St George Girls High School

Year 12

Common Test 3

May 2006

Mathematics Extension 2

Time Allowed: 75 Minutes

Instructions to Candidates

1. Both questions may be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.

<u>Question 1</u> – (25 marks) – Start a New Page **Marks**

a) Show that the curves $x^2 - y^2 = c^2$ and $xy = c^2$ intersect at right angles. Do not attempt to solve these equations simultaneously. 3

b) For the hyperbola $xy = 16$, write down the:

(i) eccentricity 1

(ii) coordinates of the vertices 2

(iii) coordinates of the foci 2

(iv) equations of the directrices 2

c) (i) Prove that the equation of the tangent to the curve $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ is $x + t^2 y = 2ct$. 2

(ii) Hence, prove that for the rectangular hyperbola $xy = c^2$, the area of the triangle bounded by a tangent and the asymptotes is a constant. 3

d) P and Q are the points on the rectangular hyperbola $H : xy = c^2$ with parameters p and q respectively $(p > 0,\ q > 0)$.

(i) Find the point of intersection, T, of the tangents to H at P and Q (you may use the result from c(i)). 2

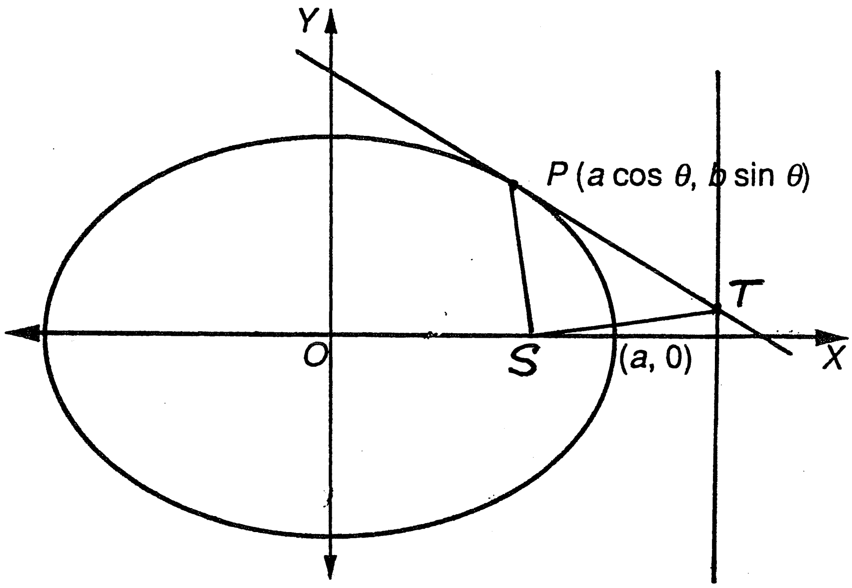
(ii) Find the locus of T as P and Q vary if $pq = k$ (k constant). 3

Part (e) continue next page

Question 1 (cont'd)

Marks

e) 5



The tangent to the ellipse $E: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$ meets the directrix at T as shown above. If S is the corresponding focus, show that PT subtends a right angle at S.

[<u>NOTE</u>: The equation of the tangent to E at P is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$]

Question 2 on next page

Question 2 – (25 marks) – Start a new page **Marks**

a) The roots of $x^3 + 2x - 1 = 0$ are α, β and γ.

(i) Find the polynomial equation with roots:

(α) $-\alpha, -\beta, -\gamma$ 1

(β) $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ 1

(γ) $\alpha^2, \beta^2, \gamma^2$ 2

(δ) $\alpha - 1, \beta - 1, \gamma - 1$ 1

(ii) Show the $(\alpha + \beta - \gamma)(\beta + \gamma - \alpha)(\gamma + \alpha - \beta) = -8$ 3

b) It is given that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(i) Use $x = \cos\theta$ to find the solutions of $16x^4 - 16x^2 + 1 = 0$ 3

(ii) Hence evaluate $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$ leaving the answers in surd form. 6

c) Consider the polynomial equation $x^3 + ax + b = 0$

(i) Show that if $a > 0$ then the equation has only one real root. 2

(ii) What condition must 'a' and 'b' satisfy if the equation has three real roots? 6

End of Paper

QUESTION 1:

(a) $$x^2 - y^2 = c^2$$

$$\Rightarrow \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = 0$$

$$\therefore 2x - 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{x}{y} \quad \text{———} \quad (1)$$

If (x_1, y_1) is point of intersection then

$$(1) \Rightarrow \frac{dy}{dx} = \frac{x_1}{y_1} = m_1$$

also $$xy = c^2$$

$$\Rightarrow y = c^2 x^{-1}$$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$$

at (x_1, y_1) $$\frac{dy}{dx} = -\frac{c^2}{x_1^2} = m_2$$

Then $$m_1 \times m_2 = \frac{x_1}{y_1} \times -\frac{c^2}{x_1^2}$$

$$= -\frac{c^2}{x_1 y_1}$$

$$= -\frac{c^2}{c^2}$$ since (x_1, y_1) lies on $xy = c^2$

$$= -1$$

$\therefore$ At points of intersection the curves are perpendicular.

(b)

$$xy = 16$$
$$= \frac{1}{2}a^2$$
$$\Rightarrow a^2 = 32$$
$$\therefore a = 4\sqrt{2} \quad (a > 0)$$

(i) $e = \sqrt{2}$

(ii) Vertices are at $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$

$$= (4, 4) \text{ and } (-4, -4)$$

(iii) Foci are at (a, a) and $(-a, -a)$

$$= (4\sqrt{2}, 4\sqrt{2}) \text{ and } (-4\sqrt{2}, -4\sqrt{2})$$

(iv) Directrices are $x + y = a$, $x + y = -a$

ie $x + y = 4\sqrt{2}$, $x + y = -4\sqrt{2}$

(c)

$$xy = c^2$$
$$\Rightarrow y = c^2 x^{-1}$$
$$\frac{dy}{dx} = -c^2 x^{-2}$$
$$= -\frac{c^2}{x^2}$$

at $P\left(ct, \frac{c}{t}\right)$ $\frac{dy}{dx} = -\frac{c^2}{c^2t^2}$

$$= -\frac{1}{t^2}$$

$\therefore$ Tangent at P is :

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$
$$t^2 y - ct = -x + ct$$

ie $\underline{x + t^2 y = 2ct}$

(ii) Tangent at P crosses

x axis at $A(2ct, 0)$

y axis at $B\left(0, \frac{2c}{t}\right)$

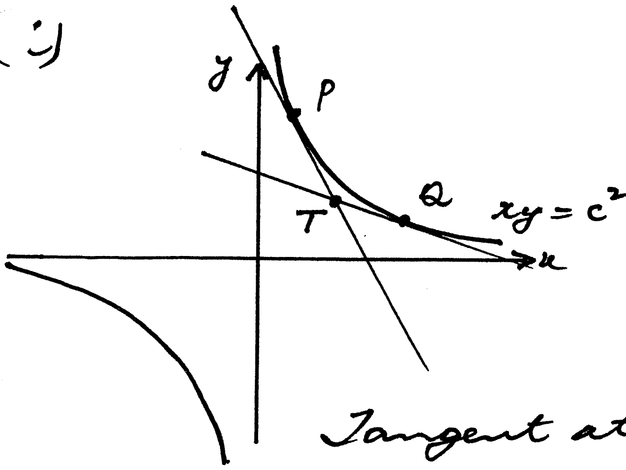
$\therefore$ area of triangle ABO is

$$\frac{1}{2} \cdot 2ct \cdot \frac{2c}{t}$$

$$= 2c^2$$

$$= \text{constant.}$$

(iii)



Tangent at P: $\quad x + p^2 y = 2cp$ —— ①

Tangent at Q: $\quad x + q^2 y = 2cq$ —— ②

$$① - ②: \quad y(p^2 - q^2) = 2c(p-q)$$

$$y(p+q)(p-q) = 2c(p-q)$$

$$\therefore y = \frac{2c}{p+q}, \quad p \neq q$$

sub in ①

$$\therefore x + \frac{2cp^2}{p+q} = 2cp$$

$$x = 2cp - \frac{2cp^2}{p+q}$$

$$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$$

$$= \frac{2cpq}{p+q}$$

$$\therefore \quad T \equiv \left(\frac{2cpq}{p+q}, \quad \frac{2c}{p+q} \right)$$

(ii) If $pq = k$ (constant)

then $T \Rightarrow \left(\frac{2ck}{p+q}, \quad \frac{2c}{p+q} \right)$

ie $x = \frac{2ck}{p+q}, \quad y = \frac{2c}{p+q}$

$$\therefore \quad \frac{y}{x} = \frac{\frac{2c}{p+q}}{\frac{2ck}{p+q}}$$

$$= \frac{1}{k}$$

ie $y = \left(\frac{1}{k}\right) x$

sub $y = \left(\frac{1}{k}\right) x$ in $xy = c^2$

$$\therefore \quad x \cdot \frac{x}{k} = c^2$$

$$x^2 = c^2 k$$

$$x = c\sqrt{k}$$ ($x > 0$ since $p > 0, q > 0$)

$\therefore$ Locus of T is the line $y = \left(\frac{1}{k}\right) x$

with $0 < x < c\sqrt{k}$

(since $\frac{2ck}{p+q} \neq 0$)

(e) Tangent at P is:

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Tangent intersects directrix $x = \frac{a}{e}$ at $T\left(\frac{a}{e}, y_1\right)$

$$\therefore \quad \frac{\cos\theta}{e} + \frac{y_1 \sin\theta}{b} = 1$$

$$b\cos\theta + y_1 . e\sin\theta = be$$

$$\therefore y_1 = \frac{be - b\cos\theta}{e\sin\theta}.$$

$$\therefore \quad T \equiv \left(\frac{a}{e}, \frac{be - b\cos\theta}{e\sin\theta}\right)$$

$$m_{TS} = \frac{\frac{be - b\cos\theta}{e\sin\theta} - 0}{\frac{a}{e} - ae}$$

$$m_{PS} = \frac{b\sin\theta - 0}{a\cos\theta - ae}$$

$$\therefore m_{TS} \times m_{PS} = \frac{\left(\frac{be - b\cos\theta}{e\sin\theta}\right)}{\left(\frac{a - ae^2}{e}\right)} \times \frac{b\sin\theta}{a(\cos\theta - e)}$$

$$= \frac{b(e - \cos\theta)}{\cancel{e\sin\theta}} \times \frac{\cancel{e}}{a(1-e^2)} \times \frac{b\cancel{\sin\theta}}{a(\cos\theta - e)}$$

$$= \frac{b^2(e - \cos\theta)}{a^2(1-e^2).(\cos\theta - e)}$$

$$= -1 \text{ since } b^2 = a^2(1-e^2).$$

$\therefore TS \perp PS$ i.e. PT subtends 90° at S

QUESTION 2:

(a) (i) (α) $P(-x) = 0$

$\Rightarrow (-x)^3 + 2(-x) - 1 = 0$

ie $-x^3 - 2x - 1 = 0$

ie $x^3 + 2x + 1 = 0$

(β) $(x^3 + 2x + 1)(x^3 + 2x - 1) = 0$

ie $x^6 + 2x^4 - x^3 + 2x^4 + 4x^2 - 2x + x^3 + 2x - 1 = 0$

ie $x^6 + 4x^4 + 4x^2 - 1 = 0$

(γ) $P(\sqrt{x}) = 0$

ie $(\sqrt{x})^3 + 2(\sqrt{x}) - 1 = 0$

$x\sqrt{x} + 2\sqrt{x} = 1$

$\sqrt{x}\,(x + 2) = 1$

squaring

$\Rightarrow x(x+2)^2 = 1$

$x(x^2 + 4x + 4) - 1 = 0$

ie $x^3 + 4x^2 + 4x - 1 = 0$

(δ) $P(x+1) = 0$

$(x+1)^3 + 2(x+1) - 1 = 0$

$x^3 + 3x^2 + 3x + 1 + 2x + 2 - 1 = 0$

ie $x^3 + 3x^2 + 5x + 2 = 0$

(ii) Roots of $x^3 + 2x - 1 = 0$ are α, β, γ

$\therefore \alpha + \beta + \gamma = 0$ —— (1)

Then $(\alpha + \beta - \gamma)(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)$

$= (-\gamma - \gamma)(-\alpha - \alpha)(-\beta - \beta)$ from (1)

$= (-2\gamma)(-2\alpha)(-2\beta)$

$= -8\alpha\beta\gamma$

8 since $\alpha\beta\gamma = 1$

(b) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ ——— ①

(i) $16x^4 - 16x^2 + 1 = 0$ ——— ②

let $x = \cos\theta$

$\Rightarrow 16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$\Rightarrow 8\cos^4\theta - 8\cos^2\theta = -\frac{1}{2}$

$\Rightarrow \cos 4\theta - 1 = -\frac{1}{2}$

$\therefore \cos 4\theta = \frac{1}{2}$

$\therefore 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \ldots$

$\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \ldots$

$\therefore x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}, \ldots$

$= \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, -\cos\frac{5\pi}{12}, -\cos\frac{\pi}{12}$

(ii) From ②: $x^2 = \frac{16 \pm \sqrt{192}}{32}$

$= \frac{16 \pm 8\sqrt{3}}{32}$

$= \frac{2 \pm \sqrt{3}}{4}$

$= \frac{2-\sqrt{3}}{4}, \frac{2+\sqrt{3}}{4}$

Since $\cos\frac{\pi}{12} > \cos\frac{5\pi}{12}$ it follows that

$\cos\frac{\pi}{12} = \left(\frac{2+\sqrt{3}}{4}\right)^{\frac{1}{2}}$ $\cos\frac{5\pi}{12} = \left(\frac{2-\sqrt{3}}{4}\right)^{\frac{1}{2}}$

(c) $p(x) = x^3 + ax + b = 0$

(i) $p'(x) = 3x^2 + a$

stationary points when $p'(x) = 0$

$$\text{ie } 3x^2 = -a$$

$$x^2 = -\frac{a}{3}$$

If $a > 0$ then there are <u>NO</u> stationary points. [1]

<u>BUT</u> $p(x) \to \infty$ as $x \to \infty$

and $p(x) \to -\infty$ as $x \to -\infty$ [1]

$\therefore$ there must be one value of x for which $p(x) = 0$

(ii) Three real roots

$\Rightarrow$ 2 stationary points whose y-values have opposite signs. [1]

$$p'(x) = 0 \Rightarrow x^2 = -\frac{a}{3}$$

$$\therefore x_1 = \left(-\frac{a}{3}\right)^{\frac{1}{2}}$$

$$x_2 = -\left(-\frac{a}{3}\right)^{\frac{1}{2}}$$

[1]

$$p(x_1) = \left(-\frac{a}{3}\right)^{\frac{3}{2}} + a\left(-\frac{a}{3}\right)^{\frac{1}{2}} + b$$

$$= \left(-\frac{a}{3}\right)^{\frac{1}{2}}\left[-\frac{a}{3} + a\right] + b$$

$$= \left(-\frac{a}{3}\right)^{\frac{1}{2}}\left(\frac{2a}{3}\right) + b$$

[1]

$$P(x_2) = -\left(-\frac{a}{3}\right)^{\frac{3}{2}} - a\left(-\frac{a}{3}\right)^{\frac{1}{2}} + b$$

$$= \left(-\frac{a}{3}\right)^{\frac{1}{2}}\left[\frac{a}{3} - a\right] + b$$

$$= \left(-\frac{2a}{3}\right)\left(-\frac{a}{3}\right)^{\frac{1}{2}} + b \qquad [1]$$

we must have $P(x_1) . P(x_2) < 0$

ie $\left[b + \frac{2a}{3}\left(-\frac{a}{3}\right)^{\frac{1}{2}}\right]\left[b - \left(\frac{2a}{3}\right)\left(-\frac{a}{3}\right)^{\frac{1}{2}}\right] < 0 \quad [1]$

ie $b^2 - \frac{4a^2}{9}.\left(-\frac{a}{3}\right) < 0$

$$27b^2 + 4a^3 < 0 \qquad [1]$$