St George Girls High School

Year 12

Common Test 3

June 2009

Mathematics Extension 2

General Instructions

- Working time – 75 minutes
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- All necessary working must be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 52

- Attempt Questions 1 – 3
- Questions are not all of equal value.

Students are advised that this is a School Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (18 marks) – Start a New Page **Marks**

a) Find $\int \frac{\sin x}{\cos^4 x}\,dx$ 2

b) Use completion of the square to find $\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$ 2

c) Using the substitution $t = \tan\frac{\theta}{2}$ (ie $\theta = 2\tan^{-1}t$) find the value of 4

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+3\sin\theta+2\cos\theta}\,d\theta$$

d) (i) Show that $\sqrt{x^2+16} - \frac{16}{\sqrt{x^2+16}} = \frac{x^2}{\sqrt{x^2+16}}$ 1

(ii) Use integration by parts to find the value of $\int_0^3 \sqrt{x^2+16}\;dx$ 4

e) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\;dx$

(i) Show that $I_n = \frac{n-1}{n}\,I_{n-2}$ 3

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} cos^5 x\;dx$ 2

<u>Question 2</u> - (17 marks) - Start a New Page **Marks**

a) Given that $\alpha + \beta + \gamma = -3$
$\alpha^2 + \beta^2 + \gamma^2 = 29$
$\alpha\beta\gamma = -6$

form the monic cubic equation whose roots are α, β and γ. 3

b) α, β, γ are the roots of the polynomial equation $3x^3 - 4x^2 + 5x + 6 = 0$

Form a polynomial equation with integer coefficients and with roots

(i) $\alpha + 3,\ \ \beta + 3,\ \ \gamma + 3$ 2

(ii) $\alpha^2,\ \beta^2,\ \gamma^2$ 2

c) (i) Show that if a polynomial equation $P(x) = 0$ has a root $x = \alpha$ of multiplicity n, then the polynomial equation $P'(x) = 0$ has $x = \alpha$ as a root of multiplicity $(n - 1)$. 3

(ii) Show that the cubic equation $ax^3 + bx^2 + c = 0$ has a double root when $27a^2c + 4b^3 = 0$ where a, b, c are all non-zero. 3

d) (i) Find real numbers A, B and C so that

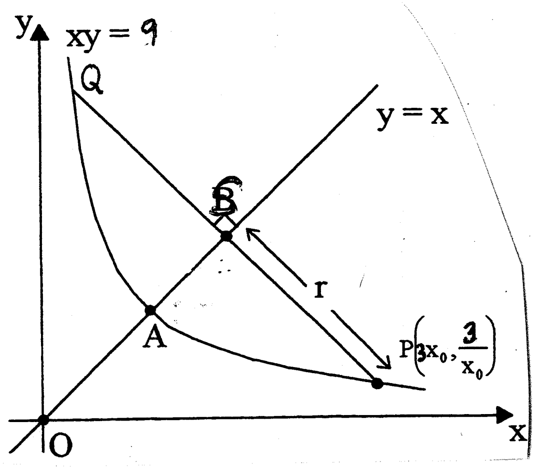
$$\frac{5x+2}{(x^2+4)(x+2)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x+2} \text{ for all } x \neq -2$$ 2

(ii) Hence find $\int \frac{5x+2}{(x^2+4)(x+2)}\, dx$ 2

Marks

Question 3 – (17 marks) – Start a New Page

a)



The diagram shows the function $xy = 9$, for $x > 0$

(i) Write the coordinates of A and find the length of OA.

Given that the eccentricity of the hyperbola is $\sqrt{2}$, find the length of OS, where S is the focus. 2

(ii) Using the result that $y = x$ is an axis of symmetry of $xy = 9$, and given that P has coordinates $\left(3x_0, \frac{3}{x_0}\right)$ write coordinates for Q in terms of x_0.

Hence show that $2r^2 = \left(3x_0 - \frac{3}{x_0}\right)^2$ 2

(iii) Using Part (ii), show that $OP^2 = 2(9 + r^2)$ 2

(iv) Using Part (iii), or otherwise, find the length of the latus rectum. 2

b) $P\left(5p, \frac{5}{p}\right)$ and $Q\left(5q, \frac{5}{q}\right)$ are points on the same branch of the hyperbola $xy = 25$.

(i) Show that the equation of the tangent at P is $x + p^2y = 10p$ and hence find the point of intersection, T, of the tangents at P and Q. 4

(ii) Show that the equation of the chord PQ is $x + pqy = 5(p + q)$ 2

(iii) If the chord PQ passes through the point $(0, -2)$ show that $\frac{p+q}{pq} = -\frac{2}{5}$. 3
Hence find the equation of the locus of T as P and Q vary under this condition.

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions 2009 EXT #CT3.

Q1

(a) $\int \cos^{-4}x \cdot \sin x\,dx = -\int (\cos x)^{-4} \cdot -\sin x\,dx$

$= -\dfrac{(\cos x)^{-3}}{-3} + C$

$= \dfrac{1}{3\cos^3 x} + C$

(b) $\displaystyle\int \frac{dx}{\sqrt{5-(x^2-4x)}} = \int \frac{dx}{\sqrt{5-(x^2-4x+4)+4}}$

$\displaystyle = \int \frac{dx}{\sqrt{9-(x-2)^2}}$

$= \sin^{-1}\left(\dfrac{x-2}{3}\right) + C$ $\left[-\cos^{-1}\left(\dfrac{x-2}{3}\right)+ \right.$

(c) $t = \tan\dfrac{\theta}{2}$

$\therefore 2\tan^{-1} t = \theta$

$\dfrac{2}{1+t^2}\,dt = d\theta$

When $\theta = 0$, $t = 0$

$\theta = \frac{\pi}{2}$, $t = 1$

$\sin\theta = \dfrac{2t}{1+t^2}$

$\cos\theta = \dfrac{1-t^2}{1+t^2}$

Substitution gives

$\displaystyle\int_0^1 \frac{1}{2+\frac{6t}{1+t^2}+\frac{2-2t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$\displaystyle = \int_0^1 \frac{\cancel{1+t^2}}{2+\cancel{2t^2}+6t+2-\cancel{2t^2}} \cdot \frac{2}{\cancel{1+t^2}}\,dt$

$\displaystyle = \int_0^1 \frac{dt}{2+3t}$

$= \frac{1}{3}\ln|2+3t|\Big]_0^1$

$= \frac{1}{3}(\ln 5 - \ln 2)$

$= \frac{1}{3}\ln\left(\frac{5}{2}\right)$

(d) (i) $\sqrt{x^2+16} - \dfrac{16}{\sqrt{x^2+16}} = \dfrac{(x^2+16)-16}{\sqrt{x^2+16}}$

$= \dfrac{x^2}{\sqrt{x^2+16}}$

(ii) $\displaystyle\int_0^3 \sqrt{x^2+16}\,dx = \sqrt{x^2+16}\cdot x\Big]_0^3 - \int_0^3 x \cdot \frac{x}{\sqrt{x^2+16}}\,dx$

let $u = \sqrt{x^2+16}$ and $\dfrac{dv}{dx} = 1$

$\dfrac{du}{dx} = \dfrac{x}{\sqrt{x^2+16}}$ $v = x$

$\displaystyle = 15 - \int_0^3 \frac{x^2}{x^2+16}\,dx$

from (i) $\displaystyle\int_0^3 \sqrt{x^2+16}\,dx = 15 - \left[\int_0^3 \sqrt{x^2+16}\,dx - \int_0^3 \frac{16}{\sqrt{x^2+16}}\,dx\right]$

$\displaystyle\therefore 2\int_0^3 \sqrt{x^2+16}\,dx = 15 + 16\int_0^3 \frac{dx}{\sqrt{x^2+16}}$

$= 15 + 16\left[\ln\left(x+\sqrt{x^2+16}\right)\right]_0^3$

$= 15 + 16\left[\ln 8 - \ln 4\right]$

$= 15 + 16\ln 2$

So $\displaystyle\int_0^3 \sqrt{x^2+16}\,dx = \frac{1}{2}(15 + 16\ln 2)$

(e) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

(i) Then

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x . \cos x \, dx$$

let $u = \cos^{n-1} x$ & $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = (n-1)\cos^{n-2} x . -\sin x$ $v = \sin x$

$$\therefore I_n = \left[\cos^{n-1} x . \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x . (n-1)\cos^{n-2} x . -\sin x \, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, (\sin^2 x) \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$I_n = (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1) I_n$$

So $(n-1) I_n + I_n = (n-1) I_{n-2}$

$$n I_n = (n-1) I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

(ii) $I_5 = \frac{4}{5} I_3$

$I_3 = \frac{2}{3} I_1$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= 1$$

So $I_3 = \frac{2}{3} \times 1$

$= \frac{2}{3}$

$I_5 = \frac{4}{5} \times I_3$

$= \frac{4}{5} \times \frac{2}{3}$

$I_5 = \frac{8}{15}$

Q(2)

(a) If α, β, γ are roots of monic cubic

then $P(x) = (x-\alpha)(x-\beta)(x-\gamma)$

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma$

Now $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = \alpha^2+\beta^2+\gamma^2$

$\therefore 2(\alpha\beta+\alpha\gamma+\beta\gamma) = (-3)^2 - 29$

$= -20$

$\alpha\beta+\alpha\gamma+\beta\gamma = -10$

So $P(x) = x^3 + 3x^2 - 10x + 6$

(b) (i) If $x = \alpha + 3$

then $x - 3 = \alpha$

Since α is a root, then

$3(x-3)^3 - 4(x-3)^2 + 5(x-3) + 6 = 0$ is true.

$3[x^3 - 9x^2 + 27x - 27] - 4[x^2 - 6x + 9] + 5x - 15 + 6 = 0$

$3x^3 - 27x^2 + 81x - 81 - 4x^2 + 24x - 36 + 5x - 15 + 6 = 0$

$$3x^3 - 31x^2 + 110x - 126 = 0$$

(ii) If $x = \alpha^2$

then $\pm\sqrt{x} = \alpha$

Since α is a root, then

$3(\pm\sqrt{x})^3 - 4(\pm\sqrt{x})^2 + 5(\pm\sqrt{x}) + 6 = 0$

$+\sqrt{x}$ $3x\sqrt{x} - 4x + 5\sqrt{x} + 6 = 0$

$\sqrt{x}\,(3x+5) = (4x-6)$

Square both sides $x\,(9x^2+30x+25) = 16x^2 - 48x + 36$

$$9x^3 + 30x^2 + 25x = 16x^2 - 48x + 36$$
$$9x^3 + 14x^2 + 73x - 36 = 0$$

[same for $-\sqrt{x}$].

(c) (i) let $P(x) = (x-\alpha)^n\, Q(x)$

α is root of multiplicity n of $P(x)$ and is not a root of polynomial $Q(x)$ ie $Q(\alpha) \neq 0$

then $P'(x) = n(x-\alpha)^{n-1}.\,Q(x) + (x-\alpha)^n.\,Q'(x)$

$$= (x-\alpha)^{n-1}\left[n.\,Q(x) + (x-\alpha).\,Q'(x)\right]$$

$$= (x-\alpha)^{n-1}\,R(x)$$, $R(x) \in$ polynomial such that $R(\alpha) \neq 0$

$R(\alpha) = n\,Q(\alpha) + (\alpha-\alpha)\,Q'(\alpha)$

$= n\,Q(\alpha) \neq 0$ since $n > 1$ & $Q(\alpha) \neq 0$

Then α is root of multiplicity $(n-1)$ of $P'(x)$.

(ii) $P(x) = ax^3 + bx^2 + c$

$P'(x) = 3ax^2 + 2bx$

If double root then $x\,[3ax + 2b] = 0$

Now $x = 0$ not a root of $P(x)$ since c is non zero!

$\therefore\ x = -\dfrac{2b}{3a}$ is a root of $P(x)$

ie $a\left(\dfrac{-2b}{3a}\right)^3 + b\left(\dfrac{2b}{3a}\right)^2 + c = 0$

$$\frac{-8b^3}{27a^2} + \frac{4b^3}{9a^2} + c = 0$$

$$-8b^3 + 12b^2 + 27a^2c = 0$$ [multiply by $27a^2$ since a non-zero]

$$27a^2c + 4b^3 = 0$$

Required condition for double root.

(d) (i) $\dfrac{5x+2}{(x^2+4)(x+2)} = \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x+2}$ $(x \neq -2$

then $5x+2 = (Ax+B)(x+2) + C(x^2+4)$

$= Ax^2 + 2Ax + Bx + 2B + Cx^2 + 4C$

$= (A+C)x^2 + (2A+B)x + 2B + 4C$

Equating coeff. (I) $A + C = 0 \Rightarrow A = -C$

(II) $2A + B = 5$

(III) $2B + 4C = 2 \Rightarrow B + 2C = 1$

Sub I into III $B + 2(-A) = 1 \Rightarrow -2A + B = 1$

Then II + III $\begin{array}{l} 2A+B=5 \\ -2A+B=1 \end{array} \Rightarrow 2B = 6$

$\underline{B = 3}$

Sub $2A + 3 = 5$ & $3 + 2C = 1$

$\underline{A = 1}$ $\underline{C = -1}$

$$\therefore\ \frac{5x+2}{(x^2+4)(x+2)} = \frac{x+3}{x^2+4} - \frac{1}{x+2}$$

(ii) $\displaystyle\int \frac{5x+2}{(x^2+4)(x+2)}\,dx = \int \frac{x+3}{x^2+4} - \frac{1}{x+2}\,dx$

$\displaystyle = \int \frac{x}{x^2+4} + \frac{3}{x^2+4} - \frac{1}{x+2}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + 3\int \frac{dx}{x^2+4} - \int \frac{dx}{x+2}$

$\displaystyle = \frac{1}{2}\ln(x^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \ln|x+2| + C$

Q3

(a) (i) $xy = 9$ $\quad c^2 = 9$, $c = 3$ $\quad$ & $\frac{1}{2}a^2 = 9$, $a = 3\sqrt{2}$ $[c\sqrt{2}]$

1

$OA = \sqrt{3^2+3^2} = 3\sqrt{2}$ $\quad$. $A(c,c) \to A(3,3)$

. given $e = \sqrt{2}$ $\quad$ (or) $\left[OS = \sqrt{(3\sqrt{2})^2 + (3\sqrt{2})^2} = 6\right]$

$OS = ae$
$= 3\sqrt{2} \times \sqrt{2}$ $\quad$ 1
$= 6$

(ii) . Q is reflection of P across $y = x$

$\therefore Q\left(\frac{3}{x_0}, 3x_0\right)$

1

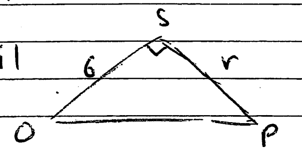
. Distance formula $\quad 2r = \sqrt{\left(3x_0 - \frac{3}{x_0}\right)^2 + \left(\frac{3}{x_0} - 3x_0\right)^2}$

$(2r)^2 = 2\left(3x_0 - \frac{3}{x_0}\right)^2$

1 $\quad \therefore 2r^2 = \left(3x_0 - \frac{3}{x_0}\right)^2$

$x - y = 0$
OR $d = \left|\dfrac{1\times 3x_0 + -1\times\frac{3}{x_0} + 0}{\sqrt{1^2+(-1)^2}}\right|$
$d = \dfrac{3x_0 - \frac{3}{x_0}}{\sqrt{2}}$ $\quad$ 1
$\downarrow$ $\left(3x_0 - \frac{3}{x_0}\right)^2 = \left(\frac{3}{x_0} - 3x_0\right)^2$
$r = \dfrac{\left(3x_0 - \frac{3}{x_0}\right)}{\sqrt{2}}$ $\quad$ 1
$r^2 = \left(3x_0 - \frac{1}{3x_0}\right)^2$
$2r^2 = \left(3x_0 - \frac{3}{x_0}\right)^2$

(iii)



$OP^2 = OS^2 + SP^2$
(Pythagoras)
$OP^2 = 6^2 + r^2$

$OP^2 = 36 + \frac{1}{2}\left(3x_0 - \frac{3}{x_0}\right)^2$ $\quad$ ①

& $OP^2 = (3x_0 - 0)^2 + \left(\frac{3}{x_0} - 0\right)^2$

$= (3x_0)^2 + \left(\frac{3}{x_0}\right)^2$

now $\left(3x_0 - \frac{3}{x_0}\right)^2 = (3x_0)^2 - 18 + \left(\frac{3}{x_0}\right)^2$ $\quad$ 1

so $(3x_0)^2 + \left(\frac{3}{x_0}\right)^2 = \left(3x_0 - \frac{3}{x_0}\right)^2 + 18$

From (ii) $\quad = 2r^2 + 18$ $\quad$ 1

$\therefore OP^2 = 2r^2 + 18$
$= 2(r^2 + 9)$.

(iv) QP is latus rectum [focal chord // to directrix
⊥ to axis]

from (iii) $2(9+r^2) = 6^2 + r^2$ | $[OP^2 = \ ]$

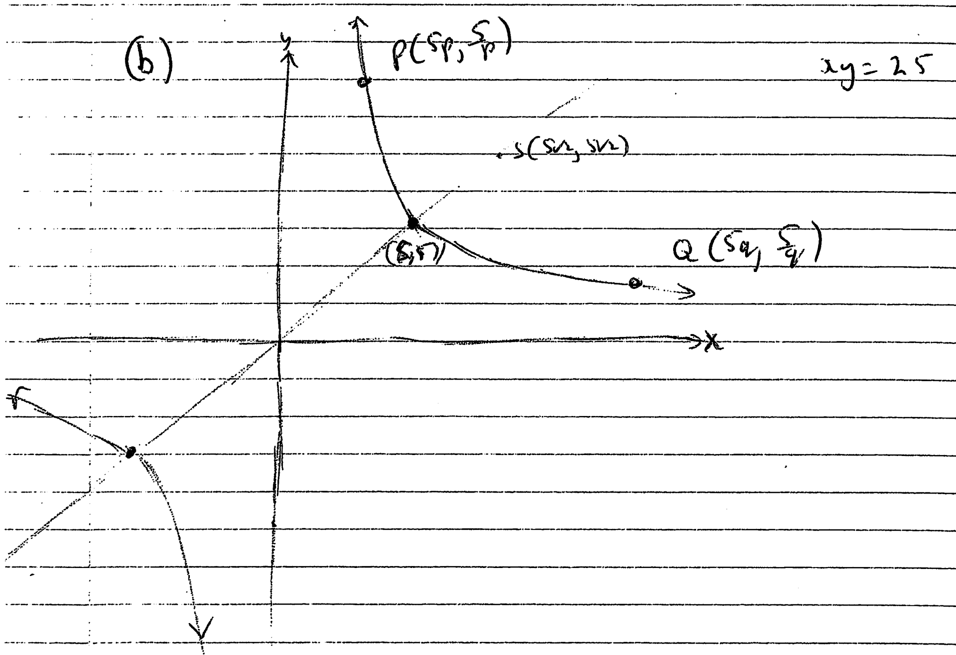
$18 + 2r^2 = 36 + r^2$

$r^2 = 18$

$r = 3\sqrt{2}$ |

$\therefore$ length of latus rectum $2r = \underline{6\sqrt{2}}$

(b)



(i) $x = 5p$; $y = \frac{5}{p}$

$\frac{dx}{dp} = 5$ $\frac{dy}{dp} = -\frac{5}{p^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{p^2}$ |

or $\frac{dy}{dx} = \frac{-25}{x^2}$ |

$x = 5p \Rightarrow m = \frac{-1}{p^2}$

Equation of tangent

$y - \frac{5}{p} = -\frac{1}{p^2}(x - 5p)$

$p^2 y - 5p = -x + 5p$

$x + p^2 y = 10p$ *

Tangent at $P(5p, \frac{5}{p}) \Rightarrow$ $x + p^2 y = 10p$ -- A

Similarly Tangent at $Q(5q, \frac{5}{q}) \Rightarrow$ $x + q^2 y = 10q$ -- B

Solve simultaneously to find T |

Ⓐ-Ⓑ & q^2Ⓐ $- p^2$(B)

$(p^2 - q^2)y = 10(p-q)$ $(q^2 - p^2)x = 10pq(q-p)$

$y = \frac{10}{p+q}$ $x = \frac{10pq}{p+q}$

$T\left[\frac{10pq}{p+q}, \frac{10}{p+q}\right]$ |

(ii) Equation of PQ • gradient $m = \frac{\frac{5}{p} - \frac{5}{q}}{5p - 5q}$

$= \frac{\frac{5q - 5p}{pq}}{5p - 5q}$

| $= -\frac{1}{pq}$

The $y - \frac{5}{p} = -\frac{1}{pq}(x - 5p)$

$pqy - 5q = -x + 5p$ |

$x + pqy = 5(p+q)$. ✱

(iii) Substitute $(0, -2)$ into ✱ (ii)

$0 - 2pq = 5(p+q)$

$\frac{pq}{p+q} = -\frac{5}{2}$ |

OR $\frac{p+q}{pq} = \frac{-2}{5}$

P.T.O.

$X = \frac{10pq}{p+q}$; $Y = \frac{10}{p+q}$.

the $X = 10 \times -\frac{5}{2}$

$x = -25$

T moves on the vertical line $x = -25$

Now yre

- to pass through $(0, -2)$ P and Q must be in 3RD quadrant since gradient PQ ; $m = \frac{-1}{pq}$ is always $-ve$.

- when $x = -25$, $xy = 25$

$-25 \times y = 25$

$y = -1$

$\therefore$ tangent can not intersect inside hyperbola, so $y > -1$

Since $p + q < 0$ the $\frac{10}{p+q} < 0$
(3rd quadrant).

So locus is line $x = -25$
where $-1 < y < 0$.