NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2024 **HSC ASSESSMENT TASK 3**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks – 38

Section I – 4 marks (pages 2–3)

- Attempt Questions 1–4
- Allow about 8 minutes for this section

Section II – 34 marks (pages 5–7)

- Attempt Questions 5–7
- Allow about 52 minutes for this section

NAME: ______________________ STUDENT NUMBER: ____________________

TEACHER: ____________________

	Multiple Choice	Question 5	Question 6	Question 7	Total
Proof	/3	/ 2	/7	/4	/16
Integration	/1	/ 8	/5	/8	/22
Total	/4	/ 10	/12	/12	/38

Section I

4 marks
Attempt Questions 1–4
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–4.

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1 Without evaluating the integrals, which one of the following will give an answer of zero?

A. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^3\theta + 1}{\cos^2\theta}\,dx$

B. $\displaystyle\int_{-1}^{1} (x^2 - 1)^2\,dx$

C. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \cos x\,dx$

D. $\displaystyle\int_{-2}^{2} |x^2 - 4|\,dx$

2 Given that $x > 0$, what is the smallest value of $\dfrac{x}{2} + \dfrac{8}{x}$?

A. 16

B. 8

C. 4

D. 2

3 If a and b are positive, which of the following is false?

A. $(a+b)^2 \geq 4ab$

B. $\dfrac{1}{a} + \dfrac{1}{b} < \dfrac{4}{a+b}$

C. $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$

D. $(a+b)^2 \leq 2(a^2+b^2)$

4 Which of the following inequalities is incorrect?

A. $\displaystyle\int_0^1 \ln(1+x)\,dx > \int_0^1 \frac{x-1}{e-1}\,dx$

B. $\displaystyle\int_0^{\frac{\pi}{4}} \sin x\,dx > \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

C. $\displaystyle\int_0^1 e^x\,dx > \int_0^1 \frac{1}{1+x^2}\,dx$

D. $\displaystyle\int_0^1 e^{-x^2}\,dx > \int_0^1 e^{-x^3}\,dx$

End of Section I

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Section II

34 marks
Attempt Questions 5–7
Allow about 52 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (10 marks) Use a SEPARATE Writing Booklet

(a) Prove for any real numbers $x,\ y,\ z$ and u that **2**

$$\left(x^2+y^2\right)\left(z^2+u^2\right) \geq \left(xz+yu\right)^2.$$

(b) Using integration by parts find $\displaystyle\int \sqrt{x}\ln x\,dx$. **2**

(c) By writing $\dfrac{9x}{(x+2)(x-1)^2}$ in the form $\dfrac{A}{x+2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, **4**

find $\displaystyle\int \frac{9x}{(x+2)(x-1)^2}\,dx$.

(d) Find $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,dx$. **2**

Question 6 (12 marks) Use a SEPARATE Writing Booklet

(a) You are given that for positive real numbers $x_1, x_2, \ldots, x_n$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

(i) Show that for positive real numbers x, y and z **1**

$$\frac{x^3}{yz} + y + z \geq 3x.$$

(ii) Hence, or otherwise show for positive real numbers x, y and z **2**

$$\frac{x^3}{yz} + \frac{y^3}{xz} + \frac{z^3}{xy} \geq x + y + z.$$

(b) Use symmetry to evaluate $\displaystyle\int_0^{\pi} \frac{\sin 4x}{1 + \sin x}\, dx$. **2**

(c) Use the substitution $t = \tan \dfrac{x}{2}$ to find **3**

$$\int_0^{\frac{2\pi}{3}} \frac{dx}{5 + 3\sin x + 4\cos x}.$$

(d) (i) Show that $\sin\left(\theta + \dfrac{\pi}{2}\right) = \cos\theta$. **1**

(ii) Consider $f(x) = \sin(ax)$ where a is a constant. **3**

Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$

where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

Question 7 (12 marks) Use a SEPARATE Writing Booklet

(a) Use a suitable substitution to find $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx$. **3**

(b) Let $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x^3}\,dx$ for $n \geq 2$.

(i) Show that for $n \geq 5$ **3**

$$I_n = \frac{2n-4}{2n+5}\,I_{n-3}.$$

(ii) Hence find I_5. **2**

(c) Prove by mathematical induction for $n \geq 2$: **4**

$$n\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right).$$

You may use the following result where $k \geq 2$:

$$\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2k-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}$$

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Q1) C Q2) C Q3) B Q4) D

Question 5

a) Consider

$(x^2+y^2)(z^2+u^2)-(xz+yu)^2$

$= x^2z^2+x^2u^2+y^2z^2+y^2u^2-(x^2z^2+2xzyu+y^2u^2)$

$= \cancel{x^2z^2}+x^2u^2+y^2z^2+\cancel{y^2u^2}-\cancel{x^2z^2}-2xzyu-\cancel{y^2u^2}$

$= x^2u^2+y^2z^2-2xzyu$

$= (xu-yz)^2 \geq 0$ as $(xu-yz)^2$ is a perfect square

$\therefore (x^2+y^2)(z^2+u^2)-(xz+yu)^2 \geq 0$

$\therefore (x^2+y^2)(z^2+u^2) \geq (xz+yu)^2$

b) $\int \sqrt{x}\ln x\,dx$

$u = \ln x \quad v' = \sqrt{x}$

$u' = \frac{1}{x} \quad v = \frac{2x^{\frac{3}{2}}}{3}$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{1}{x}\times\frac{2}{3}x^{\frac{3}{2}}\,dx$ (by parts)

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\times\frac{2x^{\frac{3}{2}}}{3}$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$

c) $\frac{9x}{(x+2)(x-1)^2} \equiv \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$9x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

sub $x=1$: $9 = 3C \quad \therefore C = 3$

sub $x=-2$: $-18 = A(-3)^2$

$-18 = 9A$

$\therefore A = -2$

coefficient of x^2: $0 = A + B$

$0 = -2 + B$

$B = 2$

$\therefore \frac{9x}{(x+2)(x-1)^2} \equiv \frac{-2}{x+2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$

$\therefore \int \frac{9x}{(x+2)(x-1)^2}\,dx = \int\left(\frac{-2}{x+2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}\right)dx$

$= -2\ln|x+2| + 2\ln|x-1| - \frac{3}{(x-1)} + C$

d) $\int \frac{1}{(1+x)\sqrt{x}}\,dx$ Let $u=\sqrt{x}$

$= 2\int \frac{1}{(1+x)} \times \frac{dx}{2\sqrt{x}}$ $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$= 2\int \frac{1}{1+u^2}\,du$ $du = \frac{dx}{2\sqrt{x}}$

$= 2\tan^{-1} u$

$= 2\tan^{-1}(\sqrt{x}) + c$

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Question 6

a) i) $\frac{1}{3}\left(\frac{x^3}{yz}+y+z\right) \geq \sqrt[3]{\frac{x^3}{yz}\times y\times z}$ using AM-GM given

$$\frac{x^3}{yz}+y+z \geq 3\sqrt[3]{x^3}$$

$$\frac{x^3}{yz}+y+z \geq 3x \quad \text{— (1)}$$

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ii) Similarly

$$\frac{y^3}{xz}+x+z \geq 3y \text{ — (2)} \quad \& \quad \frac{z^3}{xy}+x+y \geq 3z \text{ — (3)}$$

(1) + (2) + (3)

$$\frac{x^3}{yz}+y+z+\frac{y^3}{xz}+x+z+\frac{z^3}{xy}+x+y \geq 3x+3y+3z$$

$$\frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy}+2(x+y+z) \geq 3(x+y+z)$$

$$\therefore \frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy} \geq 3(x+y+z)-2(x+y+z)$$

$$\therefore \frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy} \geq x+y+z$$

b) $I=\int_0^{\pi}\frac{\sin 4x}{1+\sin x}\,dx$

$$=\int_0^{\pi}\frac{\sin 4(\pi-x)}{1+\sin(\pi-x)}\,dx$$

$$=\int_0^{\pi}\frac{\sin(4\pi-4x)}{1+\sin x}\,dx$$

$$=\int_0^{\pi}\frac{\sin(-4x)}{1+\sin x}\,dx$$

$$=\int_0^{\pi}\frac{-\sin 4x}{1+\sin x}\,dx \quad \text{as sine is odd}$$

$$=-I$$

$$2I=0$$

$$I=0$$

$$\therefore \int_0^{\pi}\frac{\sin 4x}{1+\sin x}\,dx=0$$

c) $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{dx}{5+3\sin x+4\cos x}$

$\displaystyle = \int_0^{\sqrt{3}} \frac{1}{5+\frac{3\times 2t}{1+t^2}+\frac{4(1-t^2)}{1+t^2}} \times \frac{2dt}{1+t^2}$

$\displaystyle = \int_0^{\sqrt{3}} \frac{2dt}{5(1+t^2)+6t+4(1-t^2)}$

$\displaystyle = \int_0^{\sqrt{3}} \frac{2dt}{5+5t^2+6t+4-4t^2}$

$\displaystyle = \int_0^{\sqrt{3}} \frac{2dt}{t^2+6t+9}$

$\displaystyle = 2\int_0^{\sqrt{3}} \frac{dt}{(t+3)^2}$

$\displaystyle = 2\left[\frac{(t+3)^{-1}}{-1}\right]_0^{\sqrt{3}}$

$\displaystyle = -2\left[\frac{1}{t+3}\right]_0^{\sqrt{3}}$

$\displaystyle = -2\left(\frac{1}{\sqrt{3}+3} - \frac{1}{3}\right)$

$\displaystyle = 2\left(\frac{1}{3} - \frac{1}{3+\sqrt{3}}\right)$

$\displaystyle = 2\left(\frac{1}{3} - \frac{3-\sqrt{3}}{6}\right)$

$\displaystyle = 2 \times \frac{2-3+\sqrt{3}}{6}$

$\displaystyle = \frac{\sqrt{3}-1}{3}$

let $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$x = 0 \Rightarrow t = 0$

$x = \frac{2\pi}{3} \Rightarrow t = \sqrt{3}$

d) i) $\sin\left(\theta + \frac{\pi}{2}\right)$

$= \sin\theta \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\theta$

$= \sin\theta \times 0 + 1 \times \cos\theta$

$= \cos\theta$

ii) Prove true for $n=1$:

$f(x) = \sin(ax)$

LHS $= f^{(1)}(x)$ RHS $= a^1 \sin\left(ax + \frac{\pi}{2}\right)$

$= a\cos(ax)$ $= a\sin\left(ax + \frac{\pi}{2}\right)$

$= a\cos(ax)$ (using part (i))

Assume result is true for $n=k$, $k \in \mathbb{Z}^+$

$\therefore f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$

Prove result is true for $n = k+1$

RTP: $f^{(k+1)}(x) = a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$

From the assumption:

$f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$

$f^{(k+1)}(x) = \frac{d}{dx}\left(f^k(x)\right)$

$= \frac{d}{dx}\left(a^k \sin\left(ax + \frac{k\pi}{2}\right)\right)$

$= a^k \times a\cos\left(ax + \frac{k\pi}{2}\right)$

$= a^{k+1}\cos\left(ax + \frac{k\pi}{2}\right)$

$= a^{k+1}\sin\left(\left(ax + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right)$ using part (i) where $\theta = ax + \frac{k\pi}{2}$

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$

$=$ RHS.

$\therefore$ The result is true for $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

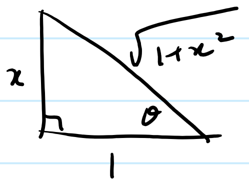
Question 7

a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx$

let $x = \tan\theta$

$\dfrac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$



$\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

$$= \int \frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta}\sec^2\theta\, d\theta$$
$$= \int \frac{\sec\theta}{\tan^4\theta}\sec^2\theta\, d\theta$$
$$= \int \sec^3\theta \times \frac{\cos^4\theta}{\sin^4\theta}\, d\theta$$
$$= \int \frac{1}{\cos^3\theta} \times \frac{\cos^4\theta}{\sin^4\theta}\, d\theta$$
$$= \int \cos\theta(\sin\theta)^{-4}\, d\theta$$
$$= \frac{(\sin\theta)^{-3}}{-3}$$
$$= -\frac{1}{\sin^3\theta}$$
$$= -\frac{1}{\left(\frac{x}{\sqrt{1+x^2}}\right)^3}$$
$$= -\frac{(1+x^2)\sqrt{1+x^2}}{x^3} + C$$

b) i) $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x^3}\,dx \quad n \geq 2$

$u = x^{n-2}$ $\qquad v' = x^2(1-x^3)^{\frac{1}{2}}$

$u' = (n-2)x^{n-3}$ $\qquad v = -\frac{1}{3}(1-x^3)^{\frac{3}{2}} \times \frac{2}{3}$

$\qquad = -\frac{2}{9}(1-x^3)^{\frac{3}{2}}$

$$I_n = \left[x^{n-2} \times -\frac{2}{9}(1-x^3)^{\frac{3}{2}}\right]_0^1 - \int_0^1 (n-2)x^{n-3} \times -\frac{2}{9}(1-x^3)^{\frac{3}{2}}\,dx \quad \text{(by parts)}$$
$$= 0 - 0 + \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)^{\frac{3}{2}}\,dx$$
$$= \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)\sqrt{1-x^3}\,dx$$
$$= \frac{2}{9}(n-2)\int_0^1 x^{n-3}\sqrt{1-x^3} - x^n\sqrt{1-x^3}\,dx$$
$$= \frac{2}{9}(n-2)\left[\int_0^1 x^{n-3}\sqrt{1-x^3}\,dx - \int_0^1 x^n\sqrt{1-x^3}\,dx\right]$$
$$= \frac{2}{9}(n-2)\left[I_{n-3} - I_n\right]$$
$$= \frac{2}{9}(n-2)I_{n-3} - \frac{2}{9}(n-2)I_n$$

$$I_n + \frac{2}{9}(n-2)I_n = \frac{2}{9}(n-2)I_{n-3}$$
$$9I_n + 2(n-2)I_n = 2(n-2)I_{n-3}$$
$$I_n(9 + 2n - 4) = (2n-4)I_{n-3}$$
$$(2n+5)I_n = (2n-4)I_{n-3}$$
$$I_n = \frac{2n-4}{2n+5}I_{n-3}$$

ii) $I_2 = \displaystyle\int_0^1 x^2\sqrt{1-x^3}\,dx$

$$= -\frac{1}{3}\left[(1-x^3)^{\frac{3}{2}} \times \frac{2}{3}\right]_0^1$$
$$= -\frac{2}{9}[0 - 1]$$
$$= \frac{2}{9}$$

$$I_5 = \frac{2(5)-4}{2(5)+5}I_2$$
$$= \frac{\cancel{6}\,2}{\cancel{15}\,5} \times \frac{2}{9}$$
$$= \frac{4}{45}$$

c) Prove true for $n=2$

$LHS = 2\left(1+\frac{1}{3}\right)$ $\quad$ $RHS = (2+1)\left(\frac{1}{2}+\frac{1}{4}\right)$

$= 2\times\frac{4}{3}$ $\quad$ $= 3\times\frac{3}{4}$ $\quad$ $\therefore$ LHS > RHS

$= \frac{8}{3}$ $\quad$ $= \frac{9}{4}$ $\quad$ $\therefore$ result is true for $n=2$

Assume result is true for $n=k$, $k\geq 2$

$$k\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right)$$

Prove result is true for $n=k+1$

RTP: $(k+1)\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}+\frac{1}{2(k+1)-1}\right) > (k+1+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)}\right)$

ie. $(k+1)\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}+\frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}+\frac{1}{2k+2}\right)$

From the given identity we can deduce

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}$$

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1} > \frac{k}{(2k+1)(2k+2)}+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}\right) \quad (*)$$

$$LHS = (k+1)\left(\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right)+\frac{1}{2k+1}\right)$$

$$= k\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right)+1\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$> \underbrace{(k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)}_{\text{using the assumption}}+\underbrace{\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{k}{(2k+1)(2k+2)}\right)}_{\text{using } (*)}+\frac{k+1}{2k+1}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k}{(2k+1)(2k+2)}+\frac{k+1}{2k+1}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k+(k+1)(2k+2)}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k+2k^2+2k+2k+2}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{2k^2+5k+2}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{\cancel{(2k+1)}(k+2)}{\cancel{(2k+1)}(2k+2)}$$

$$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{1}{2k+2}\right)$$

$= RHS$

$\therefore$ The result is true by the principle of mathematical induction for $n\geq 2$