SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:

2020

YEAR 11
TERM 4
HSC ASSESSMENT
TASK #1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- **Working time – 60 minutes**
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 6–7, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 36

Section I – 5 marks (pages 2–5)

- Attempt Questions 1–5
- Allow about 8 minutes for this section

Section II – 31 marks (pages 6–11)

- Attempt Questions 6–7
- Allow about 52 minutes for this section

Examiner:

JC

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–5.

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1 Given that $z = \dfrac{1+i\sqrt{3}}{1+i}$ which of the following is equal to z^5?

A. $2\sqrt{2}\,e^{\frac{5\pi}{6}i}$

B. $2\sqrt{2}\,e^{\frac{5\pi}{12}i}$

C. $4\sqrt{2}\,e^{\frac{5\pi}{12}i}$

D. $4\sqrt{2}\,e^{\frac{7\pi}{12}i}$

2 Let $z^2 = \sqrt{2}i$, what is the value of z^{8n+4}?

A. -2^{2n+1}

B. $(-2)^{8n+2}$

C. $(-2)^{6n+4}$

D. 2^{2n+1}

3 If $z = x + iy$, where x and y are non-real numbers, which one of the following must be a purely imaginary number?

A. $\operatorname{Im}\overline{z}$

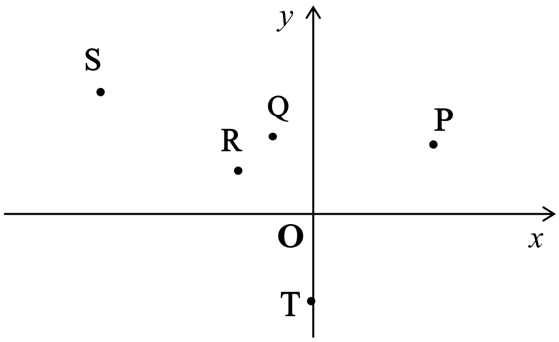
B. iz

C. $i\overline{z}$

D. $z - \overline{z}$

4 In the Argand diagram the point P represents the complex number z.

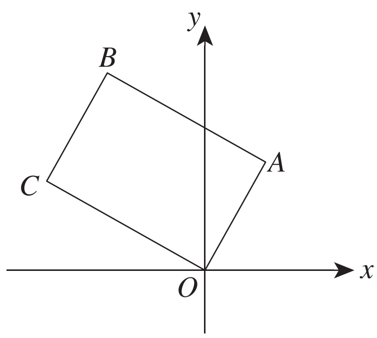
When this number is divided by $\dfrac{i}{-\sqrt{3}-i}$ it gives a new complex number.



Which of the following points best represents the new complex number?

A. Q B. R C. S D. T

5 The Argand diagram shows the rectangle $OABC$ where $OC = 2\,OA$.
Vertex A corresponds to the complex number w.



Which of the following complex numbers corresponds to vertex C?

A. $-2iw$

B. $2iw$

C. $-2w$

D. $2w$

Section II

31 marks
Attempt Questions 6–7
Allow about 52 minutes for this section

In Questions 6–7 your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (17 marks) Start a NEW Answer Booklet

(a) The complex numbers w_1 and w_2 are given by

$$w_1 = 2 - i \text{ and } w_2 = -2 + i$$

(i) Show w_1 and w_2 on a single Argand diagram. **1**

(ii) Find the value of $|\, w_1 \,|$. **1**

(iii) Find the value of $\arg w_2$, giving your answer in radians to two decimal places. **1**

(b) Given that $z_1 = 3 - i$ and $z_2 = 2 + 5i$, express the following in the form of $a + ib$, where a and b are real:

(i) $\left(\overline{z}_1\right)^2$ **1**

(ii) $\dfrac{\overline{z}_1}{z_2}$ **2**

(c) (i) Find the square roots of $16 + 30i$. **2**
Leave your answers in the form $x + iy$, where $x, y \in \mathbb{R}$.

(ii) Hence, solve the equation $z^2 - 2z - (15 + 30i) = 0$. **2**
Leave your answers in the form $x + iy$, where $x, y \in \mathbb{R}$.

Question 6 continues on page 5

Question 6 (continued)

(d) Given that $z = x + iy$, solve $z + 3i\overline{z} = -1 + 13i$. **2**
Leave any answers in the form $x + iy$, where $x, y \in \mathbb{R}$.

(e) Given that $w = \lambda - 3i$, where $\lambda \in \mathbb{R}$, and $\arg(4 - 5i + 3w) = -\dfrac{\pi}{2}$, find the value of λ. **2**

(f) It is given that $z = 1 + 2i$. **3**

The real numbers p and q are such that $pz^2 + \dfrac{q}{z^3}$ is real.

Find, in terms of p, the value of q and the value of $pz^2 + \dfrac{q}{z^3}$.

End of Question 6

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Question 7 (14 marks) Start a NEW Answer Booklet

(a) What are the integer values of n for which $\left(3-\sqrt{3}\,i\right)^n$ is purely imaginary? **2**

Leave your answer in the form $n = ak + b$, where a, k, and b are integers.

(b) The series C and S are defined for $0<\theta<\pi$ by

$$C = 1+\cos\theta+\cos 2\theta+\cos 3\theta+\cos 4\theta+\cos 5\theta$$
$$S = \quad \sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta+\sin 5\theta$$

(i) Show that $C+iS=\dfrac{e^{3i\theta}-e^{-3i\theta}}{e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta}}e^{\frac{5}{2}i\theta}$. **3**

(ii) Deduce that $C=\operatorname{cosec}\dfrac{1}{2}\theta\cos\dfrac{5}{2}\theta\sin 3\theta$ and write down the corresponding expression for S. **3**

(iii) Hence find the values of θ for which $C = S$. **2**

(c) Let $z_1, z_2, \ldots, z_n$ be arbitrary complex numbers.

(i) Prove that $z_1\overline{z}_1+z_2\overline{z}_2 \ge z_1z_2+\overline{z_1z_2}$. **2**

(ii) Hence, otherwise, show that **2**

$$|\,z_1\,|^2+|\,z_2\,|^2+\ldots+|\,z_n\,|^2 \ge \operatorname{Re}(z_1z_2+z_2z_3+\ldots+z_{n-1}z_n+z_nz_1)$$

End of paper

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Sample Solutions

NOTE: Some of you may be disappointed with your mark.
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come by heeding the advice and comments in the solutions and through further revision and practice.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.
Just because you have shown 'working' does not justify that your solution is worth any marks.

Quick MC Answers

1	C
2	A
3	D
4	C
5	B

2020 Y11 T4 X2 Multiple Choice Solutions

Mean (out of 5): 4.06

1. $z = \dfrac{1+i\sqrt{3}}{1+i}$

$= \dfrac{2e^{i\frac{\pi}{3}}}{\sqrt{2}\,e^{i\frac{\pi}{4}}}$

$= \sqrt{2}\,e^{i\frac{\pi}{12}}$

$\therefore z^5 = (\sqrt{2})^5\, e^{i\frac{5\pi}{12}}$

$= 4\sqrt{2}\,e^{i\frac{5\pi}{12}}$ C

A	0
B	9
C	110
D	2

2. $z^2 = \sqrt{2}\,i$

$\therefore z^{8n+4} = (z^2)^{4n+2}$

$= (z^2)^{4n}\cdot(z^2)^2$

$= \left((\sqrt{2}i)^4\right)^n \cdot -2$

$= (4.1)^n \cdot -2$

$= -2 \cdot 4^n$

$= -2 \cdot 2^{2n}$

$= -2^{2n+1}$ A

A	96
B	10
C	6
D	9

3. $z = x+iy = A+iB$, A, B real

A. $\bar{z} = A - iB$

$\text{Im}\,\bar{z} = B$ Real number ✗

B. $iz = iA - B$ only imaginary if $B=0$ ✗

C. $i\bar{z} = i(A-iB)$

$= iA + B$ only imaginary if $B=0$ ✗

D. $z - \bar{z} = A+iB-(A-iB)$

$= 2iB$ purely imaginary ✓

D

A	25
B	1
C	3
D	92

4. $\dfrac{i}{-\sqrt{3}-i} = \dfrac{e^{i\frac{\pi}{2}}}{2e^{-i5\pi/6}}$

$= \frac{1}{2}e^{i\frac{8\pi}{6}}$

$= \frac{1}{2}e^{i\frac{4\pi}{3}}$

z divided by $\frac{1}{2}e^{i\frac{4\pi}{3}}$

$\Rightarrow$ z multiplied by $2e^{-i\frac{4\pi}{3}}$

$\Rightarrow$ z multiplied by $2e^{i\frac{2\pi}{3}}$

$\Rightarrow$ Rotate by $\frac{2\pi}{3}$ radians and increase magnitude by 2

$\Rightarrow$ S C

A	5
B	14
C	74
D	28

5. OA → OC

Rotate by $\frac{\pi}{2}$, increase magnitude by 2

∴ Multiply by $2i$

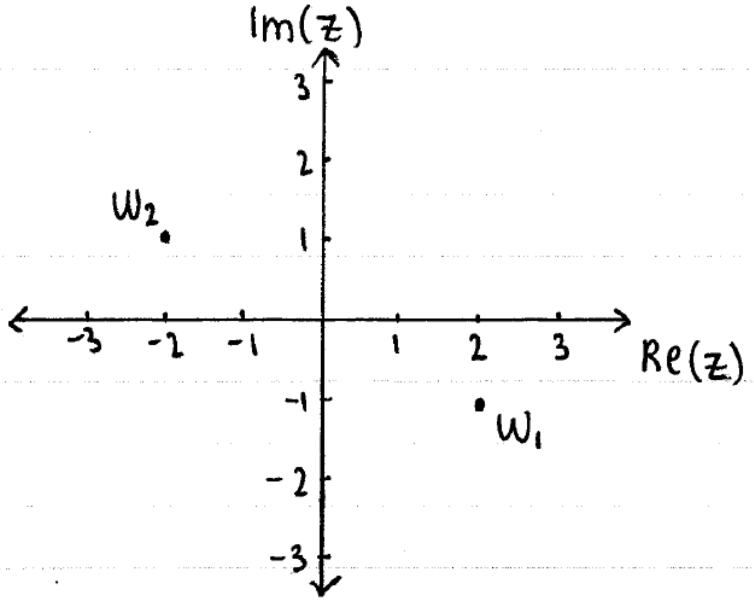
⇒ $2iw$ B

A	2
B	119
C	0
D	0

Question 6

(a) $w_1 = 2 - i$ and $w_2 = -2 + i$

(i)



[1]

(ii) $|w_1| = \sqrt{2^2 + (-1)^2}$
$= \sqrt{5}$ [1]

(iii) $\arg(w_2) = \pi - \theta$, where $\tan\theta = \frac{1}{2}$

$= \pi - \tan^{-1}\left(\frac{1}{2}\right)$

$= 2.677945045$

$= 2.68$ radians. (2.d.p). [1]

Marker's comments

- Parts (i) and (ii), generally done well.
- Some candidates had the right argument, but the wrong quadrant.

(b) $z_1 = 3 - i$ and $z_2 = 2 + 5i$

(i)
$$\begin{aligned}(\bar{z}_1)^2 &= (\overline{3 - i})^2 \\ &= (3+i)^2 \\ &= 9 + 6i + i^2 \\ &= 9 + 6i - 1 \\ &= 8 + 6i \end{aligned}$$
[1]

(ii)
$$\begin{aligned}\frac{\bar{z}_1}{z_2} &= \frac{\overline{3 - i}}{2 + 5i} \\ &= \frac{3+i}{2+5i} \times \frac{2-5i}{2-5i} \\ &= \frac{6 - 15i + 2i + 5}{2^2 + 5^2} \\ &= \frac{11}{29} - \frac{13i}{29}\end{aligned}$$
[2]

Marker's comments

- Part (i) generally done well.
- Although, some candidates forgot the realise the complex number in part (ii).

c) i) Let $x + iy = \sqrt{16+30i}$

$$(x+iy)^2 = 16+30i$$
$$x^2 - y^2 + 2xyi = 16 + 30i$$

$$x^2 - y^2 = 16 \quad ①$$

$$2xyi = 30i$$
$$y = \frac{15}{x} \quad ②$$

sub ② into ①: $x^2 - \left(\frac{15}{x}\right)^2 = 16$

$$x^4 - 16x^2 - 225 = 0$$
$$(x^2+9)(x^2-25) = 0$$

$$\therefore \ x^2 = 25$$
$$x = \pm 5$$
(since $x \in \mathbb{R}$.)

$$\therefore \ y = \pm\frac{15}{5}$$
$$= \pm 3$$

$\therefore \ x + iy = \pm(5+3i)$ [2]

Marker's comments

- Generally done well.

(ii) $z^2 - 2z - (15 + 30i) = 0$

$$\therefore\ z = \frac{2 \pm \sqrt{(-2)^2 + 4 \times 1 \times (15 + 30i)}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4 + 60 + 120i}}{2}$$

$$= \frac{2 \pm 2\sqrt{16 + 30i}}{2}$$

$$= 1 \pm \sqrt{16 + 30i}$$

$$= 1 \pm (5 \pm 3i)$$

$$= 6 + 3i,\ -4 - 3i$$

[2]

Marker's comments

- Generally done well.

(d) $z + 3i\bar{z} = -1 + 13i$

$x + iy + 3i(x - iy) = -1 + 13i$

$x + 3y + i(y + 3x) = -1 + 13i$

Equating Real and Imaginary parts:

$x + 3y = -1$ (1)

$3x + y = 13$ (2)

From (2): $y = 13 - 3x$ (3)

Sub (3) into (1): $x + 3(13 - 3x) = -1$

$x + 39 - 9x = -1$

$-8x = -40$

$x = 5$

Sub $x = 5$ into (3): $y = 13 - 3(5)$

$= -2$

$\therefore z = 5 - 2i$ [2]

Marker's comments

- Generally done well.

(e) $4 - 5i + 3w = 4 - 5i + 3(\lambda - 3i)$
$= 4 + 3\lambda - 14i$

since $\text{Arg}(4 + 3\lambda - 14i) = -\frac{\pi}{2}$
$\therefore 4 + 3\lambda - 14i$ (lies on the imaginary axis)

$\therefore 4 + 3\lambda = 0$
$\lambda = -\frac{4}{3}$ [2]

<u>Marker's comments</u>

- Generally done well.
- Although, some candidates didn't equate the real part equalling zero.

(f) $z = 1 + 2i \Rightarrow z^2 = 1 + 4i - 4$
$= -3 + 4i$

$\Rightarrow z^3 = -3 + 4i - 6i - 8$
$= -11 - 2i$

$pz^2 + \frac{q}{z^3}$ is real

$pz^2 + \frac{q}{z^3} = p(-3+4i) + \frac{q}{(-11-2i)}$
$= -3p + 4pi + \frac{q(-11+2i)}{125}$
$= -3p - \frac{11q}{125}$ (as the number is real)

Since the number is real:

$4p + \frac{2q}{125} = 0$

$\frac{2q}{125} = -4p$

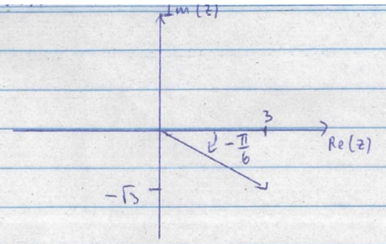
$q = -250p$

$\therefore pz^2 - \frac{q}{z^3} = -3p - \frac{11 \times -250p}{125}$
$= -3p + 22p$
$= 19p$ [3]

Marker's comments

- Some candidates forgot the completely answer the question and didn't find the actual values in terms of p.

Question 7

Solution	**Marks**	**Marker's comments**
a)  $\lvert 3-\sqrt{3}i\rvert = \sqrt{3^2+(-\sqrt{3})^2}$ $= \sqrt{12}$ $= 2\sqrt{3}$ $\therefore 3-\sqrt{3}i = 2\sqrt{3}e^{-\frac{\pi i}{6}}$ $(3-\sqrt{3}i)^n = (2\sqrt{3}e^{-\frac{\pi i}{6}})^n$ $= (2\sqrt{3})^n e^{-\frac{\pi n i}{6}}$ When purely imaginary $-\frac{\pi n}{6} = k\pi + \frac{\pi}{2}$ where $k \in \mathbb{Z}$ $-\frac{n}{6} = k + \frac{1}{2}$ $-n = 6k+3$ $n = -6k-3$ * Also $n=-6k+3$, or $n = 6k-3$ or $n = 6k+3$ (will be acceptable)	**1 mark** for correct mod arg form or exponential form. **1 mark** for correct answer.	- Candidates whom had the right argument, generally had the right answer. - Some general form answers from candidates were incorrect due to not including all the required solutions or had excess amount of incorrect solutions. Marks were deducted.
b) i) $LHS = C + iS$ $= 1 + \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta$ $+ \cos 3\theta + i\sin 3\theta + \cos 4\theta + i\sin 4\theta$ $+ \cos 5\theta + i\sin 5\theta$ $= 1 + e^{i\theta} + e^{i2\theta} + e^{3i\theta} + e^{4i\theta} + e^{5i\theta}$ which is a geometric series where $a = 1 \quad r = e^{i\theta} \quad n = 6$ $\therefore S_6 = \frac{a(r^6-1)}{r-1}$ $LHS = \frac{(e^{i\theta})^6 - 1}{e^{i\theta}-1}$ $= \frac{e^{6i\theta}-1}{e^{i\theta}-1}$ $= \frac{e^{3i\theta}(e^{3i\theta}-e^{-3i\theta})}{e^{i\theta/2}(e^{i\theta/2}-e^{-i\theta/2})}$ $= \frac{e^{\frac{5}{2}i\theta}(e^{3i\theta}-e^{-3i\theta})}{e^{i\theta/2}-e^{-i\theta/2}}$ $= RHS$	**1 mark** for establishing that $C + iS = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta} + e^{5i\theta}$ **1 mark** for establishing a geometric series. **1 mark** for showing ALL required steps to simplify to the correct result.	- Candidates who manipulated RHS were generally not able to achieve the LHS successfully. - Candidates should avoid working on both sides to reach a common result, especially this example as it is a *SHOW THAT* question. Start on one side and work your way towards the other side.

<table>
<tr>
<td>b) ii)

From (i)
$C + iS = \frac{e^{3i\theta} - e^{-3i\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \times e^{\frac{5}{2}i\theta}$
$= \frac{\cos 3\theta + i\sin 3\theta - (\cos(-3\theta) + i\sin(-3\theta))}{\cos\frac{\theta}{2} + i\sin(\frac{\theta}{2}) - (\cos(-\frac{\theta}{2}) + i\sin(-\frac{\theta}{2}))} \times \left(\cos\frac{5\theta}{2} + i\sin\frac{5\theta}{2}\right)$
$= \frac{2i\sin 3\theta}{2i\sin\frac{\theta}{2}}\left(\cos\frac{5\theta}{2} + i\sin\frac{5\theta}{2}\right)$
$= \sin 3\theta \operatorname{cosec}\frac{\theta}{2} \cos\frac{5\theta}{2} + i\sin 3\theta \operatorname{cosec}\frac{\theta}{2}\sin\frac{5\theta}{2}$

Equate real part and imaginary part
$C = \sin 3\theta \operatorname{cosec}\frac{\theta}{2} \cos\frac{5\theta}{2}$
$S = \sin 3\theta \cos\frac{\theta}{2} \sin\frac{5\theta}{2}$.</td>
<td>1 mark
Simplifying the result of b) i) showing all necessary steps.

1 mark for the result
$\frac{\sin 3\theta}{\sin\frac{\theta}{2}}\left(\cos\frac{5\theta}{2} + i\sin\frac{5\theta}{2}\right)$

1 mark for correct result of S.</td>
<td>- Generally, well done by most candidates.
- Important that candidates show all necessary working.</td>
</tr>
<tr>
<td>b) iii)

(iii) $\sin 3\theta \operatorname{cosec}\frac{\theta}{2} \cos\frac{5\theta}{2} = \sin 3\theta \operatorname{cosec}\frac{\theta}{2}\sin\frac{5\theta}{2}$
$\sin 3\theta \operatorname{cosec}\frac{\theta}{2}\left(\cos\frac{5\theta}{2} - \sin\frac{5\theta}{2}\right) = 0$
For $0 < \theta < \pi$
$\sin 3\theta = 0 \quad 0 < 3\theta < 3\pi$
$3\theta = \pi, 2\pi$
$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$
$\operatorname{cosec}\frac{\theta}{2} \neq 0$
$\cos\frac{5\theta}{2} - \sin\frac{5\theta}{2} = 0 \quad 0 < \frac{5\theta}{2} < \frac{5\pi}{2}$
$\tan\frac{5\theta}{2} = 1$
$\frac{5\theta}{2} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$
$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$
$\therefore \theta = \frac{\pi}{10}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{9\pi}{10}$</td>
<td>1 mark for correct set of solution when $\sin 3\theta = 0$.

1 mark for correct set of solution when $\tan\frac{5\theta}{2} = 1$.</td>
<td>- Done poorly by many candidates.
- This was mainly due to 2 reasons:
1) Cancelling a set of solution i.e. $\sin 3\theta = 0$.
2) Ignore the fact this is not a general solution problem i.e. the question states clearly that $0 < \theta < \pi$. Note that 0 is also not included in the boundary.</td>
</tr>
</table>

c) i)	(c) i) $z_1\bar{z}_1 + z_2\bar{z}_2 - z_1z_2 - \overline{z_1z_2}$ $= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1z_2 - \bar{z}_1\bar{z}_2$ $= \bar{z}_1(z_1 - \bar{z}_2) + z_2(\bar{z}_2 - z_1)$ $= (\bar{z}_1 - z_2)(z_1 - \bar{z}_2)$ let $z_1 = x+iy$ $z_2 = a+ib$ where $a,b,x,y \in \mathbb{R}$ $z_1\bar{z}_1 + z_2\bar{z}_2 - z_1z_2 - \bar{z}_1\bar{z}_2$ $= (x+iy)(x-iy) + (a+ib)(a-ib)$ $- (x+iy)(a+ib) - (x-iy)(a-ib)$ $= x^2+y^2+a^2+b^2 - (ax-by+i(ay+bx))$ $- (ax-by-i(ay+bx))$ $= x^2+y^2+a^2+b^2 - 2(ax-by)$ $= x^2-2ax+a^2+y^2+2by+b^2$ $= (x-a)^2+(y+b)^2$ ≥ 0 $\therefore z_1\bar{z} + z_2\bar{z}_2 \geq z_1z_2 + \overline{z_1z_2}$	**1 mark** for establishing correct result on both sides of inequality. **1 mark** for using "sum of squares" to explain that one side is positive to prove the inequality is true.	- Candidates who tried to use the triangle inequality were least successful in this question. - Candidates must give strong reasoning why the inequality is true i.e. establish a sum of squares.
c) ii)	(ii) Noting that $z_1z_2 + \overline{z_1z_2} = 2\mathrm{Re}(z_1z_2)$ $\therefore \lvert z_1\rvert^2 + \lvert z_2\rvert^2 \geq 2\mathrm{Re}(z_1z_2)$ More generally $\lvert z_p\rvert^2 + \lvert z_q\rvert^2 \geq 2\mathrm{Re}(z_pz_q)$ $LHS = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 + \cdots + \lvert z_n\rvert^2$ $= \frac{1}{2}[2\lvert z_1\rvert^2 + 2\lvert z_2\rvert^2 + 2\lvert z_3\rvert^2 + \cdots + 2\lvert z_n\rvert^2]$ $= \frac{1}{2}(\lvert z_1\rvert^2 + \lvert z_2\rvert^2) + \frac{1}{2}(\lvert z_2\rvert^2 + \lvert z_3\rvert^2)$ $+ \frac{1}{2}(\lvert z_3\rvert^2 + \lvert z_4\rvert^2) + \cdots +$ $\frac{1}{2}(\lvert z_{n-1}\rvert^2 + \lvert z_n\rvert^2) +$ $\frac{1}{2}(\lvert z_n\rvert^2 + \lvert z_1\rvert^2)$ $\geq \frac{1}{2}\times 2\mathrm{Re}(z_1z_2) + \frac{1}{2}\times 2\mathrm{Re}(z_2z_3)$ $+ \frac{1}{2}\times 2\mathrm{Re}(z_3z_4) + \cdots + \frac{1}{2}2\mathrm{Re}(z_{n-1}z_n)$ $+ \frac{1}{2}\times 2\mathrm{Re}(z_nz_1)$ (From (i)) $= \mathrm{Re}(z_1z_2 + z_2z_3 + \cdots + z_{n-1}z_n$ $+ z_nz_1)$ $= RHS$. $\therefore LHS \geq RHS$.	**1 mark** for using the result in i) and establishing $\lvert z_p\rvert^2 + \lvert z_q\rvert^2 \geq 2\mathrm{Re}(z_pz_q)$ **1 mark** for showing relevant working to establish the correct result.	- This was done poorly by many candidates as they either did not attempt it or wrote irrelevant results and try to fudge their answer.