St Aloysius' College
Year 12 Mid-Year Examinations
2009

EXTENSION 2 MATHEMATICS
(Additional paper)

Total marks: 75

Reading time - 5 minutes
Working time – 2 hours

Examination papers must NOT be removed from the examination room.

General Instructions

- Start each question in a new booklet
- Board approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Show all necessary work

SUPERVISOR'S INSTRUCTIONS:
Please issue five 4 page answer booklets.
Please collect the examination paper with the answer booklets.

Question 1 *(15 marks)*

(a) If $z = 2 - i$,

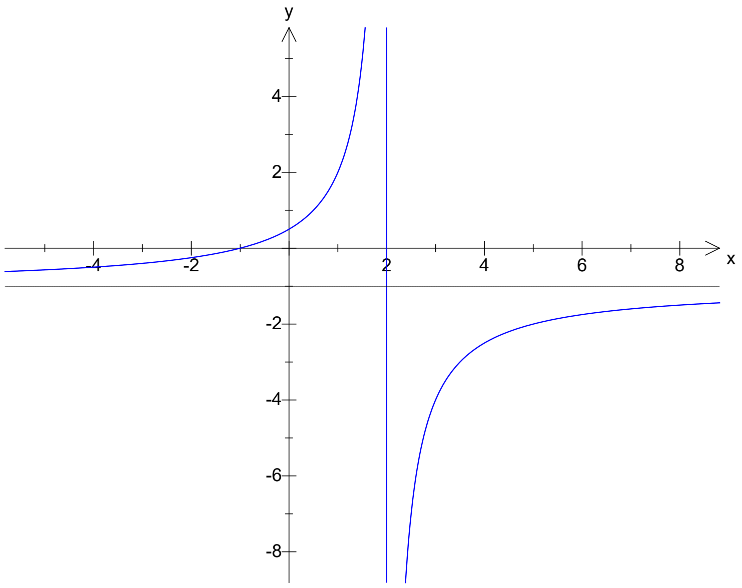
(i) Find $\frac{1}{z}$ in the form $a + ib$. **1**

(ii) Find the real numbers p and q such that $pz + \frac{q}{z} = 1$. **2**

(b) Use De Moivre's theorem to evaluate $(1+i)^7 + (1-i)^7$. **3**

(c) If $|z - 2 - i| = 2$, find the maximum and minimum values of $|z|$. **3**

(d) The graph for $y = \frac{x+1}{2-x}$ is shown below.



On the separate answer sheet provided sketch the graphs of the following:

(i) $y^2 = \frac{x+1}{2-x}$ **2**

(ii) $|y| = \frac{x+1}{2-x}$ **2**

(iii) $|y| = \frac{|x|+1}{2-|x|}$ **2**

Question 2 *(15 marks)* START A NEW BOOKLET

(a) Sketch the locus of z for each of the following:

(i) $2(z+\bar{z})-5i(z-\bar{z})=20$ **2**

(ii) $\dfrac{\pi}{4}\le\arg\left(\dfrac{1}{\bar{z}}\right)\le\dfrac{\pi}{2}$ **3**

(b) If $1, \omega, \omega^2$ are the three cube roots of 1, evaluate

(i) $(1+\omega)^9$ **3**

(ii) $\omega^2+\omega+1+\dfrac{1}{\omega}+\dfrac{1}{\omega^2}$ **2**

(c) Use De Moivre's Theorem to solve $z^4=-4i$, showing all steps in your reasoning.
Write your answers in mod-arg form and show the solutions clearly on an Argand diagram. **5**

Question 3 *(15 marks)* START A NEW BOOKLET

(a) Find the equation of the ellipse with the point (1, 2) as centre, the line $4x = 29$ as directrix, and eccentricity $\dfrac{4}{5}$. **3**

(b) For the curve $y=\dfrac{x^3-4}{x^2}$

(i) Find any stationary points and intercepts. **3**

(ii) Find the equations of any asymptotes for the curve $y=\dfrac{x^3-4}{x^2}$ **2**

(iii) Sketch the curve $y=\dfrac{x^3-4}{x^2}$. **2**

(iv) From the graph find the values of k for which the equation $x^3-kx^2-4=0$ has three distinct real roots. **2**

(c) α, β, γ are three complex numbers represented by the points A, B, C on the Argand diagram and $\dfrac{\alpha-\gamma}{\beta-\gamma}=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$.
Show that the triangle ABC is equilateral. **3**

Question 4 *(15 marks)* START A NEW BOOKLET

(a) (i) Find an expression for the derivative for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **2**

(ii) Hence show that, at the point in the first quadrant where $x=\frac{a}{2}$, the gradient of the tangent to the ellipse is $-\frac{b}{a\sqrt{3}}$. **2**

(b) For the hyperbola $4x^2-y^2=100$ find the

(i) eccentricity, **3**

(ii) coordinates of the foci, **1**

(iii) equations of the directrices, **1**

(iv) equations of the asymptotes. **1**

(c) The locus defined by the equation $|z-1|-|z+3|=2$ in the complex number plane gives a conic.
Find the cartesian equation of the conic <u>and</u> state any restrictions. **5**

Question 5 is on the next page

Question 5 *(15 marks)* START A NEW BOOKLET

(a) θ is real and constant and $0 \le \theta < \frac{\pi}{2}$. Solve the following equation for *z* over the complex numbers. Give your answer in simplest form.

$$z^2 \cos^2 \theta + z \sin 2\theta + 1 = 0$$ **3**

(b) For the curve $\ln y = \cos^{-1} x$

(i) State the domain and range, **3**

(ii) Find any intercepts, stationary points or vertical tangents, **3**

(iii) Sketch the curve showing the above features. **2**

(c) A particle is projected from a point at ground level with speed V and making an angle of α with the horizontal. After time t its horizontal and vertical displacements are x and y respectively. As a consequence the cartesian equation of the projectile's path is

$$y = x \tan \alpha - \frac{g \sec^2 \alpha}{2V^2} x^2 .$$

The projectile passes through the point $P(a, h)$.

Prove that there are two different paths for any given initial speed V if

$$\left(V^2 - gh\right)^2 > g^2\left(a^2 + h^2\right).$$ **4**

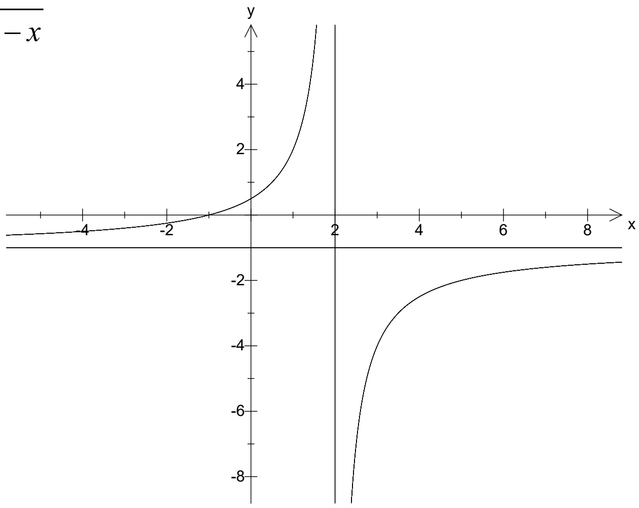
END OF EXAM

Student Number:

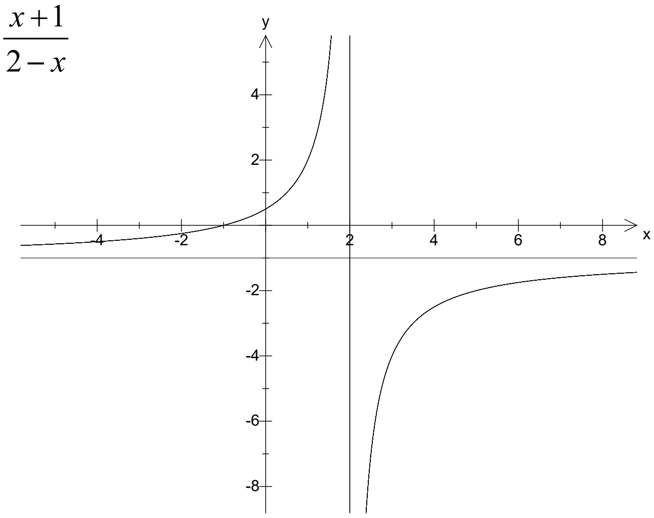
Hand in this page with your answer booklets.

Question 1 (d) Use the graphs of $y = \dfrac{x+1}{2-x}$ as a template to sketch each of the following:

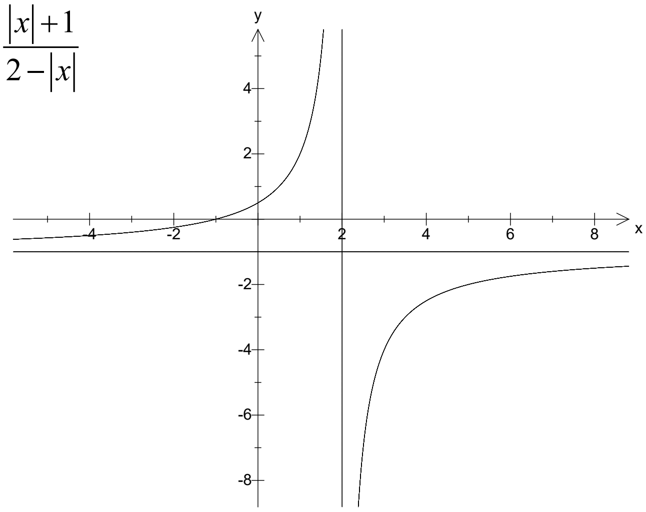
(i) $y^2 = \dfrac{x+1}{2-x}$



(ii) $|y| = \dfrac{x+1}{2-x}$



(iii) $|y| = \dfrac{|x|+1}{2-|x|}$



SAC Y12 MY Ext2 2009 Solutions

Q1 a) i) $\frac{1}{2-i} = \frac{2+i}{(2-i)(2+i)}$

$= \frac{2+i}{2^2 - i^2}$

$= \frac{2+i}{5}$

$= \frac{2}{5} + \frac{i}{5}$ ✓

ii) $pz + \frac{q}{z} = p(2-i) + \frac{q}{2-i}$

$= 2p - pi + q\left(\frac{2}{5} + \frac{i}{5}\right)$

$= 2p + \frac{2q}{5} - pi + \frac{qi}{5}$ ✓

but $pz + \frac{q}{z} = 1 + 0i$

$\therefore \quad 2p + \frac{2q}{5} = 1 \quad$ and $\quad -p + \frac{q}{5} = 0$

$\therefore \quad p = \frac{q}{5}$

$\frac{2q}{5} + \frac{2q}{5} = 1$

$\frac{4q}{5} = 1$

$q = \frac{5}{4}$

$p = \frac{1}{4}.$ ✓

Q 1 (b) $(1+i) = \sqrt{2}\ \text{cis}\,\dfrac{\pi}{4}$

$(1-i) = \sqrt{2}\ \text{cis}\left(\dfrac{-\pi}{4}\right)$

$\therefore (1+i)^7 + (1-i)^7 = (\sqrt{2})^7 \text{cis}\,\dfrac{7\pi}{4} + (\sqrt{2})^7 \text{cis}\left(-\dfrac{7\pi}{4}\right)$

$= 8\sqrt{2}\left[\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4} + \cos\left(\dfrac{-7\pi}{4}\right) + i\sin\left(\dfrac{-7\pi}{4}\right)\right]$

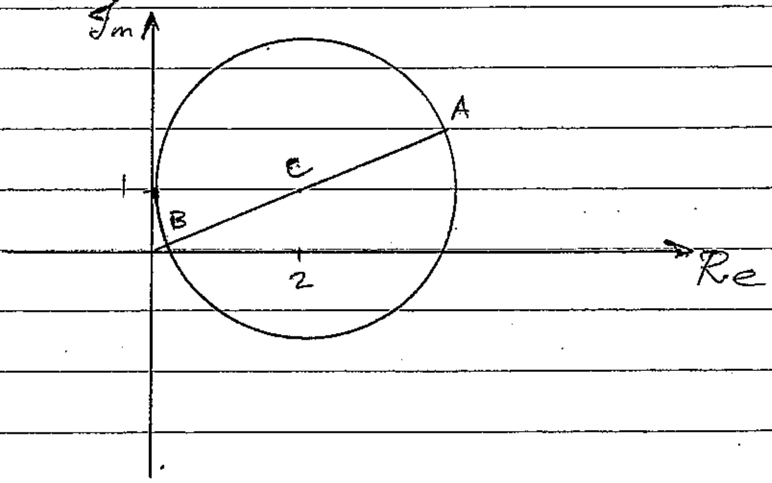
$= 8\sqrt{2}\left[\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4} + \cos\dfrac{7\pi}{4} - i\sin\dfrac{7\pi}{4}\right]$

$= 8\sqrt{2} \times 2\cos\dfrac{7\pi}{4}$

$= 8\sqrt{2} \times \dfrac{2}{\sqrt{2}}$

$= 16.$

(c) $|z-(2+i)| = 2$ gives a circle centre $(2,1)$ $r = 2$



Distance $OC = \sqrt{2^2+1^2}$

$= \sqrt{5}$

$\therefore$ At A $\max|z| = \sqrt{5}+2$

At B $\min|z| = \sqrt{5}-2.$

Q 2 (a) (i) If $z = x+iy$

$\tilde{z} = x - iy$

$z + \tilde{z} = 2x$

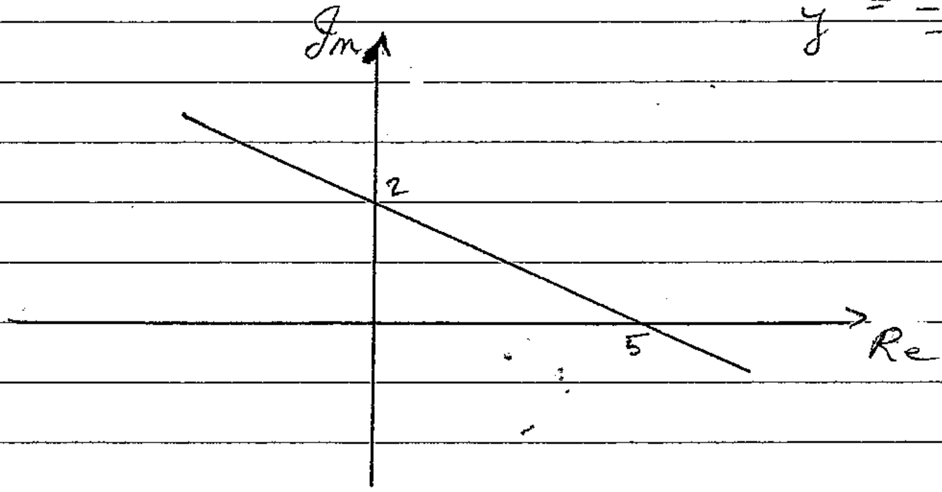
$z - \tilde{z} = 2iy$

$\therefore 2(z+\tilde{z}) - 5i(z-\tilde{z}) = 4x - 5i(2iy)$

$= 4x + 10y = 20$

$2x + 5y = 10$

$y = \frac{-2x}{5} + 2$

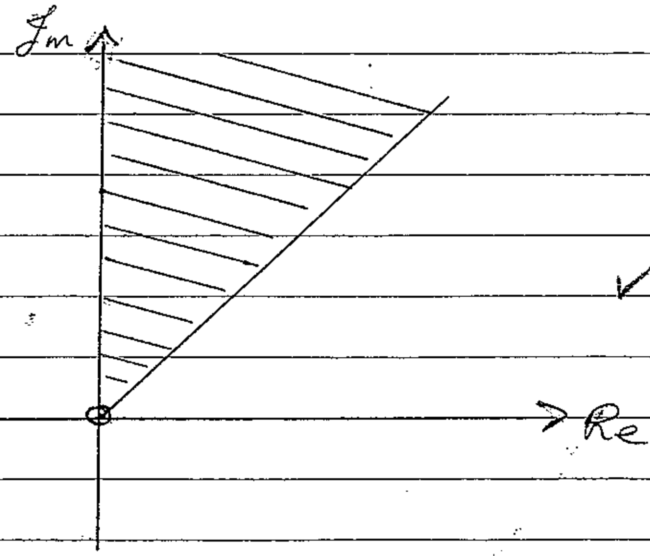


ii) $\arg\left(\frac{1}{\tilde{z}}\right) = \arg 1 - \arg \tilde{z}$

$= 0 - -\arg z$

$= \arg z$

$\therefore \frac{\pi}{4} \leq \arg z \leq \frac{\pi}{2}$.



Note $z \neq 0$.

b) i) $1 + \omega + \omega^2 = 0$

$$1+\omega = -\omega^2$$
$$(1+\omega)^9 = (-\omega^2)^9$$
$$= -\omega^{18}$$
$$= -(\omega^3)^6$$
$$= -1^6$$
$$= -1.$$

ii) $\omega^2 + \omega + 1 + \frac{1}{\omega} + \frac{1}{\omega^2} = 0 + \frac{1}{\omega} + \frac{1}{\omega^2}$

$$= 0 + \frac{\omega^2}{\omega^3} + \frac{\omega}{\omega^3}$$
$$= 0 + \frac{\omega^2}{1} + \frac{\omega}{1}$$
$$= \omega^2 + \omega$$
$$= -1.$$

c) $|z^4| = |-4i|$

$$|z|^4 = 4$$
$$|z| = \sqrt[4]{4}$$
$$= \sqrt{2} \quad \text{since } |z| \geq 0.$$

then $z = \sqrt{2} \operatorname{cis} \theta$

$$z^4 = 4 \operatorname{cis} 4\theta$$
$$= 4\cos 4\theta + 4i \sin 4\theta = 0 + -4i$$

then $\cos 4\theta = 0$ and $\sin 4\theta = -1.$

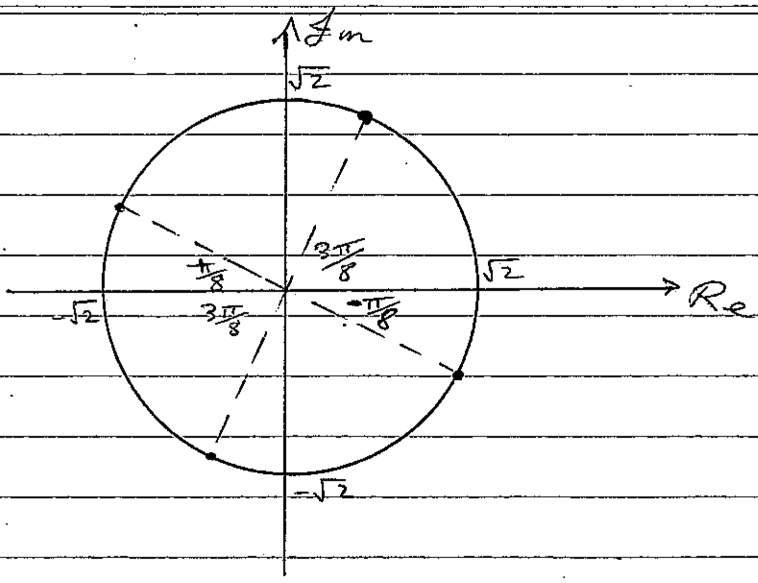
$$4\theta = \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\theta = \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}.$$

or $\theta = -\frac{5\pi}{8}, \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}.$

$$z = \sqrt{2} \operatorname{cis}\left(-\frac{5\pi}{8}\right), \sqrt{2} \operatorname{cis}\left(\frac{-\pi}{8}\right), \sqrt{2} \operatorname{cis} \frac{3\pi}{8}, \sqrt{2} \operatorname{cis} \frac{7\pi}{8}.$$

Q2 c)



Im
$\sqrt{2}$
$\frac{3\pi}{8}$
$\frac{\pi}{8}$
$\sqrt{2}$
Re
$-\sqrt{2}$
$\frac{3\pi}{8}$
$\frac{\pi}{8}$
$-\sqrt{2}$

Q 3 (a) Directrix $x = \frac{29}{4}$

$$1 + \frac{a}{e} = \frac{29}{4}$$

$$\frac{a}{e} = \frac{25}{4}$$

but $e = \frac{4}{5}$

$$\therefore \quad \frac{5a}{4} = \frac{25}{4}$$

$$a = 5$$

$$b^2 = a^2(1-e^2)$$

$$= 25\left(1 - \frac{16}{25}\right)$$

$$= 9$$

$$b = 3.$$

Ellipse $\frac{(x-1)^2}{5^2} + \frac{(y-2)^2}{3^2} = 1$

$$\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1.$$

(b) i)

$$y = x - \frac{4}{x^2}$$

$$\frac{dy}{dx} = 1 + 8x^{-3}$$

$$= 1 + \frac{8}{x^3} = 0.$$

$$x^3 = -8$$

$$x = -2.$$

$$y = -3.$$

Stat point $(-2, -3)$

(i) $x = 0$ y undef.

$y = 0$ $x = \sqrt[3]{4}$. $\doteqdot 1.59$

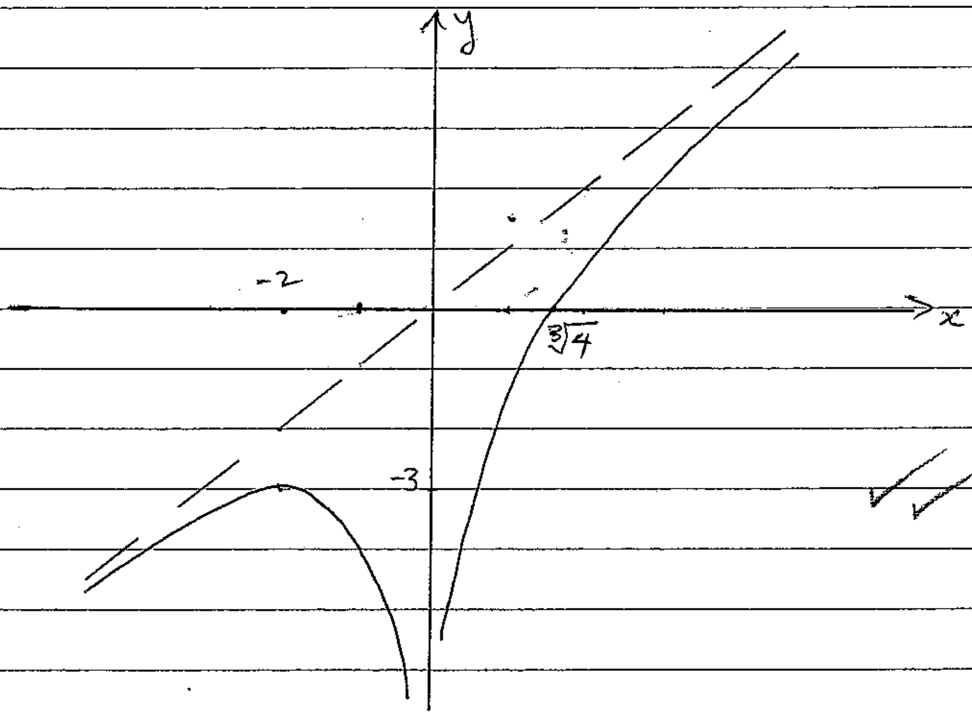
ii) $y = x - \dfrac{4}{x^2}$

$y \to x$ as $x \to \pm\infty$.

Asymptotes: $y = x$

$x = 0$.

iii)



iv) $x^3 - 4 = kx^2$

$\dfrac{x^3 - 4}{x^2} = k$.

$y = k$

$k < -3$. gives 3 intersection points

Q 3 c)

$$\arg \frac{\alpha - \gamma}{\beta - \gamma} = \frac{\pi}{3}.$$

$$\arg(\alpha - \gamma) - \arg(\beta - \gamma) = \frac{\pi}{3}.$$

$$\arg(\alpha - \gamma) = \arg(\beta - \gamma) + \frac{\pi}{3}.$$

$$\therefore \quad \angle ACB = \frac{\pi}{3}.$$

Also $\left|\frac{\alpha - \gamma}{\beta - \gamma}\right| = \left|\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right|$

$$\frac{|\alpha - \gamma|}{|\beta - \gamma|} = 1.$$

$$|\alpha - \gamma| = |\beta - \gamma|$$

$$CA = CB.$$

$\therefore$ Δ ACB is isosceles with $\angle ACB = 60^\circ$

$\therefore$ Δ ACB is equilateral.

Q4 (a) (i) $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

$$\frac{y}{b^2}\frac{dy}{dx} = \frac{-x}{a^2}$$ ✓

$$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}.$$ ✓

ii) $x = \frac{a}{2}$ $$\frac{y^2}{b^2} = 1 - \left(\frac{a}{2}\right)^2 \frac{1}{a^2}$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

$$y^2 = \frac{3b^2}{4}.$$

$$y = \pm \frac{b\sqrt{3}}{2}.$$ ✓

but in 1st quadrant point is $\left(\frac{a}{2}, \frac{b\sqrt{3}}{2}\right)$

then $m = \frac{-b^2}{a^2} \times \frac{a}{2} \times \frac{2}{b\sqrt{3}}$

$$= \frac{-b}{a\sqrt{3}}.$$ ✓

~~(b) (i)~~ $\cancel{\frac{y^2}{100} - \frac{x^2}{25}} = 1.$

Q 4 (b) i) $4x^2 - y^2 = 100$

$$\frac{x^2}{25} - \frac{y^2}{100} = 1.$$

$a = 5, \quad b = 10$ ✓

$b^2 = a^2(e^2 - 1)$

$100 = 25(e^2 - 1)$ ✓

$4 = e^2 - 1$

$e^2 = 5$

$e = \sqrt{5}.$ ✓

ii) foci $(\pm 5\sqrt{5}, 0)$ ✓

iii) directrices $x = \pm \frac{5}{\sqrt{5}}$ or $x = \pm\sqrt{5}.$ ✓

iv) asymptotes $y = \pm \frac{bx}{a}$

$= \pm 2x.$ ✓

c) Hyperbola with foci $(1,0)$ $(-3,0)$

Centre $(-1,0)$ ✓

$2a = 2$

$a = 1$ ✓

$ae = 2$

$e = 2.$ ✓

$b^2 = a^2(e^2 - 1)$

$b^2 = 1(4 - 1)$

$= 3$

$b = \sqrt{3}.$

Equation $\frac{(x+1)^2}{1^2} - \frac{y^2}{(\sqrt{3})^2} = 1$

$(x+1)^2 - \frac{y^2}{3} = 1.$ ✓

Locus gives left/negative branch only ✓

c) ALT If $z = x + iy$.

$$\sqrt{(x-1)^2+y^2} - \sqrt{(x+3)^2+y^2} = 2.$$

$$\sqrt{(x-1)^2+y^2} = 2 + \sqrt{(x+3)^2+y^2}$$

$$(x-1)^2 + y^2 = 4 + 4\sqrt{(x+3)^2+y^2} + (x+3)^2 + y^2$$

$$x^2 - 2x + 1 + y^2 = x^2 + 6x + 9 + y^2 + 4 + 4\sqrt{(x+3)^2+y^2}$$

$$-8x - 12 = 4\sqrt{(x+3)^2+y^2}$$

$$-2x - 3 = \sqrt{(x+3)^2+y^2}$$

$$4x^2 + 12x + 9 = (x+3)^2 + y^2$$

$$= x^2 + 6x + 9 + y^2$$

$$3x^2 + 6x - y^2 = 0.$$

$$x^2 + 2x - \frac{y^2}{3} = 0$$

$$x^2 + 2x + 1 - \frac{y^2}{3} = 1$$

$$(x+1)^2 - \frac{y^2}{3} = 1.$$

Restriction

Q5 a)

$$z^2\cos^2\theta + 2\sin\theta\cos\theta\, z + 1 = 0$$

$$z^2\cos^2\theta + 2\sin\theta\cos\theta\, z + \sin^2\theta = \sin^2\theta - 1$$

$$(z\cos\theta + \sin\theta)^2 = -(1-\sin^2\theta)$$

$$= -\cos^2\theta$$

$$z\cos\theta + \sin\theta = \pm i\cos\theta$$

$$z\cos\theta = -\sin\theta \pm i\cos\theta$$

$$z = -\tan\theta \pm i$$

ALT

$$z = \frac{-2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\cos^2\theta}}{2\cos^2\theta}$$

$$= -\frac{\sin\theta}{\cos\theta} \pm \frac{2\cos\theta\sqrt{\sin^2\theta - 1}}{2\cos^2\theta}$$

$$= -\tan\theta \pm \frac{\sqrt{-(1-\sin^2\theta)}}{\cos\theta}$$

$$= -\tan\theta \pm \frac{i\sqrt{\cos^2\theta}}{\cos\theta}$$

$$= -\tan\theta \pm i$$

b) i)

$$y = e^{\cos^{-1}x}$$

Domain: $-1 \le x \le 1$

Range: $0 \le \cos^{-1}x \le \pi$

$$e^0 \le e^{\cos^{-1}x} \le e^{\pi}$$

$$1 \le y \le e^{\pi}$$

ii) $x = 0 \quad y = e^{\pi/2}$

no x-intercept since $y \neq 0$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-y}{\sqrt{1-x^2}}$$

OR If $y = e^{\cos^{-1}x}$

$$\frac{dy}{dx} = e^{\cos^{-1}x} \times \frac{-1}{\sqrt{1-x^2}}$$

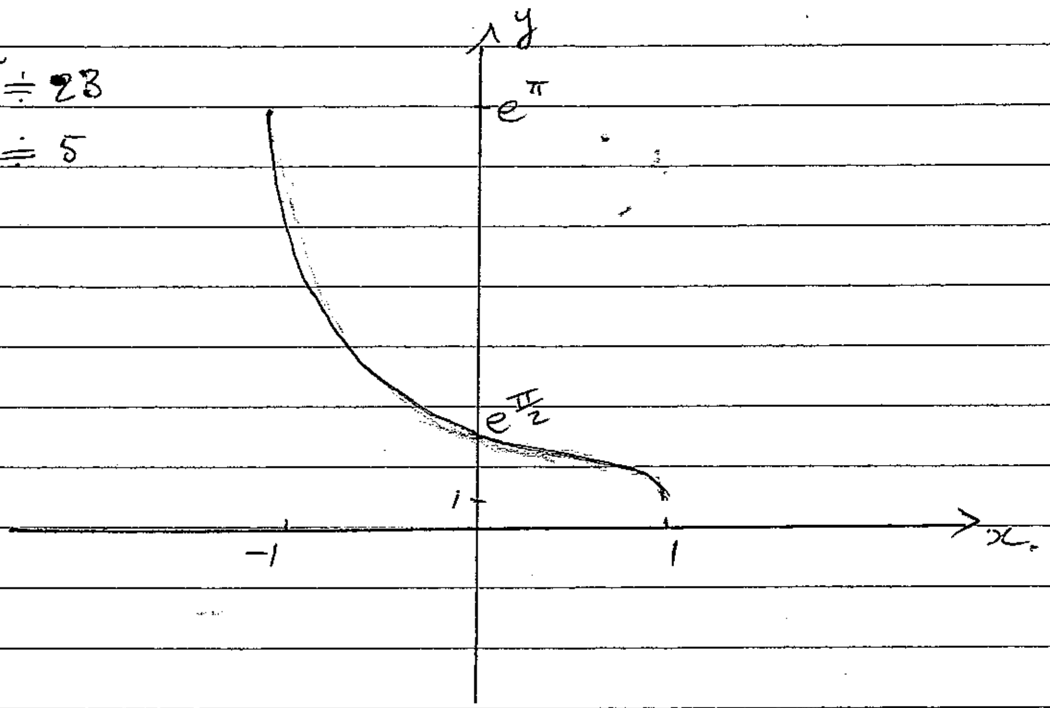
$$= \frac{-e^{\cos^{-1}x}}{\sqrt{1-x^2}}$$

Stat pts: $\frac{dy}{dx} \neq 0$ since $y \neq 0$. ✓

Vert tangents: $\frac{dy}{dx} \to -\infty$ as $x \to \pm 1$.

since $y > 0$. ✓

$e^{\pi} \doteqdot 23$
$e^{\frac{\pi}{2}} \doteqdot 5$



✓✓

c) $$h = a\tan\alpha - \frac{g\sec^2\alpha}{2V^2}a^2$$

$$= a\tan\alpha - \frac{ga^2}{2V^2}(1+\tan^2\alpha)$$ ✓

$$2V^2h = 2V^2a\tan\alpha - ga^2 - ga^2\tan^2\alpha$$

$$ga^2\tan^2\alpha - 2V^2a\tan\alpha + 2V^2h + ga^2 = 0.$$ ✓

is a quadratic in $\tan\alpha$

has 2 solutions for $\tan\alpha$ if $\Delta > 0$.

i.e. $$4V^4a^2 - 4ga^2(2V^2h + ga^2) > 0.$$ ✓

$$V^4 - 2gV^2h - g^2a^2 > 0.$$

$$V^4 - 2gV^2h + g^2h^2 > g^2a^2 + g^2h^2$$

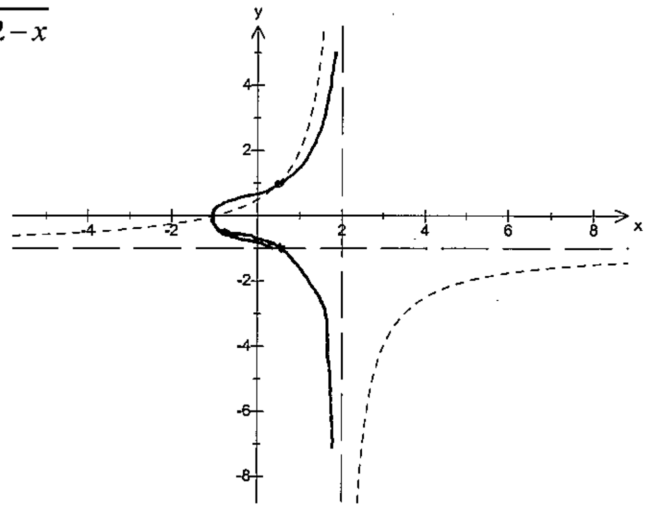
$$(V^2 - gh)^2 > g^2(a^2 + h^2)$$ ✓

Student Number:

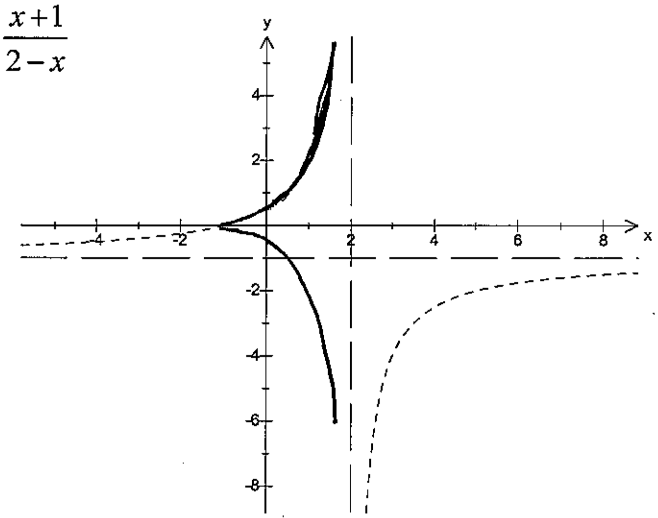
Hand in this page with your answer booklets.

Question 1 (d) Use the graphs of $y = \dfrac{x+1}{2-x}$ as a template to sketch each of the following:

(i) $y^2 = \dfrac{x+1}{2-x}$



(ii) $|y| = \dfrac{x+1}{2-x}$



(iii) $|y| = \dfrac{|x|+1}{2-|x|}$