St Aloysius' College
Year 12 Mid-Year Examinations
2010

EXTENSION 2 MATHEMATICS
(Additional paper)

Total marks: 75

Reading time - 5 minutes
Working time – 2 hours

Examination papers must NOT be removed from the examination room.

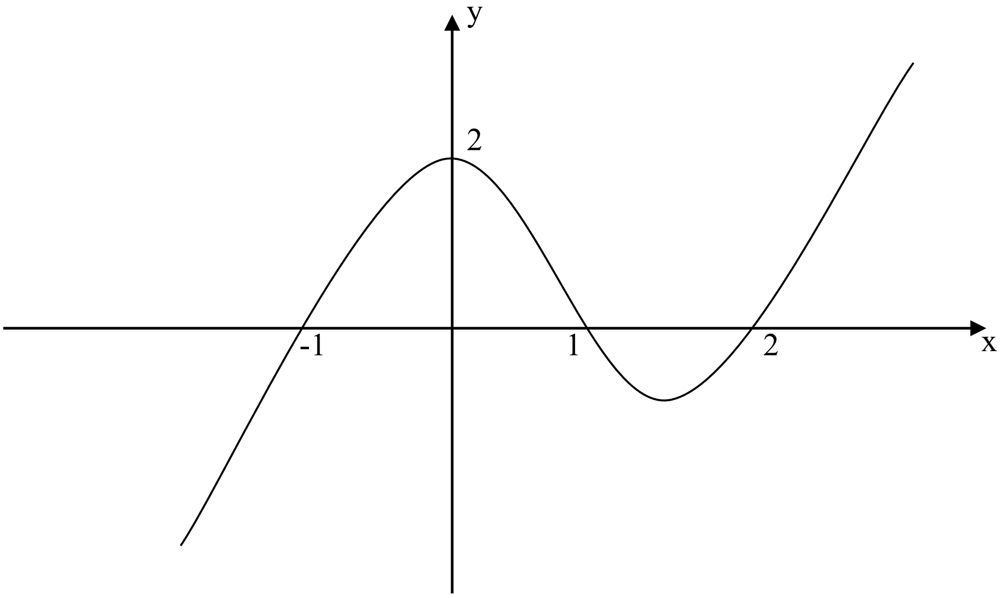
General Instructions

- Start each question in a new booklet
- Board approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Show all necessary work

SUPERVISOR'S INSTRUCTIONS:
Please issue four 8 page answer booklets.
Please collect the examination paper with the answer booklets.

Question 1 *(20 marks)*

(a)



Use the graph of $y = f(x)$ shown above to make half-page sketches of the following.

(i) $y = \dfrac{1}{f(x)}$ **(3)**

(ii) $y = f(|x|)$ **(3)**

(iii) $y^2 = f(x)$ **(3)**

(iv) $y = |f(x)|$ **(3)**

(v) $y = \ln f(x)$ **(3)**

(b) (i) Show that $3x - 5 + \dfrac{4}{x+1} = \dfrac{3x^2 - 2x - 1}{x+1}$ **(1)**

(ii) State the equations of the vertical and oblique asymptotes of the function $y = \dfrac{3x^2 - 2x - 1}{x+1}$ **(2)**

(iii) Sketch the function showing the above features. **(2)**
[Note that you are **not** required to find the coordinates of any stationary points.]

START A NEW BOOKLET

Question 2 *(20 marks)*

(a) $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ which has eccentricity e foci S and S'.

(i) Show that the normal at P is $\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta}=a^2e^2$ **(3)**

(ii) The normal at P intersects the x axis at G. **(4)**
Show that $PG^2=a^2(e^2-1)(e^2\sec^2\theta-1)$

(iii) Show that $PG^2=(e^2-1)PS\times PS'$ **(3)**

(b) $P(2p,\frac{2}{p})$ and $Q(2q,\frac{2}{q})$ are distinct points on $xy=4$.

M is the midpoint of the interval PQ, and T is the point of intersection of the tangents at P and Q.

Given that the tangent at $(2t,\frac{2}{t})$ is $x+t^2y=4t$

(i) Show that T has coordinates $\left(\dfrac{4pq}{p+q},\dfrac{4}{p+q}\right)$ **(2)**

(ii) If M always lies on the line $x=3$, find the locus of T including any restrictions. **(2)**

(c) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\varphi, b\sin\varphi)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where $a>b$. Let M be the midpoint of the chord PQ, and O be the origin.

(i) Show that the product of the gradients of OM and PQ is $-\dfrac{b^2}{a^2}$. **(3)**

(ii) Given that $\sin(\theta+\varphi)\sin(\theta-\varphi)=\cos^2\varphi-\cos^2\theta$, show that the acute angle, α, between OM and PQ is given by **(3)**

$$\tan\alpha=\frac{2ab}{a^2-b^2}\left|\operatorname{cosec}(\theta+\varphi)\right|$$

START A NEW BOOKLET

Question 3 *(15 marks)*

(a) Solve the equation $(3-i)z+(2+3i)=\overline{8+7i}$ for the complex number z. **(3)**

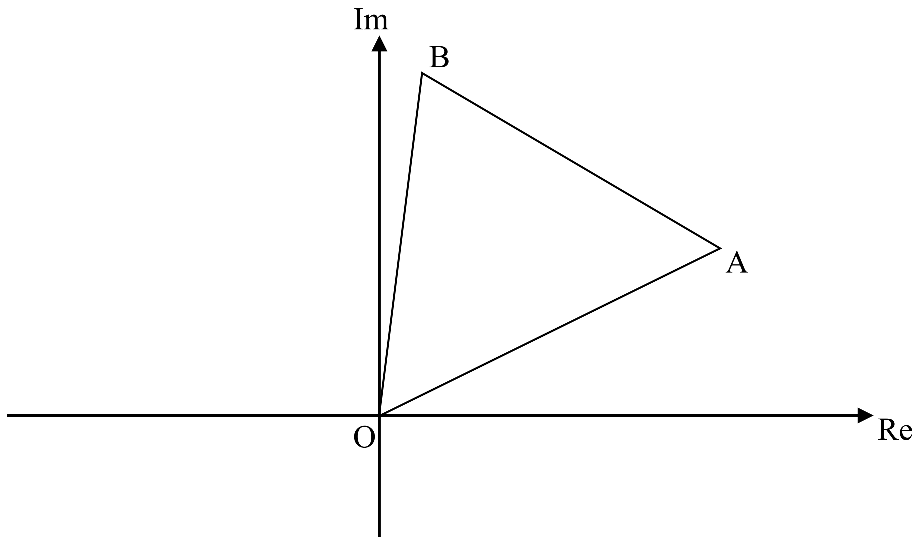
(b) Shade the region on the Argand diagram in which $|z-i|\le \operatorname{Im} z$ and $|z|\le 1$ both hold. **(3)**

(c) Find $\left(-\sqrt{3}+i\right)^7$ in the form $a+ib$ **(3)**

(d) One root of $z^2+(3-2i)z+k=0$ is $-4+2i$

(i) Find the other root, and **(1)**
(ii) Find the value of k. **(1)**

(e) The Argand diagram below shows an equilateral triangle OAB.
A complex number z is represented by the vector OA. Let $\omega=\dfrac{-1+i\sqrt{3}}{2}$



Express the complex numbers represented by AB and OB in terms of z and ω **(2)**

(f) Find the Cartesian equation of the locus of z if $\left|z^2+(\operatorname{Re} z)^2\right|=4$ **(2)**

START A NEW BOOKLET

Question 4 *(20 marks)*

(a) A sequence is defined by $T_1 = 1$, $T_2 = 4$ and $T_n = 6T_{n-2} + T_{n-1}$ for integers $n > 2$. **(5)**

Use induction to prove that $T_n = \dfrac{2 \times 3^n - (-2)^{n-1}}{5}$ for $n \geq 1$

(b) (i) Show that $x + y \geq 2\sqrt{xy}$ for all positive real numbers x and y. **(1)**

(ii) Hence, or otherwise, show that $\log_2\left(2^n + 2^m\right) \geq \dfrac{n+m+2}{2}$ for all real n and m. **(3)**

(c) (i) Use induction to prove that $(a+1)^n - a^n > na^{n-1}$ for real $a > 0$ and integers $n \geq 2$. **(3)**

In the following you may use the similar identity $(a+1)^n - a^n < n(a+1)^{n-1}$ without proof.

(ii) Hence deduce that $a < \sqrt[n-1]{\dfrac{(a+1)^n - a^n}{n}} < a+1$ **(1)**

(iii)

(α) The straight line $y = [(a+1)^n - a^n]x + a^n - [(a+1)^n - a^n]$ intersects the curve $y = x^n$ at $x = a$ and $x = a+1$ where $a > 0$. **(3)**

Using calculus and part (ii), show that the function
$f(x) = [(a+1)^n - a^n]x + a^n - [(a+1)^n - a^n] - x^n$
has a maximum stationary value in the interval $a < x < a+1$

(β) Deduce that $f(x) > 0$ for $a < x < a+1$ **(1)**

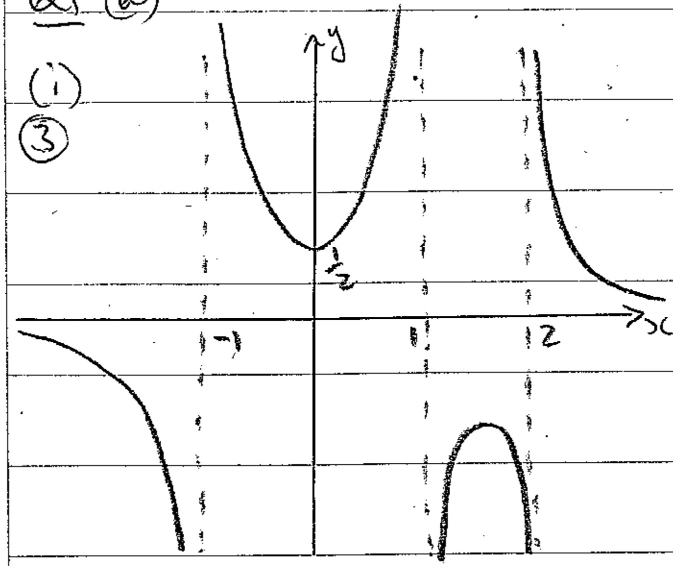
(iv) Using the above results and by comparing the area under $y = x^n$ for $0 \leq x \leq n$ where $n \geq 2$ with an approximation for that area, show that

$$\frac{n^{n+1}}{n+1} < 1 + 2^n + 3^n + \ldots + (n-1)^n + \frac{n^n}{2} \quad \text{for integers } n \geq 2$$ **(3)**

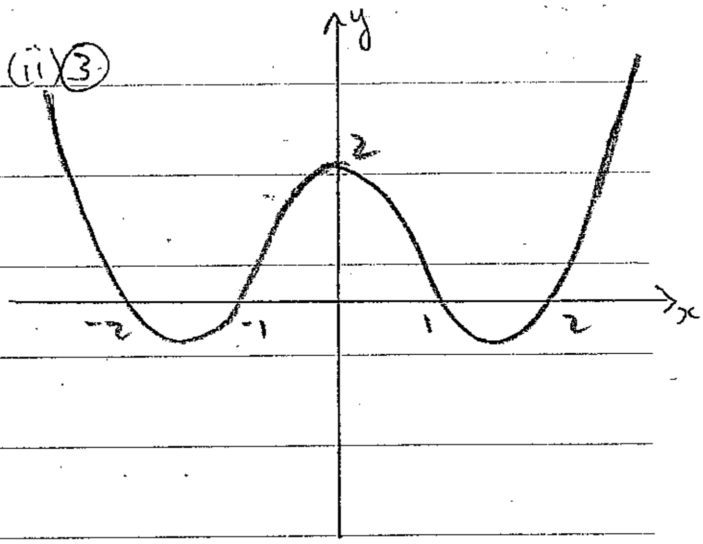
Ex 2

Q1 (a)

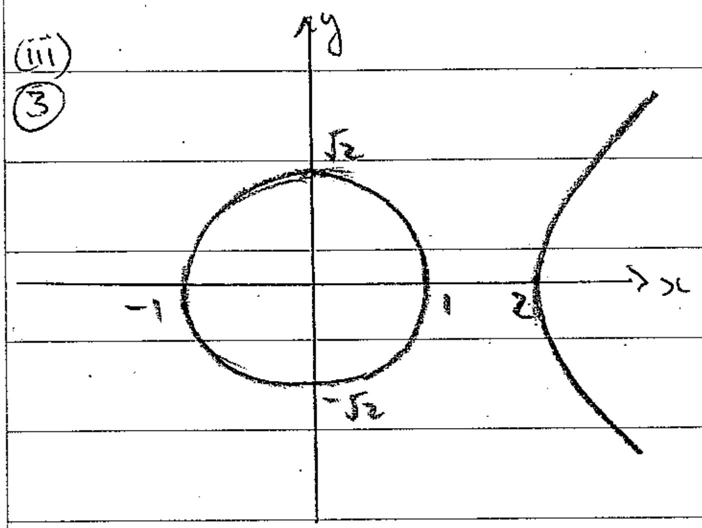
(i) 3



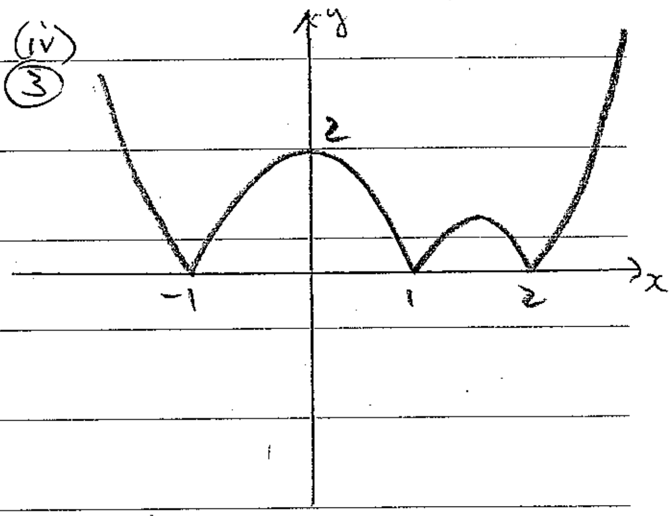
(ii) 3



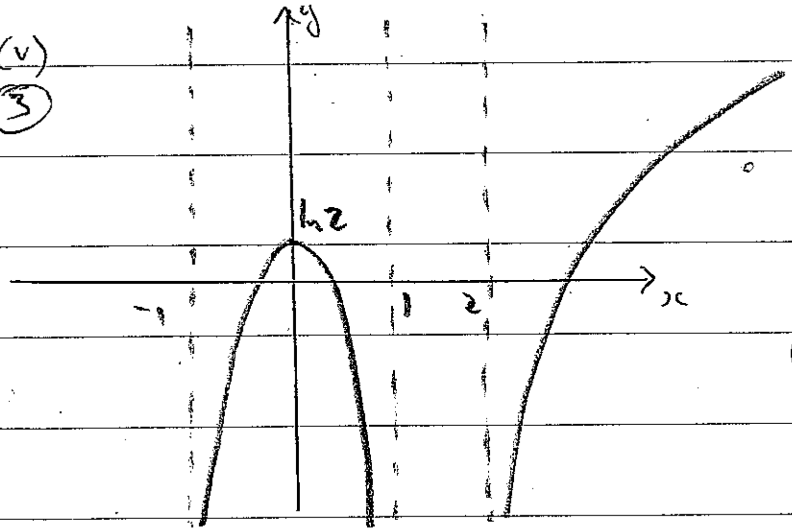
(iii) 3



(iv) 3



(v) 3

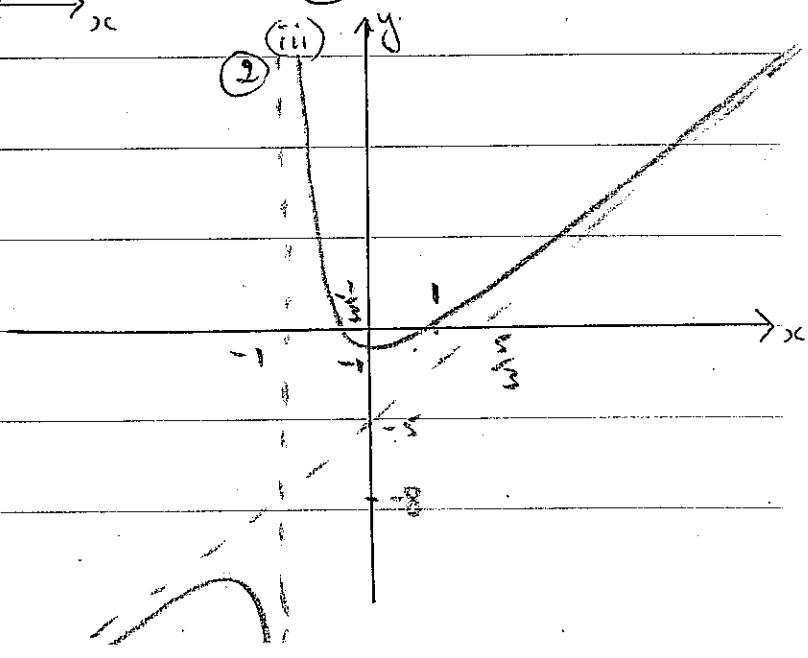


(b) (i) 1 $3x-5+\frac{4}{x+1} = \frac{(3x-5)(x+1)+4}{x+1}$

$= \frac{3x^2-2x-1}{x+1}$

(ii) 2 $x=-1$ and $y=3x-5$

(iii) 2



Q2 a) (i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b\sec\theta}{a\tan\theta}$

$\therefore$ Normal $y - b\tan\theta = -\dfrac{a\tan\theta}{b\sec\theta}\left(x - a\sec\theta\right)$

$$\frac{by}{\tan\theta} - b^2 = -\frac{ax}{\sec\theta} + a^2$$

$$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$$

$b^2 = a^2(e^2-1)$

$\therefore a^2+b^2 = a^2e^2$

(ii) $G\left(ae^2\sec\theta, 0\right)$

$$PG^2 = \left(ae^2\sec\theta - a\sec\theta\right)^2 + \left(b\tan\theta\right)^2$$

$$= a^2\sec^2\theta\,(e^2-1)^2 + a^2(e^2-1)\tan^2\theta$$

$$= a^2(e^2-1)\cancel{(\sec^2\theta + \tan^2\theta)}\left((e^2-1)\sec^2\theta + \tan^2\theta\right)$$

$$= a^2(e^2-1)\left((e^2\cancel{-1})\sec^2\theta + \cancel{\sec^2\theta} - 1\right)$$

$$= a^2(e^2-1)\left(e^2\sec^2\theta - 1\right)$$

(iii) $PS \times PS' = ePN \times ePN'$

$$= e^2\left(a\sec\theta - \frac{a}{e}\right)\left(a\sec\theta + \frac{a}{e}\right)$$

$$= a^2e^2\left(\sec^2\theta - \frac{1}{e^2}\right)$$

$$= a^2\left(e^2\sec^2\theta - 1\right)$$

$$\therefore (e^2-1)\,PS \times PS' = a^2(e^2-1)\left(e^2\sec^2\theta - 1\right)$$

$$= PG^2$$

(b) $P\left(2p, \frac{2}{p}\right)$ and $Q\left(2q, \frac{2}{q}\right)$ $\quad xy = 4$

(i) Tangents $\left.\begin{array}{l} x + p^2y = 4p \\ x + q^2y = 4q \end{array}\right\}$ $(p^2 - q^2)y = 4(p-q)$

$\therefore y = \frac{4}{p+q}$ as $p \neq q$

$\therefore x = 4p - \frac{4p^2}{p+q}$ $\qquad$ 2

$= \frac{4pq}{p+q}$ $\qquad \therefore T\left(\frac{4pq}{p+q}, \frac{4}{p+q}\right)$

(ii) $M\left(p+q, \frac{pq}{p+q}\right)$ lies on $x = 3$ $\therefore p+q = 3$

$\therefore T\left(\frac{4pq}{3}, \frac{4}{3}\right)$ $\qquad$ 1 + 1

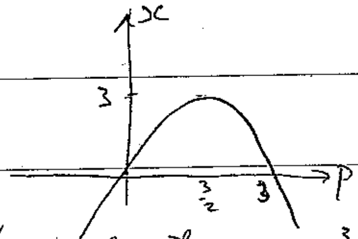
$\therefore$ Locus is $y = \frac{4}{3}$ for $x < 3$ but $x \neq 0$ ⊛

⊛ Restrictions: First $p \neq 0$ and $q \neq 0$ $\therefore x \neq 0$

Second, whether P and Q are ~~o~~ on different branches or both on the 1st quadrant branch, T can never lie above that branch. $\Rightarrow x < 3$

OR for the second part: $p + q = 3 \Rightarrow q = 3 - p$

$\therefore$ For T $\quad x = \frac{4p(\cancel{q}3-p)}{3}$



$\therefore x \leq 3$ with equality only if $p = q = \frac{3}{2}$

but $p \neq q$ $\therefore x < 3$

The $p = 3$ case is interesting as then $q = 0$ $\therefore$ The tangent at Q "is" the y axis and the tangent at P has y intercept $\left(0, \frac{4}{3}\right)$. But of course ~~q~~ $q \neq 0$, so once again $x \neq 0$.

(e) (i) $M\left(\dfrac{a(\cos\theta+\cos\phi)}{2}, \dfrac{b(\sin\theta+\sin\phi)}{2}\right)$

$$\therefore m_{OM} = \frac{b}{a}\,\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi}$$

$$m_{PQ} = \frac{b}{a}\,\frac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi}$$ 1

$$\therefore m_{OM}\cdot m_{PQ} = \frac{b^2}{a^2}\cdot\frac{\sin^2\theta-\sin^2\phi}{\cos^2\theta-\cos^2\phi}$$ 1

$$= \frac{b^2}{a^2}\,\frac{\sin^2\theta-\sin^2\phi}{(\cancel{1}-\sin^2\theta)-(\cancel{1}-\sin^2\phi)} = -\frac{b^2}{a^2}$$ 1

(ii) $$\tan\alpha = \left|\frac{\frac{b}{a}\left(\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} - \frac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi}\right)}{1 + \left(-\frac{b^2}{a^2}\right)}\right|$$ 1

$$= \frac{ab}{|a^2-b^2|}\left|\frac{\cancel{\sin\theta\cos\theta}+\sin\phi\cos\theta+\sin\theta\cos\phi-\cancel{\sin\phi\cos\phi} - \cancel{\sin\theta\cos\theta}-\sin\theta\cos\phi+\sin\phi\cos\theta+\cancel{\sin\phi\cos\phi}}{\cos^2\theta-\cos^2\phi}\right|$$ 1

$$= \frac{ab}{a^2-b^2}\left|\frac{-2\sin\cancel{(\theta-\phi)}}{\sin(\theta+\phi)\sin\cancel{(\theta-\phi)}}\right|$$ as $a^2>b^2$ 1

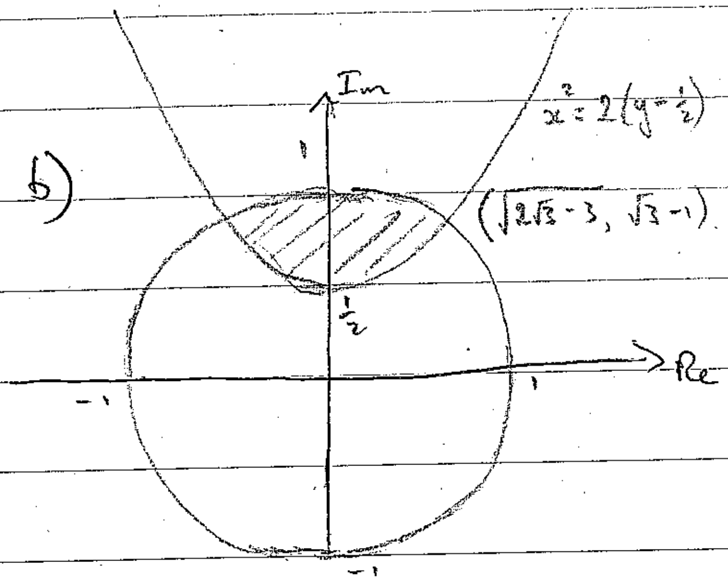
$$= \frac{2ab}{a^2-b^2}\left|\operatorname{cosec}(\theta+\phi)\right|$$

Q3 a) $(3-i)z+(2+3i)=\overline{8+7i} \Rightarrow z = \dfrac{8-7i-(2+3i)}{3-i}$ | 1

$= \dfrac{(6-10i)(3+i)}{9+1}$ | 1

$= \dfrac{14-12i}{5}$ | 1

b)



| circle
| parabola
| shading

c) $|-\sqrt{3}+i| = 2$; $\arg(-\sqrt{3}+i) = 150°$ or $\frac{5\pi}{6}$ | 1

$(-\sqrt{3}+i)^7 = 2^7 \operatorname{cis} \frac{35\pi}{6}$ [de Moivre] | 1

$= 2^7 \operatorname{cis}\left(-\frac{\pi}{6}\right) = 2^6(\sqrt{3}-i)$ | 1

d)(i) Sum of roots $= -3+2i \Rightarrow$ other root $= 1$ | 1

(ii) Product of roots $= k = -4+2i$ | 1

e) $\underline{\underline{AB}}$ $+zw$ $\underline{\underline{OB}}$ $-zw^2$ or $z(w+1)$ 1+1

(or $-z\overline{w}$)

f) $|z^2+(\text{Re } z)^2| = 4$

$|2x^2-y^2+2ixy| = 4$

$(2x^2-y^2)^2+4x^2y^2 = 16$] Progress | 1

$4x^4+y^4 = 16$ | 1

Q4 a) $T_1 = 1$; $T_2 = 4$ and $T_n = 6T_{n-2} + T_{n-1}$, $n > 2$

RTP $T_n = \dfrac{2\times 3^n - (-2)^{n-1}}{5}$ for $n \geq 1$

When $n=1$ $\quad RHS = \dfrac{2\times 3 - 1}{5} = 1$ $\qquad$ When $n=2$ $\quad RHS = \dfrac{2\times 9 + 2}{5} = 4$

$\therefore$ The proposition is true for $n=1$ and $n=2$

Assume that $T_k = \dfrac{2\times 3^k - (-2)^{k-1}}{5}$ and $T_{k+1} = \dfrac{2\times 3^{k+1} - (-2)^k}{5}$ for $k \geq 1$

$$\text{Then } T_{k+2} = 6\times \frac{2\times 3^k - (-2)^{k-1}}{5} + \frac{2\times 3^{k+1} - (-2)^k}{5}$$

$$= \frac{1}{5}\left\{2\times\left(6\times 3^k + 3^{k+1}\right) - 6\times(-2)^{k-1} - (-2)^k\right\}$$

$$= \frac{1}{5}\left\{2\times 9\times 3^k - 4\times(-2)^{k-1}\right\}$$

$$= \frac{2\times 3^{k+2} - (-2)^{k+1}}{5} \quad \text{as required}$$

Hence the proposition is proved true by induction.

b) (i) $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$ for $x, y > 0$

$\therefore x - 2\sqrt{xy} + y \geq 0$

$\therefore x + y \geq 2\sqrt{xy}$

(ii) Let $x = 2^n$ and $y = 2^m$ $\qquad$ Clearly $x, y > 0$ for real n, m

$$\therefore 2^n + 2^m \geq 2\sqrt{2^{n+m}} = 2^{\frac{n+m+2}{2}}$$

$\therefore \log_2\left(2^n + 2^m\right) \geq \dfrac{n+m+2}{2}$ $\qquad$ noting that $y = \log_2 x$ is an increasing function $\therefore$ Inequality maintains its direction.

(c) (i) RTP $(a+1)^n - a^n > n a^{n-1}$ for $a>0$, $n \geq 2$

For $n=2$ LHS $= a^2+2a+1 - a^2$

$= 2a+1$

RHS $= 2a$ $\therefore$ True for $n=2$

Assume that $(a+1)^k - a^k > k a^{k-1}$ for an integer $k \geq 2$

Now $(a+1)^{k+1} - a^{k+1} = a\left[(a+1)^k - a^k\right] + (a+1)^k$

$> k a^k + (a+1)^k$ by assumption

$> k a^k + a^k$

$= (k+1) a^k$

Hence it is true for $n=k+1$ if it is true for $n=k$.

It is true for $n=2$ $\therefore$ True for all $n \geq 2$ by induction.

(ii) $n a^{n-1} < (a+1)^n - a^n < n(a+1)^{n-1}$ from above plus given result.

$a^{n-1} < \dfrac{(a+1)^n - a^n}{n} < (a+1)^{n-1}$

$a < \left(\dfrac{(a+1)^n - a^n}{n}\right)^{\frac{1}{n-1}} < a+1$ since $a>0$

(iii) (α) $f(x) = \left[(a+1)^n - a^n\right]x + a^n - \left[(a+1)^n - a^n\right] - x^n$

$f'(x) = \left[(a+1)^n - a^n\right] - n x^{n-1}$

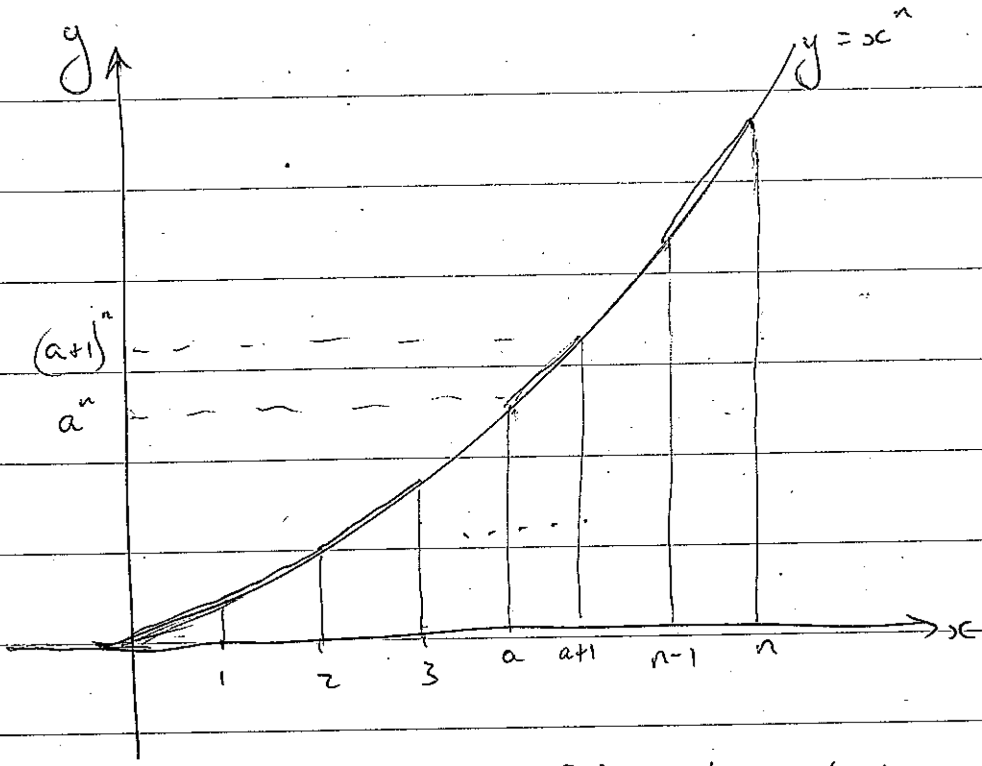
$f''(x) = -n(n-1)x^{n-2} < 0$ for all $n \geq 2$ and $x>0$ $\therefore$ Concave down

$f'(x) = 0 \Rightarrow x = \left[\dfrac{(a+1)^n - a^n}{n}\right]^{\frac{1}{n-1}}$

$y=f(x)$ is concave down $\therefore$ Maximum. From part (ii) $a < x < a+1$

(β) $f(a) = f(a+1) = 0$ and there is exactly one stationary point in $a < x < a+1$, and that point is a maximum $\therefore f(x) > 0$ for $a < x < a+1$

(iv)



Approximating the curve with straight line segments from (a, a^n) to $(a+1, (a+1)^n)$ for $a = 0, 1, 2 \ldots (n-1)$ we form n trapeziums. From part (iii)(β) we know these intervals lie above $y = x^n$ for $a < x < a+1$; $n \geq 2$

$\therefore$ The exact area $<$ The sum of the areas of the trapeziums

$$\therefore \int_0^n x^n \, dx < \tfrac{1}{2} \times 1 \times (0^n + 1^n) + \tfrac{1}{2} \times 1 \times (1^n + 2^n) + \tfrac{1}{2} \times 1 \times (2^n + 3^n) + \ldots + \tfrac{1}{2} \times 1 \times ((n-2)^n + (n-1)^n) + \tfrac{1}{2} \times 1 \times ((n-1)^n + n^n)$$

$$\therefore \frac{n^{n+1}}{n+1} < \tfrac{1}{2} \times 0^n + 1^n + 2^n + 3^n + \ldots + (n-1)^n + \tfrac{1}{2} \times n^n$$

$$\frac{n^{n+1}}{n+1} < 1 + 2^n + 3^n + \ldots + (n-1)^n + \frac{n^n}{2}$$

for $n \geq 2$