SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2018

HSC Task #1

Mathematics Extension II

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 65

Section I Pages 2-4
5 marks

- Attempt questions 1-5
- Allow about 10 minutes for this section

Section II Pages 5-10
60 Marks

- Attempt Questions 6-9
- Allow about 1 hour and 20 minutes for this section

Examiner: ***A. Wu***

1 Section I - Multiple Choice

5 Marks
Attempt questions 1-5
Allow about 10 minutes for this section
Use the multiple choice answer sheet for questions 1-5

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1. What is the correct factorisation of the $x^3 + i$?

   A. $(x - i)(-1 + ix + x^2)$

   B. $(x + i)(1 + ix - x^2)$

   C. $(x + i)(1 + x + x^2)$

   D. $(x - i)(1 + x + ix^2)$

2. The polynomial $P(x) = x^3 - ax + 2$ has a positive double zero. What is a?

   A. $-3\sqrt[3]{3}$

   B. -3

   C. $3\sqrt[3]{3}$

   D. 3

3. Let ω be a complex cube root of unity (i.e $\Im\mathrm{m}(\omega) \neq 0$). What is the sum of the roots of the following polynomial?

$$\sum_{k=1}^{2018} (x - \omega^k)^{2018}$$

   A. 0

   B. -2018

   C. 2018ω

   D. $2018(\omega^2 - 1)$

4. Suppose z, w are complex numbers. Select the inequality that is not always true.

   A. $|z+w| \leq |z| + |w|$

   B. $||z| - |w|| \leq |z - w|$

   C. $|z| - |w| \leq |z - w|$

   D. $||z| + |w|| \geq |z - w| + |w|$

5. If $\arg\left(\dfrac{z - \sqrt{3} - 2i}{z - \sqrt{3}}\right) = \dfrac{\pi}{2}$, what is the maximum value of $\arg(z)$?

   A. $\dfrac{\pi}{2}$

   B. $\dfrac{\pi}{6}$

   C. $\dfrac{\pi}{3}$

   D. $\tan^{-1}\dfrac{1}{2}$

Bonus Question: What is the value of 1+2+3+4+...

A. ∞

B. 1

C. 0

D. $-\dfrac{1}{12}$

2 Section II - Extended Response

65 Marks
Attempt questions 6-9
Allow about 1 hour and 35 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 6–9, your responses **must include** relevant mathematical reasoning and/or calculations.

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6. Question 6 (15 marks) Start a NEW writing booklet.

(a) Let $z = 1 + i$, $w = \sqrt{3} + i$.

i. $\boxed{1}$ Rewrite w in mod-arg form

ii. $\boxed{1}$ Find w^{2018} in cartesian form

iii. $\boxed{1}$ Rewrite $\dfrac{z}{w}$ in cartesian form

iv. $\boxed{2}$ Deduce that

$$\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

(b) Sketch the following loci on the Argand Diagram

i. $\boxed{2}$ $|z - 2i| = \Im\text{m}(z)$

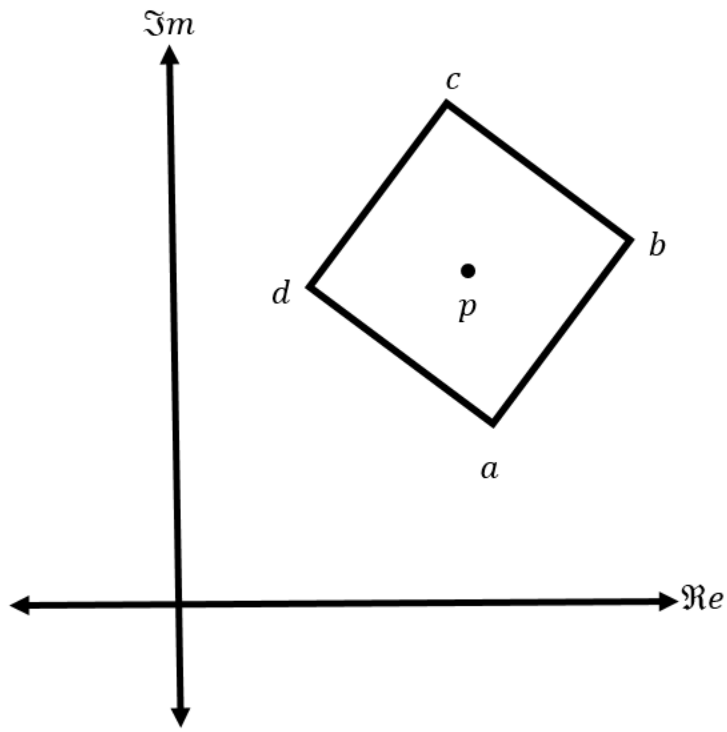
ii. $\boxed{2}$ $\Im\text{m}(z) = \tan(\arg z)$ for $0 \le \arg z \le \dfrac{\pi}{2}$

(c) i. $\boxed{1}$ Write $\dfrac{1}{(a+i)(b+i)}$ with a real denominator

ii. $\boxed{2}$ Hence, by using arguments, for $ab > 1$ and $a, b > 0$, prove that

$$\tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{1}{b}\right) = \tan^{-1}\left(\frac{a+b}{ab-1}\right)$$

(d) $\boxed{3}$



Suppose the complex numbers a, b, c, d are the vertices of a square in anti-clockwise order. Let p be the centre of the square (midpoint and intersection of diagonals). Show that

$$p = \frac{a+d+i(b-c)}{2}$$

End of Question 6

7. Question 7 (15 marks) Start a NEW writing booklet.

(a) Consider the polynomial equation $x^3 - \left(\frac{1+\sqrt{5}}{2}\right)x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x - 1 = 0$. It is known that $x = 1$ is a root.

i. $\boxed{\textbf{2}}$ Prove that at least one root is non-real.

ii. $\boxed{\textbf{2}}$ Deduce that the other two roots lie on the unit circle

(b) $\boxed{\textbf{2}}$ Let the polynomial $P(x) = x^4 - x^3 + 5x^2 - 2x + 7$ have zeroes $\alpha, \beta, \gamma, \delta$. Evaluate:

$$(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$$

Hint: Do NOT expand

(c) At the start of the year, n scholar sportsmen have the potential to achieve the Joseph Coates award by the end of the year.
Depending on procrastination habits and dedication to GPS sports, any number of students k, where $0 \leq k \leq n, \quad k \in \mathbb{Z}$ may receive the award.

Assume that all n students are equally likely to achieve or to not achieve the award and that one student's achievement is independent of anothers'.

i. $\boxed{\textbf{1}}$ Show that the probability of k students winning the Joseph Coates Award is

$$\binom{n}{k}\frac{1}{2^n}$$

ii. $\boxed{\textbf{1}}$ Deduce that

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

(d) i. $\boxed{\mathbf{2}}$ Let z, w be complex numbers with $\theta = \arg w > \arg z = \phi$. Prove that:

$$|z-w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\theta - \phi)$$

ii. $\boxed{\mathbf{1}}$ If z, w lie on the unit circle, show that they are are separated by a distance of

$$2\sin\left(\frac{\theta-\phi}{2}\right)$$

(e) $\boxed{\mathbf{4}}$ Using mathematical induction, prove the General Leibniz rule (below) for positive integers n, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

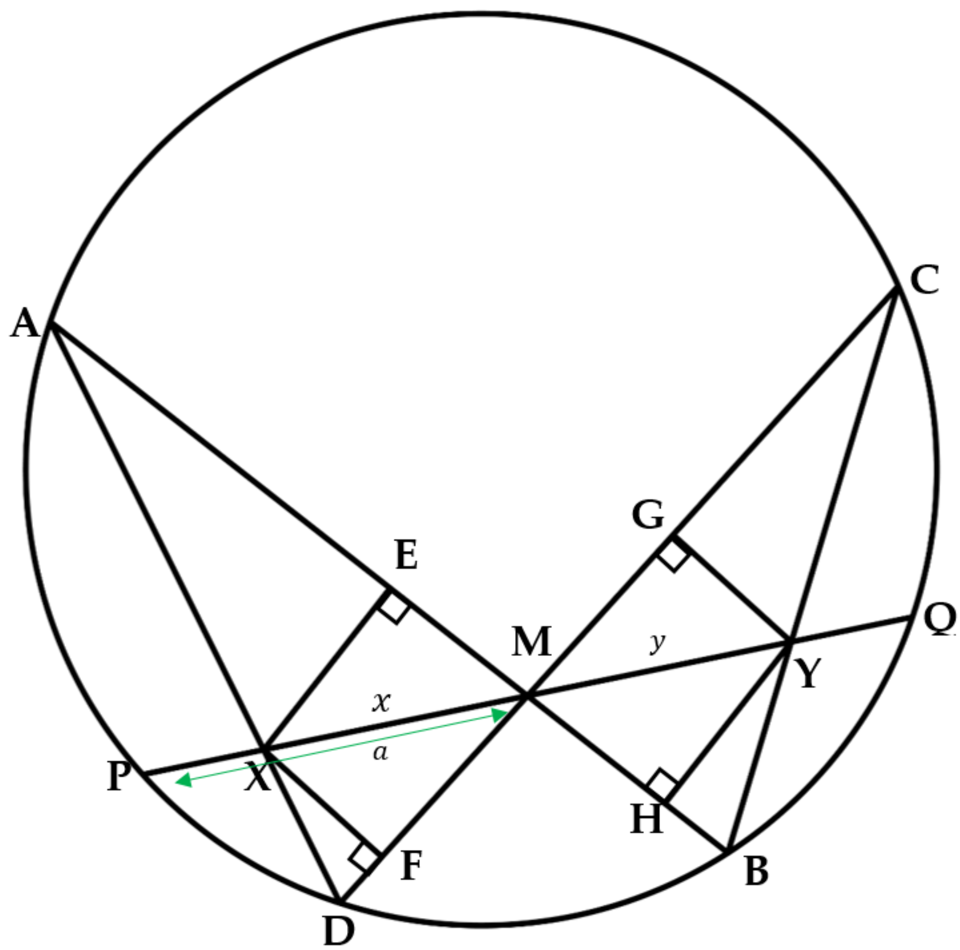
You are given that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ for positive integers $n \geq k$.

$$\frac{d^n}{dx^n}\left[f(x)g(x)\right] = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

End of Question 7

8. Question 8 (15 marks) Start a NEW writing booklet.

(a) *The Butterfly Theorem:* Let M be the midpoint of a chord PQ of a circle. Two more chords AB and CD are drawn through M. AD and BC intersect the chord PQ at points X, Y respectively. Let E, F be the feet of the perpendiculars drawn from X to AM and DM respectively. Similarly, let G, H be the feet of the perpendiculars drawn from Y to CM and BM respectively.



Let $PM = a = QM$, $MX = x$, $MY = y$.

i. $\boxed{\mathbf{1}}$ Prove that $\dfrac{x}{y} = \dfrac{EX}{HY}$

ii. $\boxed{\mathbf{1}}$ Prove that $\dfrac{EX}{GY} = \dfrac{AX}{CY}$

iii. $\boxed{\mathbf{3}}$ Hence or otherwise, show that

$$\frac{x^2}{y^2} = \frac{AX \cdot XD}{CY \cdot YB}$$

iv. $\boxed{\mathbf{2}}$ Show that

$$\frac{AX \cdot XD}{CY \cdot YB} = \frac{a^2 - x^2}{a^2 - y^2}$$

v. $\boxed{\mathbf{1}}$ Deduce that $x = y$

(b) *Chebyshev Polynomials:*

i. $\boxed{\mathbf{2}}$ Let $T_n(x) = \cos(n \cos^{-1} x)$, $U_n(x) = \sin(n \cos^{-1} x)$ where $|x| \leq 1$.

Using DeMoivre's Theorem, prove that

$$T_n(x) + i \cdot U_n(x) = \left(x + i\sqrt{1 - x^2}\right)^n$$

ii. $\boxed{\mathbf{2}}$ Hence, prove that:

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right]$$

iii. $\boxed{\mathbf{3}}$ Consider the series $S = \sum_{n=0}^{\infty} t^n (\cos\theta + i \sin\theta)^n$, where $-1 < t < 1$ and $\theta = \cos^{-1} x$.

Prove that:

$$\sum_{n=0}^{\infty} T_n(x) t^n = \frac{1 - tx}{1 - 2tx + t^2}$$

End of Question 8

9. Question 9 (15 marks) Start a NEW writing booklet.

(a) At lunchtime, 5 senior students and 6 junior students decide to line up at the canteen to purchase Chicken Pides. The canteen has three lines for students to purchase food; 2 (fixed) lines for juniors and 1 for seniors. The seniors, having much respect for the juniors, decide that they will line up wherever they want. The juniors, however, stick to their own lines.

i. $\boxed{1}$ In how many ways can the 5 seniors be distributed amongst the 3 lines?

ii. $\boxed{3}$ In how many ways can the 5 seniors and 6 juniors be distributed among the 3 lines?

(b) Consider the general polynomial cubic equation $P(x) = ax^3 + bx^2 + cx + d = 0$ for $a \neq 0$ and real constants a, b, c, d. $P(x)$ has roots α, β, γ

i. $\boxed{2}$ Show that the following polynomial has roots $\alpha + \dfrac{b}{3a}, \beta + \dfrac{b}{3a}, \gamma + \dfrac{b}{3a}$

$$ay^3 + \left(c - \frac{b^2}{3a}\right) y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0 \tag{1}$$

ii. $\boxed{1}$ Let $p = \dfrac{1}{a}\left(c - \dfrac{b^2}{3a}\right), q = \dfrac{1}{a}\left(d + \dfrac{2b^3}{27a^2} - \dfrac{bc}{3a}\right)$.

Using the substitution $y = s+t$, where s, t are real variables, show that equation (1) is equivalent to equation (2).

$$s^3 + t^3 + (3st + p)(s + t) + q = 0 \tag{2}$$

iii. $\boxed{1}$ Suppose that $3st + p = 0$. Show that

$$s^3 + t^3 = -q$$

$$s^3t^3 = -\frac{p^3}{27}$$

iv. $\boxed{\mathbf{2}}$ Assume without loss of generality that $s^3 > t^3$. Prove that:

$$s^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \qquad t^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

v. $\boxed{\mathbf{2}}$ Show that that the possible solutions of s are:

$$s = \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } k = 0, 1, 2$$

where ω is the complex cube root of unity with positive imaginary part.

vi. $\boxed{\mathbf{3}}$ Show that the roots of equation $P(x) = 0$ are

$$x = -\frac{b}{3a} + \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{2k} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } k = 0, 1, 2$$

where ω is the complex cube root of unity with positive imaginary part.

End of Examination

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$\boxed{\textbf{Section II}}$ Pages 5-10

60 Marks

- Attempt Questions 6-9
- Allow about 1 hour and 20 minutes for this section

Examiner: ***A. Wu***

Note: These are quick solutions. You may need to provide more working in an exam

1 Section I - Multiple Choice

5 Marks
Attempt questions 1-5
Allow about 10 minutes for this section
Use the multiple choice answer sheet for questions 1-5

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1. What is the correct factorisation of the $x^3 + i$?

   A. $(x - i)(-1 + ix + x^2)$

   B. $(x + i)(1 + ix - x^2)$

   C. $(x + i)(1 + x + x^2)$

   D. $(x - i)(1 + x + ix^2)$

   Explanation: Write $x^3 + i$ as $x^3 - i^3$ and use the difference of cubes factorisation.

2. The polynomial $P(x) = x^3 - ax + 2$ has a positive double zero. What is a?

   A. $-3\sqrt[3]{3}$

   B. -3

   C. $3\sqrt[3]{3}$

   D. 3

   Explanation: $P'(x) = 3x^2 - a = 0$. But the double root is positive, so take $x = \sqrt{\frac{a}{3}}$ and substitute back into $P(x)$ to give $a = 3$.

3. Let ω be a complex cube root of unity (i.e $\Im\mathrm{m}(\omega) \neq 0$). What is the sum of the roots of the following polynomial?

$$\sum_{k=1}^{2018} (x - \omega^k)^{2018}$$

   A. 0

   B. -2018

   C. 2018ω

   D. $2018(\omega^2 - 1)$

Explanation: let S be the sum of the roots of the polynomial. By the Binomial theorem (or Pascal's triangle), we have:

$$\begin{aligned} S &= 2018(\omega + \omega^2 + \omega^3 + \ldots + \omega^{2018}) \\ &= 2018(-1 + 1 + \omega + \omega^2 + \omega^3(1 + \omega + \omega^2) + \ldots + \omega^{2016}(1 + \omega + \omega^2)) \\ &= -2018 \qquad (\omega^3 = 1, 1 + \omega + \omega^2 = 0) \text{ since } \omega \text{ is a complex cube root of unity} \end{aligned}$$

4. Suppose z, w are complex numbers. Select the inequality that is not always true.

   A. $|z + w| \leq |z| + |w|$

   B. $||z| - |w|| \leq |z - w|$

   C. $|z| - |w| \leq |z - w|$

   D. $||z| + |w|| \geq |z - w| + |w|$

   Explanation: All inequalities from A to C are true for all vectors (this can be proved using geometry). D is easily seen as not true if we sub in $z = 0$.

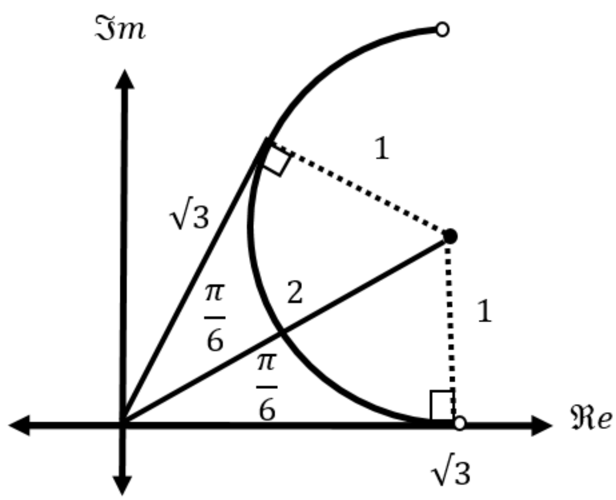
5. If $\arg\left(\dfrac{z - \sqrt{3} - 2i}{z - \sqrt{3}}\right) = \dfrac{\pi}{2}$, what is the maximum value of $\arg(z)$?

   A. $\dfrac{\pi}{2}$

   B. $\dfrac{\pi}{6}$

   C. $\dfrac{\pi}{3}$

   D. $\tan^{-1}\dfrac{1}{2}$



Explanation: The Locus is a semicircle radius 1, center $\sqrt{3}+i$. The maximum value occurs when a ray from the origin is tangent to the circle (and makes a right angle at the point of contact by circle geometry). Using trigonometry we then obtain the maximum value of $\arg z$ as $\frac{\pi}{3}$.

Bonus Question: What is the value of 1+2+3+4+...

A. ∞

B. 1

C. 0

D. $-\dfrac{1}{12}$

Explanation: No comment.

2 Section II - Extended Response

65 Marks
Attempt questions 6-9
Allow about 1 hour and 35 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 6–9, your responses **must include** relevant mathematical reasoning and/or calculations.

---

6. Question 6 (15 marks) Start a NEW writing booklet.

(a) Let $z = 1 + i$, $w = \sqrt{3} + i$.

i. $\boxed{\mathbf{1}}$ Rewrite w in mod-arg form

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

ii. $\boxed{\mathbf{1}}$ Find w^{2018} in cartesian form

$$2^{2017}(1 + i\sqrt{3})$$

iii. $\boxed{\mathbf{1}}$ Rewrite $\dfrac{z}{w}$ in cartesian form

$$\frac{\sqrt{3}+1}{4} + i\frac{\sqrt{3}-1}{4}$$

iv. $\boxed{\mathbf{2}}$ Deduce that

$$\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Using mod-arg forms we have:

$$\frac{z}{w} = \frac{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}{2\operatorname{cis}\frac{\pi}{6}} = \frac{1}{\sqrt{2}}\operatorname{cis}\frac{\pi}{12}$$

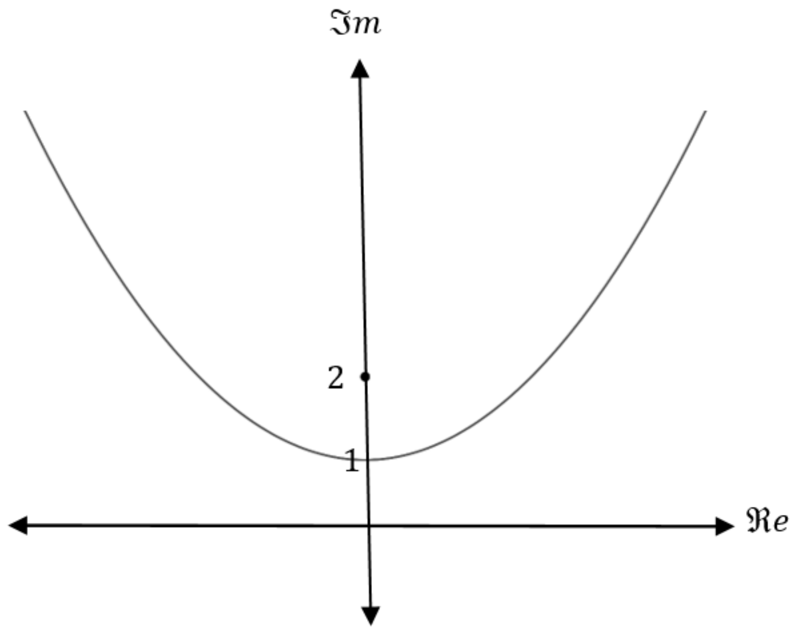
Equating real parts with iii) gives us

$$\frac{1}{\sqrt{2}}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{4}$$
$$\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

(b) Sketch the following loci on the Argand Diagram

i. $\boxed{\mathbf{2}}$ $|z - 2i| = \Im\mathrm{m}(z)$

$\Im\mathrm{m}(z)$ is the real line since the imaginary part of a complex number is real. So this locus is all points such that z is equidistant from the real line and the point $2i$, that is, a parabola with focus $2i$ and directrix $\Im\mathrm{m}(z) = 0$

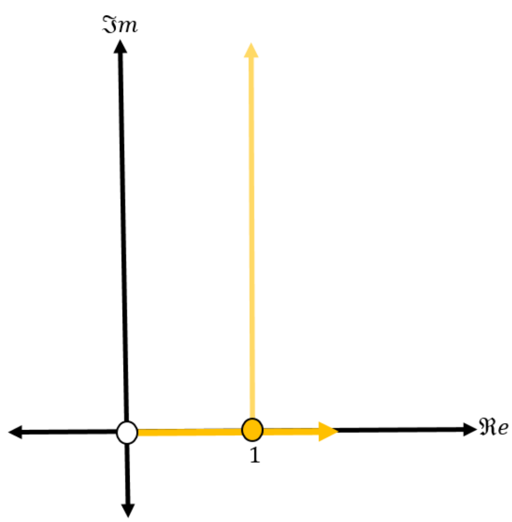


ii. $\boxed{\mathbf{2}}$ $\Im\mathrm{m}(z) = \tan(\arg z)$ for $0 \le \arg z \le \dfrac{\pi}{2}$

Since $0 \le \arg z \le \dfrac{\pi}{2}$, z lies in the first quadrant. Hence if $z = x + iy$, then the locus becomes

$$\begin{aligned} y &= \frac{y}{x} \\ y\left(1 - \frac{1}{x}\right) &= 0 \end{aligned}$$

which gives $y = 0$ or $x = 1$. But recall that z is in the first quadrant, so take only the halves of these lines that lie in the first quadrant. Open circle at origin because $\arg z$ is undefined.



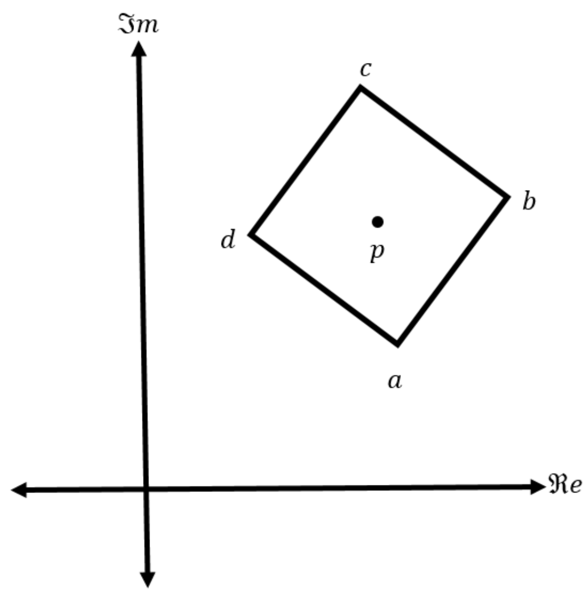
(c) i. $\boxed{\mathbf{1}}$ Write $\dfrac{1}{(a+i)(b+i)}$ with a real denominator

$$\begin{aligned}\frac{1}{(a+i)(b+i)} &= \frac{(a-i)(b-i)}{(a+i)(a-i)(b-i)(b+i)} \\ &= \frac{ab-1-i(a+b)}{(a^2+1)(b^2+1)}\end{aligned}$$

ii. $\boxed{\mathbf{2}}$ Hence, by using arguments, prove that $\tan^{-1}\left(\dfrac{1}{a}\right)+\tan^{-1}\left(\dfrac{1}{b}\right) = \tan^{-1}\left(\dfrac{a+b}{ab-1}\right)$

$$\begin{aligned}\arg\left(\frac{1}{(a+i)(b+i)}\right) &= \arg\left(\frac{ab-1-i(a+b)}{(a^2+1)(b^2+1)}\right) \\ \arg 1 - \arg(a+i) - \arg(b+i) &= \tan^{-1}\left(-\frac{a+b}{ab-1}\right) \qquad \text{since number is in first quadrant} \\ \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{1}{b}\right) &= \tan^{-1}\left(\frac{a+b}{ab-1}\right) \qquad \text{since } \tan^{-1} \text{ is odd}\end{aligned}$$

(d) $\boxed{\mathbf{3}}$



Suppose the complex numbers a, b, c, d are the vertices of a square in anti-clockwise order. Let p be the centre of the square (where its diagonals intersect). Show that

$$p = \frac{a+d+i(b-c)}{2}$$

Let the complex numbers a, b, c, d be represented by points A, B, C, D respectively.

$$\begin{aligned}
\overrightarrow{AP} + \overrightarrow{PD} &= \overrightarrow{AD} \\
p - a + i(p-a) &= d - a && \text{diagonals of square are perpendicular} \\
p &= \frac{d + ia}{1+i} \\
&= \frac{a + d + i(a-d)}{2} && \text{multiplying by } \frac{1-i}{1-i} \\
&= \frac{a + d + i(b-c)}{2} && \text{opposite sides of a square are equal}
\end{aligned}$$

7. Question 7 (15 marks) Start a NEW writing booklet.

(a) Consider the polynomial equation $x^3 - \left(\frac{1+\sqrt{5}}{2}\right)x^2 + \left(\frac{1+\sqrt{5}}{2}\right)x - 1 = 0$. It is known that $x = 1$ is a root.

i. $\boxed{\mathbf{2}}$ Prove that at least one the roots is non-real.

Let the other two roots be a,b

$$\begin{aligned}
1 + a^2 + b^2 &= (1 + a + b)^2 - 2(a + b + ab) \\
a^2 + b^2 &= -1 + \left(\frac{1+\sqrt{5}}{2}\right)^2 - 2\left(\frac{1+\sqrt{5}}{2}\right) \\
&= -\frac{1+\sqrt{5}}{2} \\
&< 0
\end{aligned}$$

As the sum of their squares is negative, at least one of roots a, b must be non-real.

ii. $\boxed{\mathbf{2}}$ Deduce that the other two roots lie on the unit circle

Let the roots be z, $\overline{z}$ (by the conjugate root theorem, real polynomials must have complex roots in conjugate pairs).
By the product of roots we have $z\overline{z} = |z|^2 = 1$ so $|z| = |\overline{z}| = 1$ and hence the roots lie on the unit circle.

(b) $\boxed{\mathbf{2}}$ Let the polynomial $P(x) = x^4 - x^3 + 5x^2 - 2x + 7$ have zeroes $\alpha, \beta, \gamma, \delta$. Evaluate:

$$(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$$

Hint: Do NOT expand

$$\begin{aligned}
P(x) &= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\\
P(-1) &= (\alpha+1)(\beta+1)(\gamma+1)(\delta+1)\\
(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) &= (-1)^4-(-1)^3+5(-1)^2-2(-1)+7\\
&= 16
\end{aligned}$$

(c) At the start of the year, n scholar sportsmen have the potential to achieve the Joseph Coates award by the end of the year.
Depending students procrastination habits and dedication to GPS sports, any number of students k, where $0 \leq k \leq n, \quad k \in \mathbb{Z}$, may receive the award. Assume that all n students are equally likely to achieve or to not achieve the award and that one student's achievement is independent of anothers'.

i. $\boxed{\mathbf{1}}$ Show that the probability of k students winning the Joseph Coates Award is

$$\binom{n}{k}\frac{1}{2^n}$$

There are $\binom{n}{k}$ ways to choose k students from n and the probability that k students achieve the award is $\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$. Multiplying these gives $\binom{n}{k}\frac{1}{2^n}$.

ii. $\boxed{\mathbf{1}}$ Deduce that

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^n} = 1 \qquad \text{sum probabilities of all possible outcomes}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} = 2^n$$

(d) i. $\boxed{\mathbf{2}}$ Let z, w be complex numbers with $\theta = \arg w > \arg z = \phi$. Prove that:

$$|z-w|^2 = |z|^2+|w|^2-2|z||w|\cos(\theta-\phi)$$

$$\begin{aligned}
|z-w|^2 &= (z-w)(\overline{z}-\overline{w})\\
&= z\overline{z}+w\overline{w}-2(w\overline{z}+\overline{(w\overline{z})})\\
&= |z|^2+|w|^2-2\Re\mathrm{e}(|w|\,\mathrm{cis}\theta|z|\,\mathrm{cis}(-\phi))\\
&= |z|^2+|w|^2-2\Re\mathrm{e}(|w||z|\,\mathrm{cis}(\theta-\phi))\\
&= |z|^2+|w|^2-2|z||w|\cos(\theta-\phi)
\end{aligned}$$

ii. $\boxed{\textbf{1}}$ If z, w lie on the unit circle, show that they are separated by a distance of $2\sin\left(\dfrac{\theta-\phi}{2}\right)$ By part i we have

$$\begin{aligned}
|\operatorname{cis}\theta - \operatorname{cis}\phi|^2 &= 2 - 2\cos(\theta-\phi) \\
&= 4\sin^2\left(\frac{\theta-\phi}{2}\right) && \text{double angle formula} \\
|\operatorname{cis}\theta - \operatorname{cis}\phi| &= 2\sin\left(\frac{\theta-\phi}{2}\right) && \text{modulus is positive}
\end{aligned}$$

(e) $\boxed{\textbf{4}}$ Using mathematical induction, prove the General Leibniz rule (below) for positive integers n, where $f^{(n)}(x)$ represents the n^{th} derivative of the $f(x)$.

You are given that $\dbinom{n}{k-1} + \dbinom{n}{k} = \dbinom{n}{k+1}$ for positive integers $n \geq k$.

$$\frac{d^n}{dx^n}\left[f(x)g(x)\right] = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

I: Test $n = 1$.

$$\begin{aligned}
LHS &= \frac{d}{dx}(f(x)g(x)) \\
&= f^{(1)}(x)g(x) + f(x)g^{(1)}(x) && \text{product rule} \\
&= \sum_{k=0}^{1} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x) && \binom{1}{0} = \binom{1}{1} = 1 \\
&= RHS
\end{aligned}$$

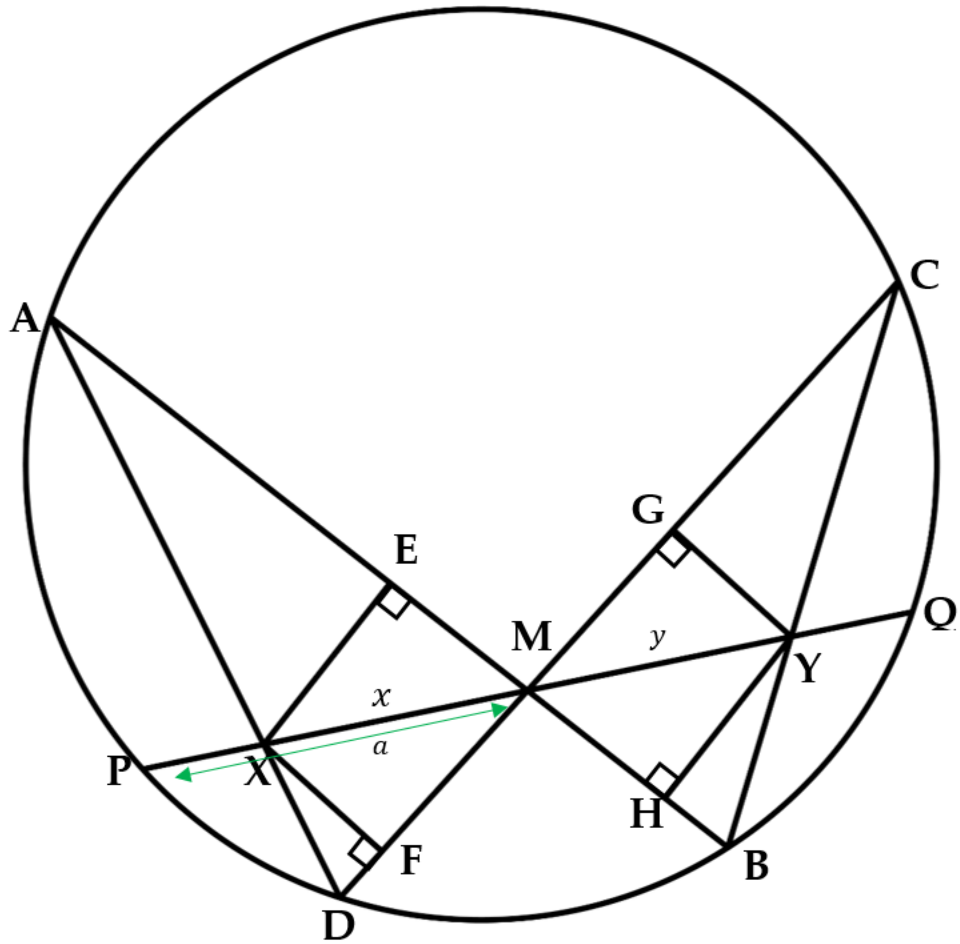
II: Assume true for $n = r$, $r \in \mathbb{Z}^+$. Prove true for $n = r + 1$.
i.e $\dfrac{d^{r+1}}{dx^{r+1}}\left[f(x)g(x)\right] = \displaystyle\sum_{k=0}^{r}\binom{r+1}{k} f^{(r+1-k)}(x)g^{(k)}(x)$.

$$\begin{aligned}
LHS &= \frac{d}{dx}\left(\frac{d^r}{dx^r}(f(x)g(x))\right)\\
&= \frac{d}{dx}\left(\sum_{k=0}^{r}\binom{r}{k} f^{(r-k)}(x)g^{(k)}(x)\right)\\
&= \sum_{k=0}^{r}\binom{r}{k} f^{(r-k+1)}(x)g^{(k)}(x) + \sum_{k=0}^{r}\binom{r}{k} f^{(r-k)}(x)g^{(k+1)}(x) \qquad \text{product rule}\\
&= \sum_{k=0}^{r}\binom{r}{k} f^{(r+1-k)}(x)g^{(k)}(x) + \sum_{k=1}^{r+1}\binom{r}{k-1} f^{(r+1-k)}(x)g^{(k)}(x)\\
&= \binom{r}{0} f^{(r+1)}(x)g(x) + \binom{r}{r} f(x)g^{(r+1)}(x) + \sum_{k=0}^{r}\binom{r+1}{k} f^{(r+1-k)}(x)g^{(k)}(x)\\
&\qquad\qquad \text{since } \binom{r}{k-1} + \binom{r}{k} = \binom{r+1}{k}\\
&= \binom{r+1}{0} f^{(r+1)}(x)g(x) + \binom{r+1}{r+1} f(x)g^{(r+1)}(x) + \sum_{k=0}^{r}\binom{r+1}{k} f^{(r+1-k)}(x)g^{(k)}(x)\\
&= \sum_{k=0}^{r+1}\binom{r+1}{k} f^{(r+1-k)}(x)g^{(k)}(x)\\
&= RHS
\end{aligned}$$

$\therefore$ true for $n = r + 1$
III: Hence the statement is true for all positive integers n

8. **Question 8 (15 marks)** Start a NEW writing booklet.

(a) *The Butterfly Theorem:* Let M be the midpoint of a chord PQ on in a circle. Two more chords AB and CD are drawn through M. AD and BC intersect the chord PQ at points X, Y respectively. Let E, F be the feet of the perpendiculars drawn from X to AM and DM respectively. Similarly, let G, H be the feet of the perpendiculars drawn from Y to CM and BM respectively.



Let $PM = a = QM$, $MX = x$, $MY = y$.

i. $\boxed{1}$ Prove that $\dfrac{x}{y} = \dfrac{EX}{HY}$

$\angle XEM = \angle YHM = 90°$ given

$\angle EMX = \angle HMY$ vertically opposite angles

$\therefore \triangle EMX$ is similar to $\triangle HMY$

$\therefore \dfrac{x}{y} = \dfrac{EX}{HY}$ correpsonding sides of similar triangles in proportion

ii. $\boxed{1}$ Prove that $\dfrac{EX}{GY} = \dfrac{AX}{CY}$

$\angle AEX = \angle CGY$ given

$\angle XAE = \angle GCY$ angles in the same segment

$\therefore \triangle XAE$ is similar to $\triangle GYC$

$\therefore \dfrac{EX}{GY} = \dfrac{AX}{CY}$ correpsonding sides of similar triangles in proportion

iii. $\boxed{\mathbf{3}}$ Hence or otherwise, show that

$$\frac{x^2}{y^2} = \frac{AX \cdot XD}{CY \cdot YB}$$

Similarly to part i and ii, we have $\dfrac{x}{y} = \dfrac{FX}{GY}, \dfrac{FX}{HY} = \dfrac{DX}{BY}$. Hence

$$\begin{aligned}\frac{x^2}{y^2} &= \frac{EX}{HY} \cdot \frac{FX}{GY} \\ &= \frac{AX \cdot DX}{CY \cdot YB}\end{aligned}$$

iv. $\boxed{\mathbf{2}}$ Show that

$$\frac{AX \cdot XD}{CY \cdot YB} = \frac{a^2 - x^2}{a^2 - y^2}$$

$$\begin{aligned}AX \cdot XD &= PX \cdot XQ \qquad \text{products of intercepts of chords} \\ &= (a+x)(a-x) \\ &= a^2 - x^2\end{aligned}$$

Similarly, $CY \cdot YB = a^2 - y^2$, so $\dfrac{AX \cdot XD}{CY \cdot YB} = \dfrac{a^2 - x^2}{a^2 - y^2}$

v. $\boxed{\mathbf{1}}$ Deduce that $x = y$

By parts iii and iv

$$\frac{x^2}{y^2} = \frac{a^2 - x^2}{a^2 - y^2}$$

which rearranges to

$$a^2(x+y)(x-y) = 0$$

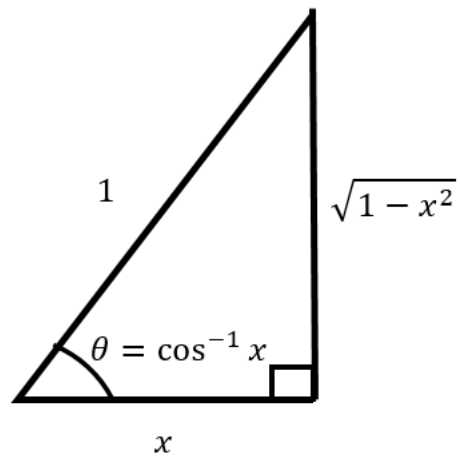
Now $x, y > 0$ so $x \neq -y$ and $a \neq 0$ so $x = y$

(b) *Chebyshev Polynomials:*

i. $\boxed{\mathbf{2}}$ Let $T_n(x) = \cos(n \cos^{-1} x)$,$U_n(x) = \sin(n \cos^{-1} x)$.
Using DeMoivre's Theorem, prove that

$$T_n(x) + i \cdot U_n(x) = \left(x + i\sqrt{1 - x^2}\right)^n$$

Drawing a right triangle:



Clearly $\sin(\cos^{-1} x) = \sqrt{1-x^2}$.

$$\begin{aligned} T_n(x) + i \cdot U_n(x) &= \cos(n\cos^{-1} x) + i\sin(n\cos^{-1} x) \\ &= \left(\cos(\cos^{-1} x) + i\sin(\cos^{-1} x)\right)^n \qquad \text{De Moivre's Theorem} \\ &= \left(x + i\sqrt{1-x^2}\right)^n \end{aligned}$$

ii. $\boxed{\mathbf{2}}$ Hence, prove that:

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]$$

$$\begin{aligned} T_n(x) - i \cdot U_n(x) &= \cos(-n\cos^{-1} x) + i\sin(-n\cos^{-1} x) \qquad \text{sine is odd and cosine is even} \\ &= \left(\cos(-\cos^{-1} x) + i\sin(-\cos^{-1}(x))\right)^n \qquad \text{De Moivre's Theorem} \\ &= \left(x - i\sqrt{1-x^2}\right)^n \end{aligned}$$

Adding the two equations, dividing by 2 and putting i inside the square root gives the result.

iii. $\boxed{\mathbf{3}}$ Consider the series $S = \sum_{n=0}^{\infty} t^n (\cos\theta + i\sin\theta)^n$, where $-1 < t < 1$ and $\theta = \cos^{-1} x$.
Prove that:

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}$$

By the limiting sum of a G.P we have

$$\begin{aligned}
S &= \frac{1}{1 - t\,\text{cis}\theta} \\
&= \frac{1 - t\,\text{cis}(-\theta)}{1 - t(\,\text{cis}\theta + \,\text{cis}(-\theta)) + t^2} \\
&= \frac{1 - t\cos\theta + i\sin\theta}{1 - 2t\cos\theta + t^2} \\
&= \frac{1 - tx + i\sqrt{1-x^2}}{1 - 2tx + t^2}
\end{aligned}$$

Equating real parts and using demoivres gives

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \cos n\theta &= \frac{1 - tx}{1 - 2tx + t^2} \\
\sum_{n=0}^{\infty} T_n(x) t^n &= \frac{1 - tx}{1 - 2tx + t^2}
\end{aligned}$$

9. Question 9 (15 marks) Start a NEW writing booklet.

(a) At lunchtime, 5 senior students and 6 junior students decide to line up at the canteen to purchase Chicken Pides. The canteen has three lines for students to purchase food; 2 (fixed) lines for juniors and 1 for seniors. The seniors, having little respect to the juniors, decide that they will line up wherever they want. The juniors, however, stick to their own lines.

i. $\boxed{1}$ In how many ways can the 5 seniors be distributed amongst the 3 lines?

Using spaces and dividers method we have $\dfrac{7!}{2! \cdot 5!} = 21$ ways

ii. $\boxed{3}$ In how many ways can the 5 seniors and 6 juniors be distributed among the 3 lines?

Ways to distribute the juniors $= \dfrac{7!}{1! \cdot 6!} = 7$ using spaces and dividers. Multiplying these two together we get 147 ways.

(b) Consider the general polynomial cubic equation $P(x) = ax^3 + bx^2 + cx + d = 0$ for $a \neq 0$ and a, b, c, d.
$P(x)$ has roots $\alpha, \beta, \gamma \in \mathbb{R}$.

i. $\boxed{2}$ Show that the following polynomial has roots $\alpha + \dfrac{b}{3a}, \beta + \dfrac{b}{3a}, \gamma + \dfrac{b}{3a}$

$$ay^3 + \left(c - \frac{b^2}{3a}\right) y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0 \qquad (1)$$

Let $y = x + \frac{b}{3a}$. Subbing this into (1) gives:

$$a\left(x + \frac{b}{3a}\right)^3 + \left(c - \frac{b^2}{3a}\right)\left(x + \frac{b}{3a}\right) + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

$$a\left(x^3 + \frac{bx^2}{a} + \frac{xb^2}{3a^2} + \frac{b^3}{27a^3}\right) + \left(cx + \frac{bc}{3a} - \frac{b^2x}{3a} - \frac{3b^3}{27a^2}\right) + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

which simplifies to $ax^3 + bx^2 + cx + d = 0$

ii. $\boxed{\mathbf{1}}$ Let $p = \dfrac{1}{a}\left(c - \dfrac{b^2}{3a}\right), q = \dfrac{1}{a}\left(d + \dfrac{2b^3}{27a^2} - \dfrac{bc}{3a}\right)$. Using the substitution $y = s + t$, where s, t are real variables, show that equation (1) is equivalent to equation (2).

$$s^3 + t^3 + (3st + p)(s + t) + q = 0 \tag{2}$$

Dividing both sides of (1) by a we have,

$$y^3 + py + q = 0$$

Substituting $y = s + t$, we have

$$\begin{aligned} s^3 + t^3 + 3st(s + t) + p(s + t) + q = 0 \\ s^3 + t^3 + (3st + p)(s + t) + q = 0 \end{aligned}$$

iii. $\boxed{\mathbf{1}}$ Suppose that $3st + p = 0$. Show that

$$s^3 + t^3 = -q$$

$$s^3t^3 = -\frac{p^3}{27}$$

$$\begin{aligned} s^3 + t^3 + 0 + q &= 0 \\ s^3 + t^3 &= -q \end{aligned}$$

$$\begin{aligned} 3st + p &= 0 \\ st &= -\frac{p}{3} \\ s^3t^3 &= -\frac{p^3}{27} \end{aligned}$$

iv. $\boxed{\mathbf{2}}$ Assume without loss of generality that $s^3 > t^3$. Prove that:

$$s^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \qquad t^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Form a quadratic using the expressions in iii with roots s^3 and t^3. We have:

$$\begin{aligned} x^2 + q - \frac{p^3}{27} &= 0 \\ x &= \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \\ x &= -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \end{aligned}$$

Now if $s^3 > t^3$, take s^3 as the solution with the positive root and the result is proved.

v. $\boxed{\mathbf{2}}$ Show that that the possible solutions of s are:

$$s = \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } k = 0, 1, 2$$

where ω is the complex cube root of unity with positive imaginary part.

Let $s = Sz$, where $S = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$.

$$\begin{aligned} S^3 z^3 &= -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \\ z^3 &= 1 \end{aligned}$$

But the solutions to $z^3 = 1$ are of course the cubic roots of unity $1, \omega$ and $\overline{\omega} = \omega^{-1} = \omega^3\omega^{-1} = \omega^2$ since solutions to a real polynomial occur in conjugate pairs and the conjugate of complex number with modulus one is equal to its inverse. Hence we have $s = S\omega^k$ for $k = 0, 1, 2$.

vi. $\boxed{\mathbf{3}}$ Show that the roots of equation (1) are

$$x = \frac{b}{3a} + \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{2k} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } k = 0, 1, 2$$

where ω is the complex cube root of unity with positive imaginary part.

Similarly to part v), we have:

$$t = \omega^j \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } j = 0, 1, 2$$

Recall the condition $3st = -p$. Letting $T = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, we have:

$$3\omega^{j+k} ST = -p$$

Now;

$$\begin{aligned} ST &= \sqrt[3]{\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} \\ &= \sqrt[3]{\frac{q^2}{4} - \frac{q^2}{4} - \frac{p^3}{27}} \\ &= -\frac{p}{3} \end{aligned}$$

Since p is real, ST must also be real. Hence for $3\omega^{j+k}ST = -p$ to be true, we require both sides to be real i.e ω^{j+k} must be real. One way to ensure this condition is to set $j+k = 3k$ since $\omega^{3k} = (\omega^3)^k = 1$ for $k = 0, 1, 2$. i.e $j = 2k$. Hence by part ii), the roots of (1) are given by $y = \omega^k S + \omega^{2k} T$. But by part i, $y = x + \dfrac{b}{3a}$. Hence we have

$$x = -\frac{b}{3a} + \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{2k} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad \text{for } k = 0, 1, 2$$