SYDNEY BOYS HIGH SCHOOL

2019

YEAR 11
TERM 4
HSC ASSESSMENT
TASK #1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–13, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 53

Section I – 10 marks (pages 2–6)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 43 marks (pages 7–19)
- Attempt Questions 11-13
- Allow about 1 hour and 15 minutes for this section

Examiner:

External Examiner

NESA Number									

Name: ______________________________ **Class: 11Ma** ___

Section I – Multiple Choice

10 Marks
Attempt questions 1−10
Allow approximately 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1−10

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1 Which one of the following is an exponential form of $\sqrt{3}i-1$?

A. $2e^{-\frac{2i\pi}{3}}$

B. $2e^{-\frac{i\pi}{6}}$

C. $2e^{\frac{2i\pi}{3}}$

D. $2e^{\frac{5i\pi}{6}}$

2 If $z^2=4\left(\cos\dfrac{5\pi}{3}-i\sin\dfrac{5\pi}{3}\right)$, then which of the following is z equal to?

A. $1+i\sqrt{3}$ or $-1-i\sqrt{3}$

B. $1-i\sqrt{3}$ or $-1+i\sqrt{3}$

C. $\sqrt{3}+i$ or $-\sqrt{3}-i$

D. $\sqrt{3}-i$ or $-\sqrt{3}+i$

3 If $z=\dfrac{\cos 4\theta-i\sin 4\theta}{\cos 4\theta+i\sin 4\theta}$, then which of the following is an expression for z?

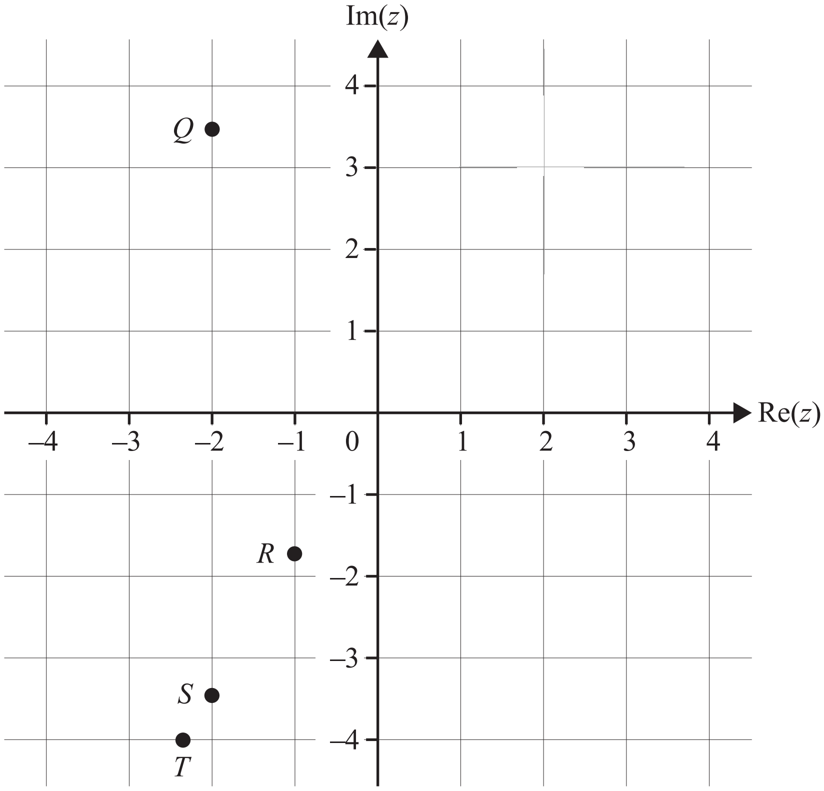
A. $\cos 8\theta+i\sin 8\theta$

B. $\cos 8\theta-i\sin 8\theta$

C. $\sin 8\theta+i\cos 8\theta$

D. $\sin 8\theta-i\cos 8\theta$

4 Which of the points on the Argand diagram below best represents the complex number $4\cos\left(-\frac{2\pi}{3}\right)+4i\sin\left(-\frac{2\pi}{3}\right)$?



A. Q

B. R

C. S

D. T

5 If $z=a+ib$ where $a\neq 0$ and $b\neq 0$.

Which of the following statements is false?

A. $z-\overline{z}=2bi$

B. $|z|^2=|z|\times|\overline{z}|$

C. $|z|+|\overline{z}|=|z+\overline{z}|$

D. $\arg z+\arg\overline{z}=0$

6 For any complex number z, which of the following describes how to find $u = i^5 z$?

A. Reflect z in the Imaginary axis.

B. Reflect z in the Real axis.

C. Rotate z by $\frac{\pi}{2}$ in a clockwise direction about the origin.

D. Rotate z by $\frac{\pi}{2}$ in an anti-clockwise direction about the origin.

7 If $z = x + iy$, where x and y are non-zero real numbers, which of the following is a real number?

A. $\frac{1}{z}$

B. $\frac{1}{\overline{z}}$

C. $\frac{1}{z} - \frac{1}{\overline{z}}$

D. $\frac{1}{z} + \frac{1}{\overline{z}}$

8 The points A and B represent the numbers $z_1 = \alpha e^{\frac{2i\pi}{3}}$ and $z_2 = \beta e^{\frac{\pi i}{6}}$ respectively, where $\alpha, \beta \in \mathbb{R}$ and $0 < \beta < \alpha$ then which of the following is a true statement about the distance AB?

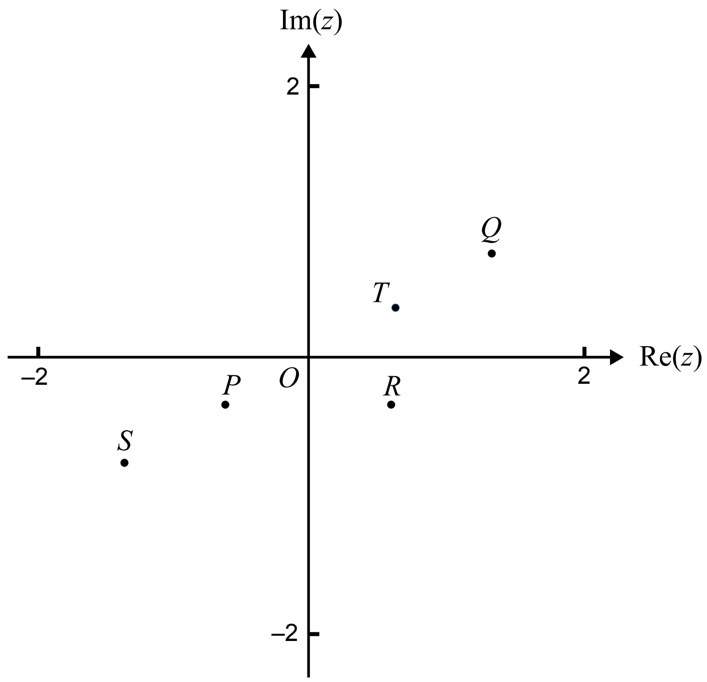
A. $AB = \alpha + \beta$

B. $AB = \alpha - \beta$

C. $AB = \sqrt{\alpha^2 + \beta^2}$

D. $AB = \sqrt{\alpha^2 - \beta^2}$

9 The point R on the Argand diagram below represents a complex number r where $|r| = \dfrac{2}{3}$.



Which of the following points best represents the complex number r^{-1}?

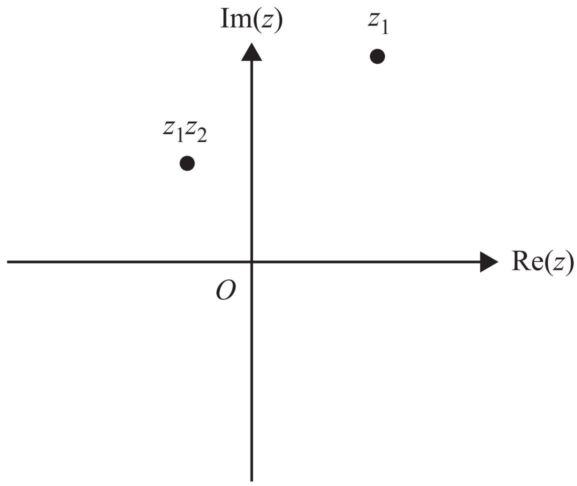
A. P

B. Q

C. S

D. T

10 Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where z_1 and $z_1 z_2$ are shown in the Argand diagram below and θ_1 and θ_2 are acute angles.



Which one of the following statements is necessarily true?

A. $\frac{r_2}{r_1} < 1$

B. $\frac{r_2}{r_1} > 1$

C. $\frac{r_2}{r_1} = 1$

D. The relationship between r_1 and r_2 cannot be determined.

NESA Number									

Q11

Section II

43 marks
Attempt Questions 11–13
Allow about 1 hour and 15 minutes for this section

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

In Questions 11-13, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)

(a) Let $u = 3 - i$ and $w = 4 + 3i$

(i) Find $\operatorname{Im}(uw)$. **1**

(ii) Find $-iw$. **1**

(iii) Evaluate $|u + w|^2$. **2**

(iv) Express $\frac{u}{w}$ in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

Question 11 continued on page 8

Question 11 (continued)

(b) (i) Find the two square roots of $16+30i$. **2**

Leave your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.

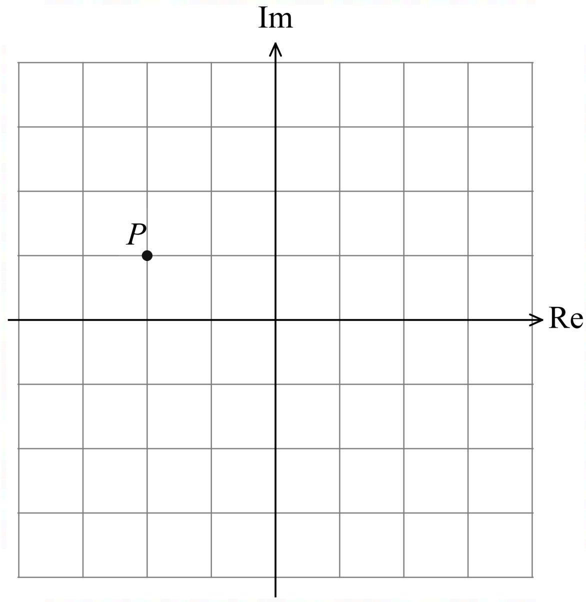
(ii) Hence solve $z^2+(1+i)z-(4+7i)=0$. **3**

Leave your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.

Question 11 continued on page 9

Question 11 (continued)

(c) On the Argand diagram below, P represents the complex number z. **1**
Mark in the point that represents the complex number $\overline{z} - z$.



(d) Let $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $w = -2 + 2i\sqrt{3}$.

(i) Express w in modulus-argument form. **1**

(ii) Hence, or otherwise, express $z^3 w$ in modulus-argument form. **2**

Additional space for Question 11 on page 10

Use this space to re-write any questions for Question 11.

End of Question 11

NESA Number									

Q12

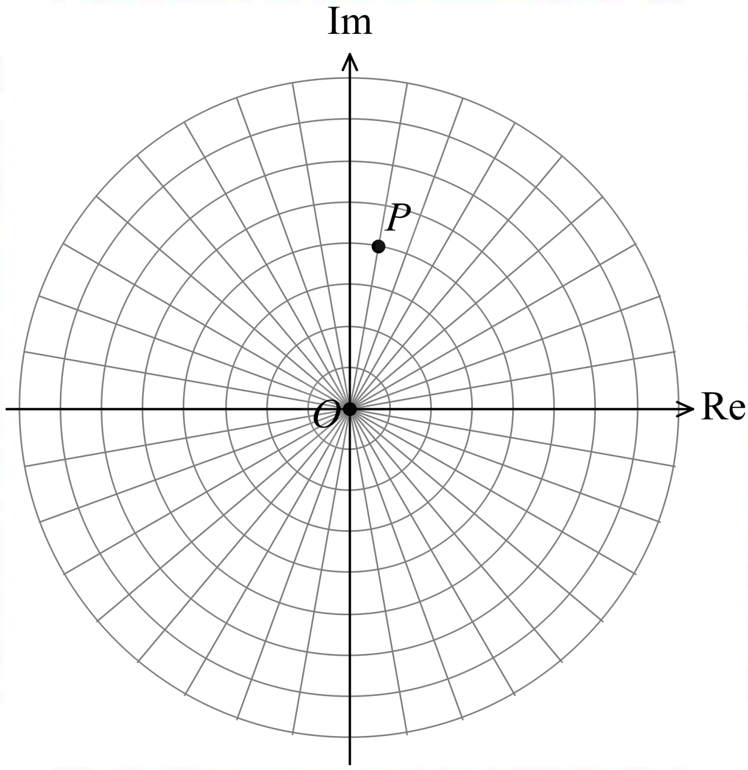
Question 12 (15 Marks)

(a) The complex number z is given by $z = 2\left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$.

It is represented by the point P on the Argand diagram below.

(i) Let Q represent the number z^2 and R represent the number $\frac{1}{z}$. **2**

Mark the points Q and R on the diagram below.



(ii) Now, P' is a point representing $w = 2(\cos\theta + i\sin\theta)$, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. **2**

If the points Q', O and R' are collinear, where Q' represents the point w^2 and R' represents the point $\frac{1}{w}$, find w in the form $a + ib$, where $a, b \in \mathbb{R}$.

Question 11 continued on page 12

Question 12 (continued)

(b) (i) Express $(8+7i)(5+4i)$ in the form $a+ib$. **1**

(ii) Hence, write $(8-7i)(5-4i)$ in the form $a+ib$. **1**

(iii) Hence find the prime factorisation of 12^2+67^2. **2**

(c) Given that $(4-i)^2+(8\theta+i)(3\phi-i)+8i=43$, find the exact real values of θ and ϕ. **3**

Question 12 continued on page 13

Question 12 (continued)

(d) The complex number q is given by $q = \dfrac{e^{i\theta}}{1 - e^{i\theta}}$, where $0 < \theta < 2\pi$. **4**

Show that $\operatorname{Re} q = -\dfrac{1}{2}$ and that $\operatorname{Im} q = \dfrac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$.

Additional space for Question 12 on page 14

Use this space to re-write any questions for Question 12.

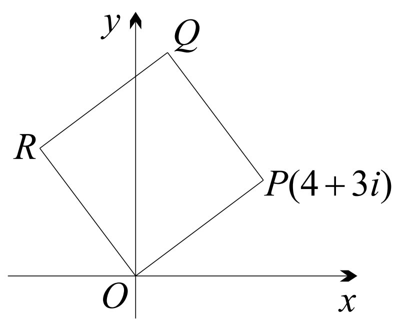
End of Question 12

NESA Number									

Q13

Question 13 (13 Marks)

(a) In the Argand diagram below, *OPQR* is a square with *P* representing the number 4 + 3*i*. Note that $\angle POQ = 45°$ and $OQ = \sqrt{2}OP$.



(i) By considering a rotation of *P* about *O*, find the number corresponding to the point *Q*. Leave your answer in the form in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

(ii) Write down the number corresponding to *R*. **1**
Leave your answer in the form in the form $a + ib$, where $a, b \in \mathbb{R}$.

(iii) *OPQR* is rotated about *O* so that the image of *P*, *P*′, is on the imaginary axis. **1**

Find the complex number(s) corresponding to the point *P*′.

Question 13 continued on page 16

Question 13 (continued)

(b) Let $u = \cos\phi + i\sin\phi$, where $0 < \phi < \pi$.
It is given that the complex number

$$u^2 + \frac{5}{u} - 2$$

is purely imaginary.

(i) Show that $2\cos^2\phi + 5\cos\phi - 3 = 0$. **2**

(ii) Hence find u. **3**

Question 13 continued on page 17

Question 13 (continued)

(c) Let $z = \cos\theta + i\sin\theta$, where $0 < \theta < 2\pi$. **4**

You may assume the following identity. Do NOT prove!

$$\sum_{r=1}^{n}\left(z + z^2 + ... + z^r\right) = \frac{nz}{1-z} - \frac{z^2}{\left(1-z\right)^2}\left(1-z^n\right),\ z \neq 1$$

Deduce that $\displaystyle\sum_{r=1}^{n}\left(\sin\theta + \sin 2\theta + ... + \sin r\theta\right) = \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\dfrac{\theta}{2}}$

[You may make use of the result in Question 12 (d)]

Additional space for Question 13 (c) on page 18

(c) Extra writing space for part (c) **ONLY**.

Additional space for Question 13 on page 19

Use this space to re-write any questions for Question 13.

End of paper

SYDNEY
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2019

YEAR 11 TERM 4
HSC ASSESSMENT TASK 1

Mathematics Extension 2

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1. C
2. C
3. B
4. C
5. C
6. D
7. D
8. C
9. B
10. D

X2 Y11 T4 Task 1 2019 Multiple choice solutions

Mean (out of 10): 7.85

1. $\sqrt{3}\, i - 1$

$= -1 + \sqrt{3}\, i$

$= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= 2 \operatorname{cis} \frac{2\pi}{3}$

$= 2 e^{i\frac{2\pi}{3}}$ C

A	0
B	7
C	91
D	20

2. $z^2 = 4 e^{-i\frac{5\pi}{3}}$

$z = \pm 2 e^{-i\frac{5\pi}{6}}$

$= \pm 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$

$= \pm(-\sqrt{3} - i)$ C

A	5
B	7
C	74
D	32

3. $z = \dfrac{\operatorname{cis}(-4\theta)}{\operatorname{cis} 4\theta}$

$= \dfrac{e^{-i4\theta}}{e^{i4\theta}}$

$= \dfrac{1}{e^{i8\theta}}$

$= e^{-i8\theta}$

$= \cos 8\theta - i \sin 8\theta$ B

A	5
B	107
C	3
D	3

4. $-\frac{2\pi}{3} \Rightarrow R, S$ or T

$4\cos\left(-\frac{2\pi}{3}\right) = 4 \times -\frac{1}{2}$

$= -2$

$\Rightarrow S$ C

A	0
B	2
C	115
D	1

5. A. $z - \bar{z} = a + ib - (a - ib)$

$= 2ib$ ✓

B. $|z|^2 = \left(\sqrt{a^2+b^2}\right)^2$

$= a^2 + b^2$

$|z||\bar{z}| = \sqrt{a^2+b^2} \times \sqrt{a^2+(-b)^2}$

$= a^2 + b^2$ ✓

C. $|z|+|\bar{z}| = \sqrt{a^2+b^2}+\sqrt{a^2+(-b)^2}$
$= 2\sqrt{a^2+b^2}$

$|z+\bar{z}| = |a+ib+a-ib|$
$= |2a|$
$= 2|a|$ ✗

D. Let $z = a+ib = r\operatorname{cis}\theta$
$\therefore \bar{z} = r\operatorname{cis}(-\theta)$

$\therefore \arg z + \arg \bar{z} = \theta + -\theta$
$= 0$ ✓

C

A	2
B	14
C	97
D	5

6. $u = i^5 z$
$= i^4 . iz$
$= iz$
$= \operatorname{cis}\frac{\pi}{2} z$

Rotation through $\frac{\pi}{2}$ in an anticlockwise direction

D

A	2
B	0
C	2
D	114

7. $z = x+iy$

A. $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$
$= \frac{\bar{z}}{|z|^2}$
$= \frac{x-iy}{x^2+y^2}$ ✗

B. $\frac{1}{z} = \frac{z}{z\bar{z}}$
$= \frac{z}{|z|^2}$
$= \frac{x+iy}{x^2+y^2}$ ✗

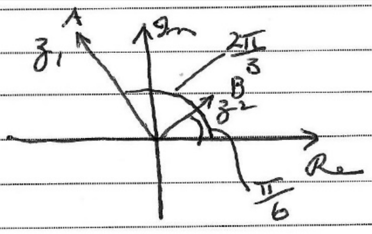
C. $\frac{1}{z} - \frac{1}{\bar{z}} = \frac{\bar{z}-z}{z\bar{z}}$
$= \frac{x-iy-(x+iy)}{x^2+y^2}$
$= \frac{-2iy}{x^2+y^2}$ ✗

D. $\frac{1}{z}+\frac{1}{\bar{z}}$
$= \frac{\bar{z}+z}{z\bar{z}}$
$= \frac{x-iy+x+iy}{x^2+y^2}$
$= \frac{2x}{x^2+y^2}$ ✓

D

A	0
B	0
C	6
D	112

8.



$\angle AOB = \frac{2\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{2}$

$\therefore AB^2 = \alpha^2 + \beta^2$

$\therefore AB = \sqrt{\alpha^2 + \beta^2}$ C

A	2
B	9
C	102
D	5

9. $r^{-1} = \frac{1}{r}$

$= \frac{\bar{r}}{|r|^2}$

$= \frac{\bar{r}}{\left(\frac{4}{9}\right)}$

$= \frac{9}{4}\bar{r}$

If $r = \frac{2}{3}\operatorname{cis}\theta$

$r^{-1} = \frac{9}{4} \times \frac{2}{3}\operatorname{cis}(-\theta)$

$= \frac{3}{2}\operatorname{cis}(-\theta)$

$\Rightarrow$ T or Q

$|r^{-1}| = \frac{3}{2} > \frac{2}{3}$

T would be $\frac{2}{3}\operatorname{cis}(-\theta)$

r^{-1} would be Q B

A	3
B	93
C	11
D	11

10. From the diagram

$|z_1 z_2| < |z_1|$

$\therefore |z_1||z_2| < |z_1|$

$\therefore |z_2| < 1$

$\therefore r_2 < 1$

If $r_1 = 1$, $\frac{r_2}{r_1} < 1$

If $r_1 > 1$, $\frac{r_2}{r_1} < 1$

If $r_1 < 1$, $\frac{r_2}{r_1} > r_2$

e.g. If $r_2 = \frac{2}{3}$ and $r_1 = \frac{3}{4}$

$\frac{r_2}{r_1} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{3}{4}\right)}$

$= \frac{2}{3} \times \frac{4}{3}$

$= \frac{8}{9} < 1$

If $r_2 = \frac{2}{3}$ and $r_1 = \frac{2}{3}$

$\frac{r_2}{r_1} = 1$

e.g. If $r_2 = \frac{2}{3}$ and $r_1 = \frac{3}{4}$

$$\frac{r_2}{r_1} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{3}{4}\right)}$$

$$= \frac{2}{3} \times \frac{4}{3}$$

$$= \frac{8}{9} < 1$$

If $r_2 = \frac{2}{3}$ and $r_1 = \frac{2}{3}$

$$\frac{r_2}{r_1} = 1$$

If $r_2 = \frac{2}{3}$ and $r_1 = \frac{3}{5}$

$$\frac{r_2}{r_1} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{3}{5}\right)} = \frac{2}{3} \times \frac{5}{3}$$

$$= \frac{10}{9}$$

$$> 1$$

$\therefore$ The relationship between r_1 and r_2 cannot be determined. D

A	92
B	9
C	1
D	16

Question 11
(general very well done).

Question 11 (15 Marks)

(a) Let $u = 3 - i$ and $w = 4 + 3i$

(i) Find $\text{Im}(uw)$.

$$uw = (3 - i)(4 + 3i)$$
$$= 12 + 9i - 4i - 3i^2$$
$$= 15 + 5i$$

$$\therefore \text{Im}(uw) = 5$$ $\boxed{1}$

(ii) Find $-iw$.

$$-iw = -i(4 + 3i)$$
$$= -4i - 3i^2$$
$$= 3 - 4i$$ $\boxed{1}$

(iii) Evaluate $|u + w|^2$

$$|u + w|^2 = |3 - i + 4 + 3i|^2$$
$$= |7 + 2i|^2$$
$$= \left(\sqrt{7^2 + 2^2}\right)^2$$
$$= 53$$ $\boxed{2}$

(iv) Express $\dfrac{u}{w}$ in the form $a + ib$, where $a, b \in \mathbb{R}$.

$$\frac{u}{w} = \frac{3 - i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i}$$
$$= \frac{12 - 9i - 4i - 3}{16 - 12i + 12i + 9}$$
$$= \frac{9 - 13i}{25}$$
$$= \frac{9}{25} - \frac{13i}{25}$$ $\boxed{2}$

Question 11 (continued)

(b) (i) Find the two square roots of $16+30i$.
Leave your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.

$$(x+iy)^2 = 16+30i$$
$$(x^2-y^2)+2ixy = 16+30i$$

$$\therefore \quad x^2 - y^2 = 16 \quad ①$$
$$2xy = 30 \Rightarrow xy = 15 \quad ②$$
$$y = \frac{15}{x}$$

Sub $y = \frac{15}{x}$ into ①: $x^2 - \left(\frac{15}{x}\right)^2 = 16$

$$x^4 - 16x^2 - 225 = 0$$
$$(x^2-25)(x^2+9) = 0$$

as x is a real part

$$x = \pm 5$$

when $x = 5$, $y = 3$ and when $x = -5$, $y = -3$

$$\therefore \sqrt{16+30i} = 5+3i \text{ or } -5-3i \quad \boxed{2}$$

(ii) Hence solve $z^2+(1+i)z-(4+7i)=0$.
Leave your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.

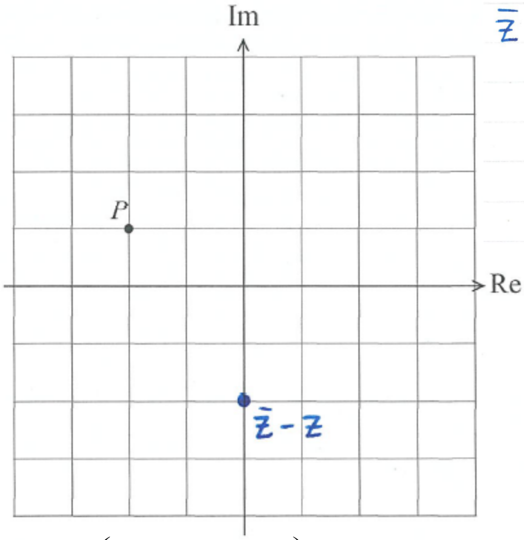
$$z^2 + (1+i)z - (4+7i) = 0$$

$$b^2 - 4ac = (1+i)^2 + 4(1)(4+7i)$$
$$= 1 + 2i - 1 + 16 + 28i$$
$$= 16 + 30i$$

$$\therefore \quad z = \frac{-(1+i) \pm \sqrt{16+30i}}{2}$$
$$= \frac{-1-i \pm (5+3i)}{2}$$
$$= 2+i \quad \text{or} \quad -3-2i \quad \boxed{3}$$

Question 11 (continued)

(c) On the Argand diagram below, P represents the complex number z.
Mark in the point that represents the complex number .



$\bar{z} - z$

(Let $z = x + iy$)

$= (x - iy) - (x + iy)$

$= -2iy$

$\therefore x = 0,\ y = -2$

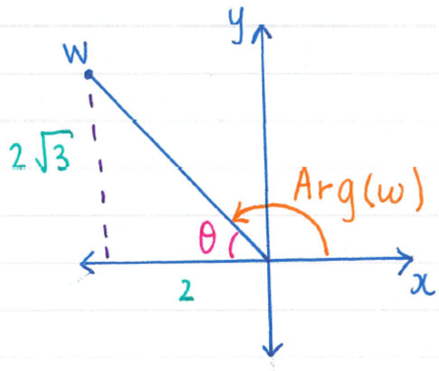
$(0, -2)$

$\boxed{1}$

(d) Let $z = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ and $w = -2 + 2i\sqrt{3}$.

(i) Express w in modulus-argument form.

$w = -2 + 2i\sqrt{3}$



$|w| = \sqrt{(2)^2 + (2\sqrt{3})^2}$

$= 4$

$\theta = \tan^{-1}\left(\dfrac{2\sqrt{3}}{2}\right)$

$= \dfrac{\pi}{3}$

$\therefore \text{Arg}(w) = \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$

$\therefore -2 + 2i\sqrt{3} = 4\left(\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right)$ $\boxed{1}$

(ii) Hence, or otherwise, express z^3w in modulus-argument form.

$z^3w = \left[2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)\right]^3\left(4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)\right)$

$= 2^3\left(\cos\dfrac{3\pi}{3} + i\sin\dfrac{3\pi}{3}\right)4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$= 32\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$

$= 32\left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$ $\boxed{2}$

Note: $-\pi \le \theta \le \pi$.

Question 12 SOLUTIONS

(a) The complex number z is given by $z = 2\left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$.

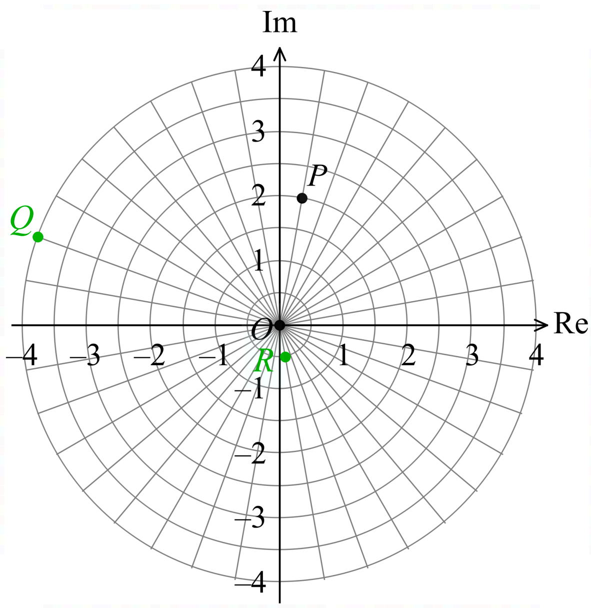
It is represented by the point P on the Argand diagram below.

(i) Let Q represent the number z^2 and R represent the number $\frac{1}{z}$. **2**

Mark the points Q and R on the diagram below.

Given $\arg\left(z^n\right) = n\arg z$ and $|\,z^n\,| = |\,z\,|^n$ then:

Q corresponds to $4\text{cis}\left(\frac{8\pi}{9}\right)$ and R corresponds to $\frac{1}{2}\text{cis}\left(-\frac{4\pi}{9}\right)$



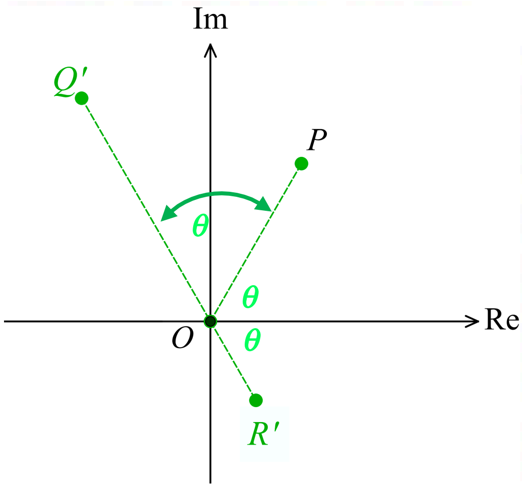
Comment:

This should have been a 'gift' question.
Yet many students did not take any care with working out the scales needed to answer this question correctly.

(a)	(ii)	Now, P' is a point representing $w = 2(\cos\theta + i\sin\theta)$, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. If the points Q', O and R' are collinear, where Q' represents the point w^2 and R' represents the point $\frac{1}{w}$, find w in the form $a + ib$, where $a, b \in \mathbb{R}$.	**2**

$Q' \Leftrightarrow 4\operatorname{cis}2\theta$ and $R' \Leftrightarrow \frac{1}{2}\operatorname{cis}(-\theta)$

As $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ then $\frac{\pi}{2} < 2\theta < \pi$ and $-\frac{\pi}{2} < -\theta < -\frac{\pi}{4}$



As points Q', O and R' are collinear then $2\theta + \theta = \pi$

$\therefore \theta = \frac{1}{3}\pi$

$\therefore w = 2\operatorname{cis}\frac{\pi}{3} = 1 + i\sqrt{3}$

Comment:

Many students are using notation that doesn't make sense e.g. $\arg(Q')$ – no penalty this time.

Too many students ignored the simple geometric solution that one would expect to use given that the diagram was provided.

Students should make sure they make sure that they know in what form the answer needs to be expressed BEFORE starting the question. Some left it as $2\operatorname{cis}\frac{\pi}{3}$ or just $\theta = \frac{\pi}{3}$. That said some found Q' instead of P.

Question 12 SOLUTIONS (CONT'D)

(b)	(i)	Express $(8+7i)(5+4i)$ in the form $a+ib$.	**1**

$$(8+7i)(5+4i) = 40+32i+35i-28 = 12+67i$$

	(ii)	Hence, write $(8-7i)(5-4i)$ in the form $a+ib$.	**1**

$\overline{zw} = \overline{z} \times \overline{w}$ and so $(8-7i)(5-4i) = 12-67i$

	(iii)	Hence find the prime factorisation of 12^2+67^2.	**2**

$z\overline{z} = |z|^2$

Let $w = (8+7i)(5+4i)$.

$\therefore |w|^2 = 12^2+67^2$ [from part (i)]

As $w = (8+7i)\times(5+4i)$ then $|w|^2 = |8+7i|^2 \times |5+4i|^2 = 113\times 41$

$\therefore 12^2+67^2 = 113\times 41$

Clearly a product of primes.

Comment:

Parts (i) and (ii) were well done, all but one or two students showed working.
Students need again to be reminded that just because a question is worth 1 mark doesn't mean that working is not required.
What working is required?
In Questions 11-13, your responses should include relevant mathematical reasoning and/or calculations.

Part (iii) is a "Hence …" and so students who did not show how they got their final answer were not able to score the maximum marks available.

(c)	Given that $(4-i)^2+(8\theta+i)(3\phi-i)+8i=43$, find the exact real values of θ and ϕ.	**3**

$$\therefore 16-8i-1+24\theta\phi-8\theta i+3\phi i+1+8i=43$$

$$\therefore 16+24\theta\phi+i\left(3\phi-8\theta\right)=43$$

By equating real and imaginary parts:

$$\begin{cases} 16+24\theta\phi=43 \\ \quad 24\theta\phi=27 \qquad -(1) \\ \\ 3\phi-8\theta=0 \\ \quad 3\phi=8\theta \qquad -(2) \end{cases}$$

Re-arranging (1): $3\phi\times 8\theta=27$

Substituting (2): $(3\phi)^2=27$

$$\therefore 9\phi^2=27 \Rightarrow \phi^2=3$$

$$\therefore \phi=\pm\sqrt{3}$$

$$\begin{aligned}\therefore \theta &= \frac{3}{8}\times\phi \\ &= \frac{\pm 3\sqrt{3}}{8}\end{aligned}$$

From (2), θ and ϕ have the same sign.

$$\therefore \theta=\frac{3\sqrt{3}}{8}, \phi=\sqrt{3} \text{ or } \theta=-\frac{3\sqrt{3}}{8}, \phi=-\sqrt{3}$$

Comment:

This question was well done by most, though many students didn't realise that θ and ϕ have the same sign.

If students have issues with writing the Greek variables θ and ϕ then they should start the solution with something like:

Let $x=\phi$ and $y=\theta$.

(d) The complex number q is given by $q = \dfrac{e^{i\theta}}{1-e^{i\theta}}$, where $0 < \theta < 2\pi$. **4**

Show that $\operatorname{Re} q = -\dfrac{1}{2}$ and that $\operatorname{Im} q = \dfrac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$.

$$q = \frac{e^{i\theta}}{1-e^{i\theta}} \times \frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}}$$

$$= \frac{e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}$$

$$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}} \qquad \left[w - \overline{w} = 2i \operatorname{Im} w \right]$$

$$= -\frac{1}{2} + \frac{\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}} \qquad \left[\frac{1}{i} = -i \right]$$

$$= -\frac{1}{2} + i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$\therefore \operatorname{Re} q = -\frac{1}{2} \text{ and } \operatorname{Im} q = \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

Alternative proof on next page

(d) $$q=\frac{\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\times\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$$

$$=\frac{(\cos\theta+i\sin\theta)(1-\cos\theta+i\sin\theta)}{(1-\cos\theta)^2+\sin^2\theta}$$

$$=\frac{\cos\theta-\cos^2\theta-\sin^2\theta+i\sin\theta}{1-2\cos\theta+\cos^2\theta+\cos^2\theta}$$

$$=\frac{\cos\theta-1+i\sin\theta}{1-2\cos\theta+\cos^2\theta+\cos^2\theta}$$

$$=\frac{\cos\theta-1+i\sin\theta}{2(1-\cos\theta)}$$

$$=-\frac{1}{2}+i\times\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\left(2\sin^2\frac{\theta}{2}\right)} \qquad \left[\text{Reference sheet: } 1-\cos 2A=2\sin^2 A\right]$$

$$=-\frac{1}{2}+i\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

Comment:

Most students who tried the first solution (or equivalent) were generally more successful.

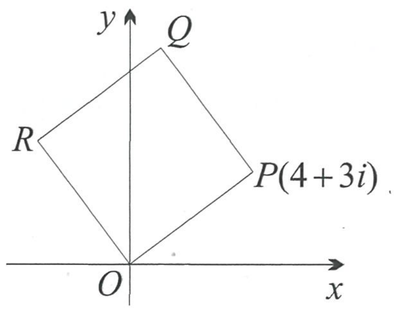
Students who tried to tried to take shortcuts generally were not able to finish the proof without fudging.

Many errors were caused by a poor understanding (and application) of complex conjugates.

NESA Number										Q13	

Question 13 (13 Marks)

(a) In the Argand diagram below, $OPQR$ is a square with P representing the number $4 + 3i$. Note that $\angle POQ = 45°$ and $OQ = \sqrt{2}OP$.



(i) By considering a rotation of P about O, find the number corresponding to the point Q. Leave your answer in the form in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

$(4+3i).\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$= (4+3i).\sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

$= (4+3i)(1+i)$

$= 4 + 4i + 3i - 3$

$= 1 + 7i$

(ii) Write down the number corresponding to R. Leave your answer in the form in the form $a + ib$, where $a, b \in \mathbb{R}$. **1**

$(4+3i).i = -3 + 4i$

(iii) $OPQR$ is rotated about O so that the image of P, P', is on the imaginary axis. **1**

Find the complex number(s) corresponding to the point P'.

$|4+3i| = 5$

$P' = 5i$ or $-5i$

Question 13 continued on page 16

Question 13 (continued)

(b) Let $u = \cos\phi + i\sin\phi$, where $0 < \phi < \pi$.
It is given that the complex number

$$u^2 + \frac{5}{u} - 2$$

is purely imaginary.

(i) Show that $2\cos^2\phi + 5\cos\phi - 3 = 0$. **2**

$Re\left((\cos\phi + i\sin\phi)^2 + 5(\cos\phi + i\sin\phi)^{-1} - 2\right) = 0$

$Re\left(\cos 2\phi + i\sin 2\phi + 5\cos(-\phi) + 5i\sin(-\phi) - 2\right) = 0$

$\cos 2\phi + 5\cos(-\phi) - 2 = 0$

$2\cos^2\phi - 1 + 5\cos\phi - 2 = 0$

$2\cos^2\phi + 5\cos\phi - 3 = 0$

(ii) Hence find u. **3**

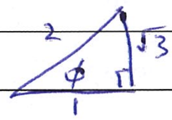
$\times\ -6$
$+\ 5$

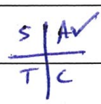
$2\cos^2\phi + 6\cos\phi - \cos\phi - 3 = 0$

$2\cos\phi(\cos\phi + 3) - (\cos\phi + 3) = 0$

$(\cos\phi + 3)(2\cos\phi - 1) = 0$

$\cos\phi = -3$ OR $\cos\phi = \frac{1}{2}$ since $0 < \phi < \pi$





Note: $\phi = \frac{\pi}{3}$

$u = \cos\phi + i\sin\phi$

$u = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

Question 13 continued on page 17

13) c)

$$\frac{z}{1-z} = -\frac{1}{2} + \frac{1}{2}\cot\frac{\theta}{2} i$$

$$= -\frac{1}{2} + \frac{i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$\frac{z^2}{(1-z)^2} = \left[\frac{i\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}\right]^2$$

$$= \frac{-(\cos\theta + i\sin\theta)}{4\sin^2\frac{\theta}{2}}$$

$$\operatorname{Im}\left[\sum_{r=1}^{n}\left(\cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^2 + \dots + (\cos\theta + i\sin\theta)^r\right)\right]$$

$$= \operatorname{Im}\left[n\left(\frac{-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right) - \frac{-(\cos\theta + i\sin\theta)}{4\sin^2\frac{\theta}{2}}\left(1 - (\cos\theta + i\sin\theta)^n\right)\right]$$

$$\operatorname{Im}\left[\sum_{r=1}^{n}\left(\cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \dots + \cos r\theta + i\sin r\theta\right)\right]$$

$$= \operatorname{Im}\left[\frac{n\left(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}} + \frac{(\cos\theta + i\sin\theta)}{4\sin^2\frac{\theta}{2}}\left(1 - \cos n\theta - i\sin n\theta\right)\right]$$

$$\sum_{r=1}^{n}\left(\sin\theta + \sin 2\theta + \dots + \sin r\theta\right)$$

$$= \frac{n\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{\cos\theta(-\sin n\theta)}{4\sin^2\frac{\theta}{2}} + \frac{\sin\theta(1 - \cos n\theta)}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{n.2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\theta - \sin n\theta\cos\theta - \cos n\theta\sin\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{n\sin\theta + \sin\theta - \sin(n\theta + \theta)}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

COMMENTS:

13) a) i) By considering a rotation of P about O (not vector addition).

ii) Generally done well

iii) Shouldn't be concerned with angle of rotation (just the modulus).

b) i) Don't skip lines of working for a show that

ii) Remember we are finding α (not ϕ)

c) Very few students gained full marks.

We are told to let $z = \cos\theta + i\sin\theta$

some students let $z = \sin\theta$.

Equate imaginary parts when in the form $x+iy$.