A. $|a+b| < |a| + |b|$

B. $\frac{a}{b} + \frac{b}{a} \geq 2$

C. $a^2 + b^2 \geq 2ab$

D. $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$

A not true for $a = b = 0$. B not true when a and b are opposite in sign. D not true for a or $b = 0$.

3 Which of the following is equivalent to $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta+\sin\theta}$ using the substitution $t=\tan\frac{\theta}{2}$?

A. $\int_0^1 \frac{2dt}{(t+1)^2}$

B. $\int_0^1 \frac{dt}{(t+1)^2}$

C. $\int_0^1 \frac{dt}{2-(t-1)^2}$

D. $\int_0^1 \frac{2\,dt}{2-(t-1)^2}$

$$t=\tan\frac{\theta}{2}$$

$$dt=\frac{1}{2}\sec^2\theta\,d\theta$$

$$\frac{2\,dt}{1+t^2}=d\theta\left(\because \sec^2\theta=1+\tan^2\theta=1+t^2\right)$$

$$\text{when }\theta=\frac{\pi}{2},\ t=\tan\frac{\left(\frac{\pi}{2}\right)}{2}=1$$

$$\text{when }\theta=0,\ t=\tan\frac{0}{2}=0$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta+\sin\theta}=\int_0^1 \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)+\left(\frac{2t}{1+t^2}\right)}\left(\frac{2\,dt}{1+t^2}\right)$$

$$=\int_0^1 \frac{2\,dt}{1-t^2+2t}$$

$$=\int_0^1 \frac{2\,dt}{2-(t-1)^2}$$

4 Of the following expressions, which one need NOT contain a term involving a logarithm in its anti-derivative?

A. $\dfrac{x+3}{x^2+6x+13}$

B. $\dfrac{x^2-1}{x^3-2x^2-x+2}$

C. $\dfrac{x^2-1}{x+2}$

D. $\dfrac{x^3+x+1}{1+x^2}$

$$\begin{aligned}\int \frac{x^3+x+1}{1+x^2}\,dx &= \int \frac{x\left(x^2+1\right)+1}{1+x^2}\,dx \\ &= \int x+\frac{1}{1+x^2}\,dx \\ &= \frac{x^2}{2}+\tan^{-1}x+C\end{aligned}$$

End of Section I

Section II

34 marks
Attempt Questions 5-7
Allow about 54 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 5-7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (11 marks) Use a SEPARATE writing booklet

(a) Use integration by parts to evaluate $\int xe^{-x}dx$. **2**

$$\begin{aligned}&\int xe^{-x}dx\\&=-\int x\,d\left(e^{-x}\right)\\&=-\left[xe^{-x}-\int e^{-x}dx\right]\\&=-\left[xe^{-x}+e^{-x}\right]+C\\&=-(x+1)e^{-x}+C\end{aligned}$$

Marker's Comments
Well done. The major issue was with sign.

(b) Find $\int \frac{x}{\sqrt{6-x}}dx$. **2**

$$\begin{aligned}&\int \frac{x}{\sqrt{6-x}}dx\\&=-\int \frac{-x}{\sqrt{6-x}}dx\\&=-\int \frac{6-x-6}{\sqrt{6-x}}dx\\&=-\int (6-x)^{\frac{1}{2}}-6(6-x)^{-\frac{1}{2}}\,dx\\&=\frac{2(6-x)^{\frac{3}{2}}}{3}-12(6-x)^{\frac{1}{2}}+C\end{aligned}$$

Marker's Comments
Generally well done. Many students didn't use the most efficient method - see solutions.

(c) Find $\int \cos^2 x \sin^2 x \, dx$. **2**

$$\begin{aligned}
&\int \cos^2 x \sin^2 x \, dx \\
&= \int (\sin x \cos x)^2 \, dx \\
&= \int \left(\frac{1}{2}\sin 2x\right)^2 dx \\
&= \frac{1}{4}\int \sin^2 2x \, dx \\
&= \frac{1}{8}\int 1 - \cos 4x \, dx \\
&= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C
\end{aligned}$$

Marker's Comments

Again, many students did not use an efficient method. Otherwise well done.

(d) Prove that $16a^4 + b^2 \geq 8a^2 b \ \forall \ a, b \in \mathbb{R}$. **2**

Consider $(4a^2 - b)^2$

$$\begin{aligned}
(4a^2 - b)^2 &\geq 0 \\
16a^4 - 8a^2 b + b^2 &\geq 0 \\
16a^4 + b^2 &\geq 8a^2 b \quad \text{(as required)}
\end{aligned}$$

Marker's Comments

Well done. You may not begin by quoting the AM-GM inequality.

(e) Use Mathematical Induction to prove that $4^n < (n+2)!$ for all positive integers n. **3**

When $n = 1$,

$$\text{LHS} = 4^1$$

$$\text{RHS} = (1+2)! = 3! = 6$$

$$\text{LHS} < \text{RHS}$$

$\therefore$ it is true for $n = 1$.

Assume that it is true for $n = k, k \geq 1$

$$4^k < (k+2)!$$

When $n = k+1$,

$$\begin{aligned} 4^{k+1} &= 4 \times 4^k \\ &< 4(k+2)! \quad \text{(by assumption)} \\ &\leq (k+3)(k+2)! \quad (\because 4 \leq (k+3) \text{ for } k \geq 1) \\ &= (k+3)! \\ &= ((k+1)+2)! \end{aligned}$$

$\therefore$ it is true for $n = k+1$ if $n = k$ is true.

Hence $4^k > (k+2)!$ for all positive integers n by Mathematical Induction.

Marker's Comments

Generally well done

Question 6 (11 marks) Use a SEPARATE writing booklet

(a) Find $\int \frac{1}{\sqrt{8+4x-4x^2}}\,dx$. **2**

$$\int \frac{1}{\sqrt{8+4x-4x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{2+x-x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2\left(x-\frac{1}{2}\right)}{3}\right)+C$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$$

or

$$\int \frac{1}{\sqrt{8+4x-4x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{9-(2x-1)^2}}\,dx$$
$$= \int \frac{1}{\sqrt{3^2-(2x-1)^2}}\,dx$$
$$= \frac{1}{2}\sin^{-1}\frac{2x-1}{3}+C$$

Marker's comments:

Generally well done.

The most common error was not completing the square properly. Take care with your fractions.

(b) (i) Express $\dfrac{x}{(x^2+9)(x-1)}$ in the form of $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. **2**

$$\frac{x}{(x^2+9)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$$

$$\frac{x}{(x^2+9)(x-1)}=\frac{A\left(x^2+9\right)+\left(Bx+C\right)\left(x-1\right)}{\left(x-1\right)\left(x^2+9\right)}$$

$$x=A\left(x^2+9\right)+\left(Bx+C\right)\left(x-1\right)$$

When $x=1$,

$$1=A\left(1^2+9\right)$$

$$A=\frac{1}{10}$$

When $x=0$,

$$0=\frac{1}{10}\left(0^2+9\right)+C\left(0-1\right)$$

$$C=\frac{9}{10}$$

By equating coefficient of x^2,

$$A+B=0$$

$$B=\frac{-1}{10}$$

$$\therefore\frac{x}{(x^2+9)(x-1)}=\frac{\left(\frac{1}{10}\right)}{\left(x-1\right)}+\frac{\left(-\frac{1}{10}\right)x+\left(\frac{9}{10}\right)}{\left(x^2+9\right)}$$

Marker's Comments:

Well done.

Generally, you should answer the question in the form requested. Some students did not write their final answer and almost lost a mark. It was returned if you wrote the fractions correctly in part (ii).

Hence evaluate $\int \frac{x}{(x^2+9)(x-1)}\,dx$. **2**

$$\int \frac{x}{(x^2+9)(x-1)}\,dx = \int \frac{1}{10(x-1)} + \frac{-x+9}{10(x^2+9)}\,dx \qquad (\text{from (i)})$$

$$= \int \frac{1}{10(x-1)} + \frac{-2x}{20(x^2+9)} + \frac{9}{10(x^2+3^2)}\,dx$$

$$= \frac{1}{10}\ln(x-1) - \frac{1}{20}\ln(x^2+9) + \frac{3}{10}\tan^{-1}\left(\frac{x}{3}\right) + C$$

Marker's Comments

Well done.

Most common error was factorising out 1/10 and then not distributing it back. Again, be careful with your fractions.

(c) You are given that $\dfrac{1}{1+\sin 2x} = \dfrac{\sec^2 x}{(1+\tan x)^2}$. (Do NOT prove this.) **2**

Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{1+\sin 2x}\,dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{1+\sin 2x}\,dx &= \int_0^{\frac{\pi}{4}} \frac{1+\sin 2x-1}{1+\sin 2x}\,dx \\
&= \int_0^{\frac{\pi}{4}} 1-\frac{1}{1+\sin 2x}\,dx \\
&= \int_0^{\frac{\pi}{4}} 1-\frac{\sec^2 x}{(1+\tan x)^2}\,dx\,(\text{given}) \\
&= \int_0^{\frac{\pi}{4}} 1\,dx - \int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan x)^2}\,d(\tan x) \\
&= [x]_0^{\frac{\pi}{4}} + \left[(1+\tan x)^{-1}\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} + \left(\frac{1}{2}-1\right) \\
&= \frac{\pi}{4} - \frac{1}{2}
\end{aligned}$$

Marker's Comments

Progress on this question was mixed.

Students who noticed how to manipulate the numerator (see solutions) were generally successful. Students who subbed in first had much more difficulty.

It was possible to sub in, then use integration by parts followed by partial fractions or t-formula. However, only one student successfully managed to solve it using this method. This is a much more time-consuming method.

(d) A sequence is given by the following recurrence relation, **3**

$$T_1 = 1,\ T_n = \frac{3T_{n-1} - 1}{4T_{n-1} - 1} \text{ for } n \geq 2.$$

Use Mathematical Induction to prove that the general formula for the sequence is $T_n = \dfrac{n}{2n-1}$.

When $n = 1$,

$$T_1 = 1$$

$$T_1 = \frac{1}{2 \times 1 - 1} = 1$$

$\therefore$ the general formula is true for $n = 1$.

Assume that the general formula is true for $n = k$, $k \geq 1$

$$T_k = \frac{k}{2k-1}$$

When $n = k+1$,

$$T_{k+1} = \frac{3T_k - 1}{4T_k - 1}$$

$$= \frac{3\left(\frac{k}{2k-1}\right) - 1}{4\left(\frac{k}{2k-1}\right) - 1} \quad \text{(by assumption)}$$

$$= \frac{3k - (2k-1)}{4k - (2k-1)}$$

$$= \frac{k+1}{2k+1}$$

$$= \frac{k+1}{2(k+1) - 1}$$

$\therefore$ the general formula is true for $n = k+1$ if it is true for $n = k$.

Hence $T_n = \dfrac{n}{2n-1}$ is the general formula of the sequence by Mathematical Induction.

Marker's Comments

Well done.

The sequence start at n=1, which is your base case. Those who successfully proved the general term for n=2 as their base case were given half a mark.

In a proof, make sure that you sub into formulas directly. Do not simplify when subbing in, you are skipping lines of the proof.

Some students are still losing half a mark for not noting when they use the assumption. This is the third test in a row where students have lost this mark.

Question 7 (12 marks) Use a SEPARATE writing booklet

(a) Consider $I_n = \int_1^e x^{-3}(\ln x)^n \, dx$, $n \geq 0$.

(i) Prove that for $n \geq 1$, $I_n = \dfrac{-1}{2e^2} + \dfrac{n}{2} I_{n-1}$. **3**

$$I_n = \int_1^e x^{-3}(\ln x)^n \, dx$$

$$= \int_1^e (\ln x)^n \, d\left(\frac{-x^{-2}}{2}\right)$$

$$= \left[\frac{-x^{-2}}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{-x^{-2}}{2} d(\ln x)^n$$

$$= \left(\frac{-e^{-2}}{2}(\ln e)^n - \left(\frac{-1^{-2}}{2}(\ln 1)^n\right)\right) + \frac{1}{2}\int_1^e x^{-2} \times n(\ln x)^{n-1} \times \frac{1}{x} dx$$

$$= -\frac{1}{2e^2} + \frac{n}{2}\int_1^e x^{-3}(\ln x)^{n-1} \, dx$$

$$= -\frac{1}{2e^2} + \frac{n}{2} I_{n-1}$$

Marker's comments

This part was done well. Most students were able to apply integration by parts to obtain the reduction formula. Since this is a "prove that" question, it is important that you don't skip steps as the answer is provided to you.

(ii) Hence, evaluate $\int_1^e x^{-3}(\ln x)^2\,dx$. **2**

$$I_0=\int_1^e x^{-3}(\ln x)^0\,dx$$
$$=\int_1^e x^{-3}dx$$
$$=\left[-\frac{1}{2x^2}\right]_1^e$$
$$=\left(-\frac{1}{2e^2}-\left(-\frac{1}{2}\right)\right)$$
$$=-\frac{1}{2e^2}+\frac{1}{2}$$

$$I_2=\frac{-1}{2e^2}+\frac{2}{2}I_1$$
$$=\frac{-1}{2e^2}+\left(\frac{-1}{2e^2}+\frac{1}{2}I_0\right)$$
$$=\frac{-1}{2e^2}+\left(\frac{-1}{2e^2}+\frac{1}{2}\left(-\frac{1}{2e^2}+\frac{1}{2}\right)\right)$$
$$=\frac{-1}{2e^2}+\left(\frac{-1}{2e^2}-\frac{1}{4e^2}+\frac{1}{4}\right)$$
$$=\frac{-1}{2e^2}+\left(\frac{-3}{4e^2}+\frac{1}{4}\right)$$
$$=\frac{-5}{4e^2}+\frac{1}{4}$$

Marker's comments

This part was done well. Most students were able to apply the reduction formula to evaluate the integral. Common mistakes general involves algebraic error and/or not substituting the appropriate value of *n* when applying the reduction formula.

(b) Prove that $t\ln t-(1+t)\ln(1+t)<0$ for all $t>0$. **2**

Method 1:

$$t\ln t-(1+t)\ln(1+t)$$
$$=\ln t^t-\ln(1+t)^{1+t}$$
$$=\ln\left(\frac{t^t}{(1+t)^{1+t}}\right)$$
$$\because t<1+t$$
$$t^t<(1+t)^t$$
$$t^t<(1+t)^t<(1+t)^{1+t}$$
$$\therefore t^t<(1+t)^{1+t}$$
$$0<\frac{t^t}{(1+t)^{1+t}}<1$$
$$\therefore \ln\left(\frac{t^t}{(1+t)^{1+t}}\right)<0$$

Hence, $t\ln t-(1+t)\ln(1+t)<0$

Method 2:

$$t\ln t-(1+t)\ln(1+t)$$
$$=t\ln t-\ln(1+t)-t\ln(1+t)$$
$$=t\left(\ln t-\ln(1+t)\right)-\ln(1+t)$$
$$=t\ln\frac{t}{1+t}-\ln(1+t)$$

$$\because t<1+t$$
$$0<\frac{t}{1+t}<1 \text{ for } t>0$$
$$\ln\frac{t}{1+t}<0$$
$$\therefore t\ln\frac{t}{1+t}<0$$

$$\because (1+t)>1$$
$$\ln(1+t)>0$$
$$-\ln(1+t)<0$$

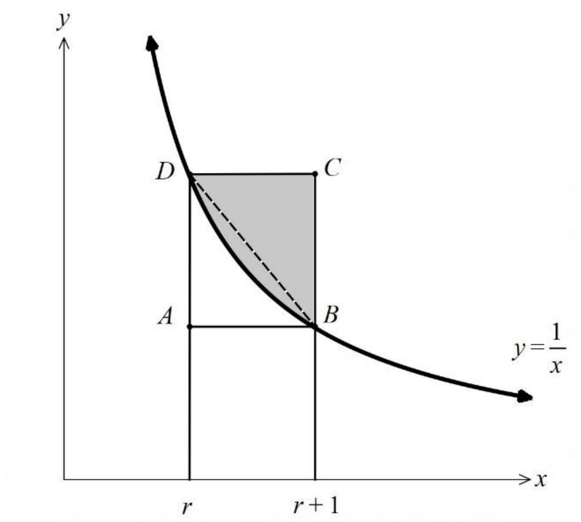
$$\therefore t\ln\frac{t}{1+t}-\ln(1+t)<0$$

Hence $t\ln t-(1+t)\ln(1+t)<0$

Marker's comments

This part was not done well. Some students have included incorrect and/or incomplete inequality arguments in their proof. Marks were not awarded or have been deducted for inadequate reason. Students who tried to use calculus/ derivative were generally unsuccessful.

(c) Consider the graph of $y = \dfrac{1}{x}$ shown below.



(i) The area of the shaded region is denoted by A_r. **2**

Show that $\dfrac{1}{2}\left(\dfrac{1}{r}-\dfrac{1}{r+1}\right) \le A_r \le \dfrac{1}{r}-\dfrac{1}{r+1}$.

By considering the area of triangle BCD and A_r,

$$A_{\Delta BCD} < A_r$$

$$\frac{1}{2} \times BC \times CD < A_r$$

$$\frac{1}{2} \times \left(\frac{1}{r}-\frac{1}{r+1}\right) \times 1 < A_r$$

$$\frac{1}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right) < A_r$$

By considering the area of rectangle $ABCD$ and A_r,

$$A_r < A_{ABCD}$$

$$A_r < AB \times BC$$

$$A_r < 1 \times \left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$A_r < \left(\frac{1}{r}-\frac{1}{r+1}\right)$$

Combining the two inequalities gives

$$\frac{1}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right) \le A_r \le \frac{1}{r}-\frac{1}{r+1}$$

Marker's comments

This part was done well. Most students were able to use area arguments to derive the inequality. Similar to an earlier question, your working out should be clear and include enough detail as the inequality is given. Marks were deducted for incorrect working out e.g. using area of a trapezium rather than a triangle to obtain the left inequality. Please also take note that the 'upper rectangle' is not the same as rectangle *ABCD*. The upper rectangle goes from *DC* all the way down to the *x* axis.

(ii) Let the sum S_n be defined as $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. **3**

Using part (i), show that $\frac{1}{2}\left(1 + \frac{1}{n}\right) \le S_n - \ln n \le 1$.

From the diagram, it can be deduced that

$A_{\text{under graph}} = A_{\text{upper rectangle}} - A_r$

$A_{\text{under graph}} = \frac{1}{r} - A_r$

Now consider the area under the graph of $y = \frac{1}{x}$ between $x = 1$ and $x = n$

$$A_{\text{under graph}} = \sum_{r=1}^{n-1}\left(\frac{1}{r} - A_r\right)$$

$$\int_1^n \frac{1}{x}\,dx = \sum_{r=1}^{n-1}\frac{1}{r} - \sum_{r=1}^{n-1} A_r$$

$$\sum_{r=1}^{n-1} A_r = \sum_{r=1}^{n-1}\frac{1}{r} - \ln n$$

$$\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) \le A_r \le \frac{1}{r} - \frac{1}{r+1}$$

$$\sum_{r=1}^{n-1}\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) \le \sum_{r=1}^{n-1} A_r \le \sum_{r=1}^{n-1}\frac{1}{r} - \frac{1}{r+1}$$

$$\sum_{r=1}^{n-1}\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) \le \sum_{r=1}^{n-1}\frac{1}{r} - \ln n \le \sum_{r=1}^{n-1}\frac{1}{r} - \frac{1}{r+1}$$

$$\frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)\right] \le S_{n-1} - \ln n \le \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$$

$$\frac{1}{2}\left(1 - \frac{1}{n}\right) \le S_{n-1} - \ln n \le 1 - \frac{1}{n}$$

$$\frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{n} \le S_{n-1} - \ln n + \frac{1}{n} \le 1 - \frac{1}{n} + \frac{1}{n}$$

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) \le S_n - \ln n \le 1$$

Marker's comments

Majority of students attempted this question. Some students tried to integrate the inequality in i) but unfortunately this would not lead to the inequality in ii). Some students recognised that the $\ln n$ represents the area under the graph of $y = \frac{1}{x}$ from 1 to n but was not able to progress any further. Some students also misinterpreted that the sum of A_r gives S_n which is incorrect. As this is the last question of the exam, only a handful of students received the full three marks.