SYDNEY
BOYS
HIGH
SCHOOL

2020

YEAR 12

TERM 2 HSC Task #2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 10–12, show ALL relevant mathematical reasoning and/or calculations
  The amount of working needed is commensurate with the marks available.

Total Marks: 50

Section I – 9 marks (pages 2–4)
- Attempt Questions 1–9
- Allow about 15 minutes for this section

Section II – 41 marks (pages 5–9)
- Attempt Questions 10–12
- Allow about 1 hour and 15 minutes for this section

Examiner:

JJ & BK

Name: ____________________ Class: 12 Ma _

NESA Number									

Section I

9 marks
Attempt Questions 1–9
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–9.

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1 Which of the following shows $-2+2\sqrt{3}i$ expressed in modulus-argument form?

A. $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

B. $4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

C. $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

D. $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

2 One of the roots of $z^3+bz^2+cz=0$ is $3-2i$, where b and c are real numbers. What are the values of b and c respectively?

A. $b=6, c=13$

B. $b=-6, c=13$

C. $b=3, c=-2$

D. $b=-3, c=2$

3 Let $z=3-4i$ and $w=\sqrt{3}+i$. What is the value of $\frac{z}{w}$?

A. $\frac{3\sqrt{3}+4}{4}+\frac{\left(-4\sqrt{3}-3\right)i}{4}$

B. $\frac{3\sqrt{3}-4}{4}+\frac{\left(-4\sqrt{3}-3\right)i}{4}$

C. $\frac{3\sqrt{3}+4}{2}+\frac{\left(-4\sqrt{3}-3\right)i}{2}$

D. $\frac{3\sqrt{3}-4}{2}+\frac{\left(-4\sqrt{3}-3\right)i}{2}$

4 Which of the following complex numbers equals $\left(\sqrt{3}+i\right)^4$?

A. $-2+\dfrac{2}{\sqrt{3}}i$

B. $-8+\dfrac{8}{\sqrt{3}}i$

C. $-2+2\sqrt{3}i$

D. $-8+8\sqrt{3}i$

5 What are the roots of the equation $x^4+x^2+6x+4=0$?

A. $-1,\ 1+\sqrt{5}i$ and $1-\sqrt{5}i$

B. $-1,\ 1+\sqrt{3}i$ and $1-\sqrt{3}i$

C. $1,\ 1+\sqrt{5}i$ and $1-\sqrt{5}i$

D. $1,\ 1+\sqrt{3}i$ and $1-\sqrt{3}i$

6 Let $z=a+bi$ where a and b are non-zero real numbers.

If $z+\dfrac{1}{z}$ is a real number, which of the following must be true?

A. $\operatorname{Arg}(z)=\dfrac{\pi}{4}$

B. $a=b$

C. $|z|=1$

D. $z^2=1$

7 Which of the following relations has a graph that passes through the point $1+2i$ in the complex plane?

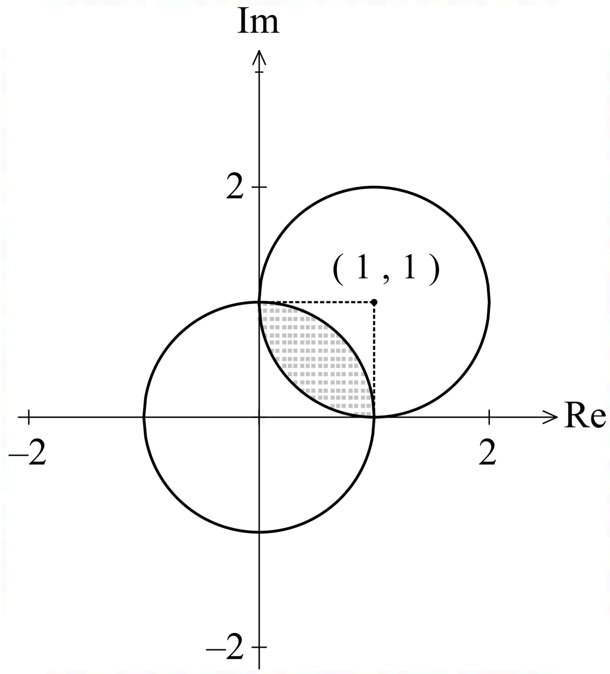
A. $z\bar{z}=5$

B. $|z-1|=|z-2i|$

C. $|z-i|=|z+2|$

D. $|z|+|\bar{z}|=2$

8 Consider the Argand diagram below.
Which of the following inequalities define the shaded area?



A. $|z| \le 1$ and $|z-1+i| \ge 1$

B. $|z| \le 1$ and $|z-1-i| \ge 1$

C. $|z| \le 1$ and $|z-1+i| \le 1$

D. $|z| \le 1$ and $|z-1-i| \le 1$

9 Which of the following relations does NOT represent a circle in the Argand diagram?

A. $z\overline{z} = 4$

B. $|z+3i| = 2|z-i|$

C. $|z-2i| = |z-2|$

D. $|z| + 2|\overline{z}| = 4$

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Section II

41 marks
Attempt Questions 10-12
Allow about 1 hour and 15 minutes for this section.

In Questions 10-12, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 10 (14 marks)   Use a SEPARATE writing booklet

(a) If $z = 4 - 3i$, evaluate

(i) $z\overline{z}$. **1**

(ii) $\dfrac{1}{z}$, leaving your answer in the form $a + ib$. **1**

(iii) $\text{Im}(z)$. **1**

(b) Consider the polynomial $P(z) = z^2 + kz + 2 - i$.

Find the complex number, k, given that i is a zero of $P(z)$. **2**

(c) Find all pairs of integers x and y that satisfy $(x + iy)^2 = 33 + 56i$. **2**

(d) Evaluate $\dfrac{\left(\sqrt{3} - i\right)^{10}}{\left(1 + i\right)^{12}}$. **3**

Leave your answer in the form $a + ib$ where $a, b \in \mathbb{R}$.

Question 10 continues on page 7

Question 10 (continued)

(e) If ω is a complex cube root of unity, evaluate $\left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)$. **2**

(f) Given $\dfrac{1-3i}{1+2i}=re^{i\theta}$ where $r>0$ and $-\pi<\theta\leq\pi$, find the values of r and θ. **2**

End of Question 10

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) (i) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, write down two different expressions for $e^{5i\theta}$. **2**

(ii) Hence write down an expression for $\cos 5\theta$, in terms of powers of cosines only. **3**

(b) An n-gon is a polygon with n sides. **3**

Show, using induction, that any convex n-gon, $n > 3$, can be subdivided into exactly $(n - 2)$ triangles so that every triangle vertex is one of the original vertices of the n-gon.

(c) (i) Show that the roots of $z^6 - z^3 + 1 = 0$ are also roots of $z^9 + 1 = 0$. **1**

(ii) Show that **3**

$$z^6 - z^3 + 1 = \left(z^2 - 2z\cos\frac{\pi}{9} + 1\right)\left(z^2 + 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 + 2z\cos\frac{2\pi}{9} + 1\right).$$

(iii) By considering the coefficients of z^2, or otherwise, show that **2**

$$\cos\frac{\pi}{9}\cos\frac{5\pi}{9} + \cos\frac{5\pi}{9}\cos\frac{7\pi}{9} + \cos\frac{7\pi}{9}\cos\frac{\pi}{9} = -\frac{3}{4}$$

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Question 12 (13 marks) Use a SEPARATE writing booklet

(a) Find and *describe* the Cartesian equation of the locus represented by **2**

$$\left(z+\overline{z}\right)^2 = i\left(z-\overline{z}\right)$$

(b) (i) Show that $\dfrac{\sin\left(\dfrac{3x}{2}\right)-\sin\left(\dfrac{x}{2}\right)}{2\sin\left(\dfrac{x}{2}\right)} = \cos x$ **2**

(ii) Hence, prove by induction that, for all $n \in \mathbb{Z}^+$, the following statement is true. **3**

$$\cos x + \cos 2x + \cos 3x + \ldots + \cos nx = \frac{\sin\left(\dfrac{2n+1}{2}x\right)}{2\sin\left(\dfrac{x}{2}\right)} - \frac{1}{2}$$

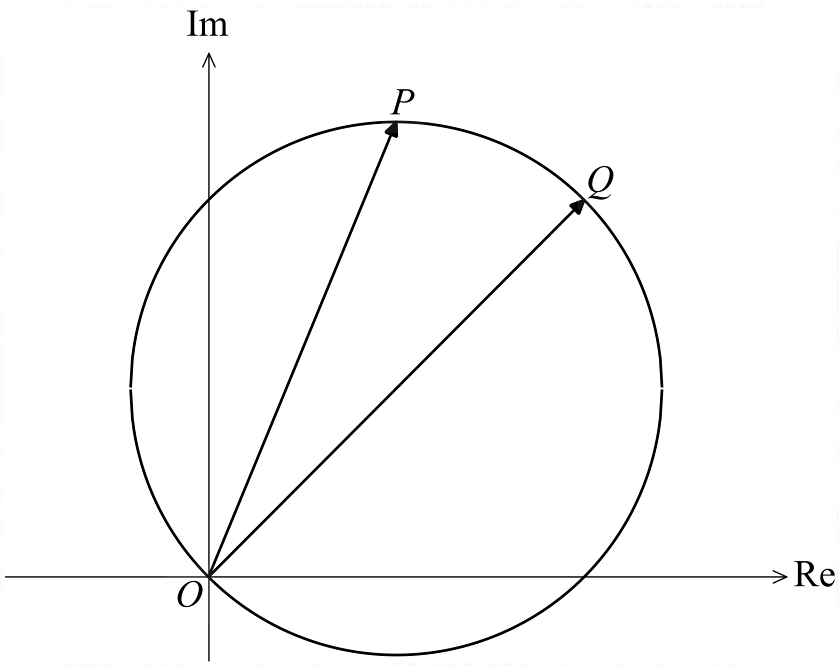
Question 12 continues on page 11

Question 12 (continued)

(c) P is a variable point represented by the complex number z and point Q is a fixed point represented by the complex number α, where $\alpha = p + iq$.

(i) If z satisfies $\text{Re}(\alpha z) = 1$, show that the locus of the point P in the Argand diagram is the line $px - qy = 1$. **1**

In the diagram below, P lies on the circle with diameter OQ $(z \neq 0)$.



(ii) Copy the diagram into your writing booklet and draw the vector $z - \alpha$. **1**

(iii) Show that $\text{Re}\left(\dfrac{z-\alpha}{z}\right) = 0$ and hence $\text{Re}\left(\dfrac{\alpha}{z}\right) = 1$. **2**

(iv) Deduce that if z is a non-zero number such that the point P representing z in the Argand diagram lies on the circle with diameter OQ, where Q has coordinates (p, q) then the point representing $\dfrac{1}{z}$ in the same Argand diagram lies on the line $px - qy = 1$. **2**

End of paper

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TERM 2 HSC TASK 2

Mathematics Extension 2

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1 B

2 B

3 B

4 D

5 B

6 C

7 A

8 D

9 C

Question 10

(a) (i) $z\bar{z} = (4-3i)(4+3i)$
$= 4^2 - 3^2 i^2$
$= 25$

* Very well done. Small number of students had $\sqrt{25} = 5$ for no marks.

(ii) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$
$= \frac{4+3i}{25}$
$= \frac{4}{25} + \frac{3}{25}i$

* Very well done. Small number left it as $\frac{4+3i}{25}$ for no penalty.

(iii) $\text{Im}(z) = -3$

* Very well done. Students with $-3i$ received $\frac{1}{2}$

(b) Method A

$P(i) = i^2 + ki + 2 - i = 0$
$ik = i - 1$
$k = 1 + i$

Method B

Let the other zero be α.
By product of roots $i\alpha = 2 - i$
$\alpha = -1 - 2i$
By sum of roots $(-1-2i) + i = -k$
$k = 1 + i$

* Very well done. Some simple arithmetic errors with + and - signs using both methods.

(c) $(x+iy)^2 = 33+56i$

$x^2-y^2+2xyi = 33+56i$

Equating real and imaginary parts:

$\left.\begin{aligned} x^2-y^2 &= 33 \\ xy &= 28 \end{aligned}\right]$ $\frac{1}{2}$ mark if only this is shown

$y = \frac{28}{x}$

$x^2 - \left(\frac{28}{x}\right)^2 = 33$

$x^4 - 33x^2 - 784 = 0$ (1) If student gets here, or equivalent for y.

$(x^2-49)(x^2+16) = 0$

Since x is real, $x^2 \geqslant 0$

$x^2 = 49,$

$x = \pm 7.$ ($1\frac{1}{2}$) If student gets here

When $x = 7, y = 4$

When $x = -7, y = -4$

OR $x+iy = \pm(7+4i)$

* Well done. Small number of students had the solution $x = \pm 7, y \pm 4$. However, this suggests that $7-4i$ and $-7+4i$ are solutions, but they're not.

(d) $\sqrt{3}-i = 2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (1) if student gets here

$$\frac{(\sqrt{3}-i)^{10}}{(1+i)^{12}} = \frac{\left[2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)\right]^{10}}{\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{12}}$$

$$= \frac{2^{10}\left(\cos\frac{-5\pi}{3} + i\sin\frac{-5\pi}{3}\right)}{2^6\,(\cos 3\pi + i\sin 3\pi)}$$ (2) if student gets here or equivalent

$$= 16\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$$

$$= 16\cos\frac{-2\pi}{3} + 16i\sin\frac{-2\pi}{3}$$ ($2\frac{1}{2}$) if student gets here or equivalent

$$= -8-8\sqrt{3}i$$

* Well done, but relatively many transcription errors and issues simplifying angle values.

(e)

Since w is a complex cube root of unity,

$\boxed{w^3=1}$, $\therefore w^3-1=0$

$(w-1)(w^2+w+1)=0$

$\boxed{w^2+w+1=0}$, since $w\neq 1$

Method A

$(1-3w+w^2)(1+w-8w^2)$

$=(1+w+w^2-4w)(1+w+w^2-9w^2)$

$=(-4w)\times(-9w^2)$

$=36w^3=36$

Method B

$(1-3w+w^2)(1+w-8w^2)$

$=-8w^4+25w^3-10w^2-2w+1$

$=-8w+25-10w^2-2w+1$

$=-10w^2-10w+26$

$=-10(w^2+w+1)+36$

$=36$

*Well done. A minority used Method A, although it led to fewer errors. Some students used $w=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ and attempted to substitute it in and evaluate. However this was arduous, and led to many errors. Also, students should not assume $w\neq\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}$. Using Method B, some students stopped after the 1st line of working, without incorporating the results of w being a complex root of unity (boxed above).

(f) **Method A**

$\frac{1-3i}{1+2i}=\frac{(1-3i)(1-2i)}{1^2+2^2}$

$=-1-i$

$r=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$

$\theta=\frac{-3\pi}{4}$

Method B

$r=\frac{|1-3i|}{|1+2i|}$

$=\frac{\sqrt{10}}{\sqrt{5}}=\sqrt{2}$

$\theta=\arg(1-3i)-\arg(1+2i)$

$=\tan(-3)-\tan(2)=\frac{-3\pi}{4}$

*Well done. However many students got a value of $\theta=\tan^{-1}\left(\frac{-1}{-1}\right)=\frac{\pi}{4}$. Plotting the point $-1-i$ will show this is incorrect.

Question 11 Solution: Extension 2

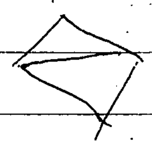
Solution	Marking Scheme	Marker's Comment
a)(i) $e^{5i\theta} = (e^{i\theta})^5$ $\therefore (e^{i\theta})^5 = \cos 5\theta + i\sin 5\theta$ (DeMoivre's Theorem) $(e^{i\theta})^5 = (\cos\theta + i\sin\theta)^5$ $= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ $= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$	2 marks for 2 correct expressions	* Done well by majority of the candidates. * Mark given for writing $(\cos\theta + i\sin\theta)^5$ without expansion.
(ii) Equating real parts from (i) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ $= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	1 mark for equating real parts 1 mark showing correct working towards answer 1 mark showing correct answer	* Done well by majority of the candidates.

Solution

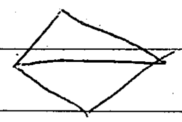
1) let S_n be the convex n-gon where $n > 3$.

Test for $n=4$ i.e. $S_4 = (n-2)$ triangles

S_4 is a quadrilateral



or



By drawing 1 diagonal, only 2 triangles can be formed where every triangle vertex is one of the original vertices of the n-gon.

$(4-2) = 2$ triangles.

$\therefore$ LHS = RHS

$\therefore$ True for $n=4$.

Assume true for $n=k$ (A convex k-gon)

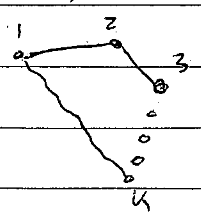
i.e $S_k = (k-2)$ triangles

Hence to prove true for $n=k+1$ (A convex $k+1$-gon)

i.e $S_{k+1} = (k+1-2)$ triangles

$= (k-1)$ triangles.

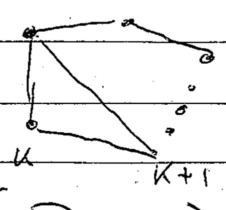
let a convex k-gon be shown.



where vertices are labelled

(A k-gon has k verticies.

A convex $k+1$-gon is a k-gon with one extra vertex as shown.



(PTO)

Marking Scheme

1 mark to correctly prove true for first case i.e $n=4$

1 mark using the assumption to prove true for k+1 statement

1 mark using geometry diagram / reasoning to prove true for k+1 statement.

Marker's Comments

* Significant number of candidates did not get full marks in this question.

* Some candidates need to read the question carefully as it states for "n>3". Mark lost if their first statement to prove is $n=3$.

* Since this is a geometry induction geometry reasoning / diagram is essential. Candidates need to give geometrical reasoning for why $S_k = S_{k+1}$ otherwise marks will be deducted. A diagram that supports the reasoning is highly effective.

* Good to see most candidates wrote the last statement but some still need to take care what they write.

Solution	Marking Scheme	Marker's Comments
As shown in the diagram, a $(k+1)$-gon polygon has an extra vertex and one extra side to form another side.		
$\therefore$ LHS $= S_{k+1}$		
$= S_k + 1$ triangles		
$= (k-2)+1$ triangles (from assumption)		
$= k-1$ triangles		
$=$ RHS		
$\therefore$ True for $n=k+1$		
$\therefore$ Hence by mathematical induction, the statement is true for all positive integers, $n \geq 3$		
c) i) $z^9+1=0$	① mark for showing the factorisation AND write a correct conclusion	* Candidates need to take more care with their solution. z^6-z^3+1 is NOT a root of z^9+1 but a factor. Precise language is important and necessary for full marks.
$(z^3)^3+1^3=0$		
$(z^3+1)(z^6-z^3+1)=0$		
Since z^6-z^3+1 is a factor of z^9+1		
$\therefore$ The roots of $z^6-z^3+1=0$ are also roots of $z^9+1=0$		* Candidates need to be efficient with their solution as this is only 1 mark and does not require a whole page of working

Solution	Marking scheme	Marker's comments

Solution

c) (ii) let $z = \text{cis}\,\theta$

$z^9 = \text{cis}\,9\theta$ (De Moivre's Theorem)

$z^9 + 1 = 0$

$\therefore \text{cis}\,9\theta + 1 = 0$

$\text{cis}\,9\theta = -1$

$\text{cis}\,9\theta = \text{cis}(\pi + 2k\pi)$ where $k \in \mathbb{Z}$

$9\theta = \pi + 2k\pi$

$\theta = \frac{\pi}{9} + 2k\pi$ where $k = 0, \pm 1, \pm 2, \pm 3, \pm 4$

$\therefore \theta = \pi, \pm\frac{\pi}{9}, \pm\frac{3\pi}{9}, \pm\frac{5\pi}{9}, \pm\frac{7\pi}{9}$

Since $z^9 + 1 = (z^3+1)(z^6 - z^3 + 1)$

$z^3 + 1 = 0$

$\text{cis}\,3\theta + 1 = 0$

$\text{cis}\,3\theta = -1$

$\text{cis}\,3\theta = \text{cis}(\pi + 2k\pi$

$3\theta = \pi + 2k\pi$

$\theta = \frac{\pi}{3} + \frac{2k\pi}{3}$ where $k = 0, \pm 1$

$\therefore \theta = \frac{\pi}{3}, \pi, -\frac{\pi}{3}$

$= \frac{3\pi}{9}, \frac{9\pi}{9}, -\frac{3\pi}{9}$

$\therefore$ For $(z^6 - z^3 + 1)$, the roots are

$\theta = \pm\frac{\pi}{9}, \pm\frac{5\pi}{9}, \pm\frac{7\pi}{9}$

$\therefore z = \text{cis}\left(\frac{\pi}{9}\right), \text{cis}\left(-\frac{\pi}{9}\right)$

$\text{cis}\left(\frac{5\pi}{9}\right), \text{cis}\left(-\frac{5\pi}{9}\right)$

$\text{cis}\left(\frac{7\pi}{9}\right), \text{cis}\left(-\frac{7\pi}{9}\right.$

Marking scheme

1 mark for identifying the six roots of $z^6 - z^3 + 1$ giving reasons. i.e. why the other 3 roots of $z^9 + 1 = 0$ is not valid.

2 marks for correct working towards results, giving all required reasons.

Marker's comments

- Candidates lost marks for the following: Not give reasons for why the 6 roots has to be $\theta = \pm\frac{\pi}{9}, \pm\frac{5\pi}{9}$ and $\pm\frac{7\pi}{9}$.
- Not explaining $\cos\theta = -\cos(\pi - \theta)$

* A show question means all steps needs to be shown and explained.

Solution	Marking Scheme	Marker's Comments
Combining conjugate pairs. $(z - cis(\frac{\pi}{9}))(z - cis(-\frac{\pi}{9}))(z - cis(\frac{5\pi}{9}))$ $(z - cis(\frac{5\pi}{9}))(z - cis(\frac{7\pi}{9}))(z - cis(\frac{7\pi}{9}))$ $\therefore z^6 - z^3 + 1 = [z^2 - 2z\cos\frac{7\pi}{9} + 1]$ $[z^2 - 2z\cos\frac{5\pi}{9} + 1]$ $[z^2 - 2z\cos\frac{\pi}{9} + 1)$ As $(z - x)(z + \bar{x}) = z^2 + 2Re(x) + \|x\|^2$ And since $\cos\theta = -\cos(\pi - \theta)$ $\therefore z^6 - z^3 + 1 = (z^2 - 2z\cos\frac{\pi}{9} + 1)(z^2 + 2z\cos\frac{4\pi}{9} + 1)$ $(z^2 + 2z\cos\frac{2\pi}{9} + 1)$		
(c)(iii) Comparing coefficients of $z^6 - z^3 + 1 = (z^2 - 2z\cos\frac{\pi}{9} + 1)(z^2 - 2z\cos\frac{5\pi}{9} + 1)$ $(z^2 - 2z\cos\frac{7\pi}{9} + 1)$ from (ii) $0 = 1 + 1 + 1 + 4\cos\frac{7\pi}{9}\cos\frac{5\pi}{9} +$ $4\cos\frac{7\pi}{9}\cos\frac{\pi}{9} + 4\cos\frac{5\pi}{9}\cos\frac{\pi}{9}$ $4(\cos\frac{\pi}{9}\cos\frac{5\pi}{9} + \cos\frac{5\pi}{9}\cos\frac{7\pi}{9} + \cos\frac{7\pi}{9}\cos\frac{\pi}{9}) = -3$ $\therefore \cos\frac{\pi}{9}\cos\frac{5\pi}{9} + \cos\frac{5\pi}{9}\cos\frac{7\pi}{9} + \cos\frac{7\pi}{9}\cos\frac{\pi}{9} = \frac{-3}{4}$ $\therefore$ LHS = RHS.	(1) mark for coefficients of z^2 (1): mark for show correct working towards result.	* Candidates whom used the result given (ii) must justify why $\cos\frac{4\pi}{9} = \cos\frac{5\pi}{9}$ and $\cos\frac{2\pi}{9} = \cos\frac{7\pi}{9}$ if not done so in (ii). * If using sum/product of roots and coefficients all steps need to be shown. Not many were successful using this method.

Question 12

a) $$\left(z+\bar{z}\right)^2 = i\left(z-\bar{z}\right)$$

let $z = x+iy$

$$(2x)^2 = i(2iy)$$
$$4x^2 = -2y$$
$$y = -2x^2$$

concave down parabola with vertex at origin.

COMMENT:

Most common mistake was $z-\bar{z} = 2\,\mathrm{Im}(z)$ instead of $z-\bar{z} = 2i\,\mathrm{Im}(z)$

b) i) $$\text{LHS} = \frac{\sin\left(\frac{3x}{2}\right) - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{2\left[\frac{1}{2}\sin\left(x+\frac{x}{2}\right) - \sin\left(x-\frac{x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\cancel{2}\cos x \cancel{\sin\left(\frac{x}{2}\right)}}{\cancel{2}\cancel{\sin\left(\frac{x}{2}\right)}}$$

$$= \cos x$$

$$= \text{RHS}$$

ii) Prove true for $n=1$

LHS $= \cos x$

$$\text{RHS} = \frac{\sin\left(\frac{2(1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}$$

$$= \frac{\sin\left(\frac{3}{2}x\right) - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$= \cos x$ using (i)

$\therefore$ true for $n=1$

Assume true for $n=k$ where $k \in \mathbb{N}$

$$\cos x + \cos 2x + \cos 3x + \ldots + \cos kx = \frac{\sin\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}$$

Prove true for $n=k+1$

ie $$\cos x + \cos 2x + \cos 3x + \ldots + \cos kx + \cos(k+1)x = \frac{\sin\left(\frac{2k+3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}$$

$$\begin{aligned}
LHS &= \underbrace{\cos x + \cos 2x + \cos 3x + \ldots + \cos kx}\; + \cos(k+1)x \\
&= \frac{\sin\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2} + \cos(k+1)x \\
&= \frac{\sin\left(\frac{2k+1}{2}x\right) + 2\cos(k+1)x\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2} \\
&= \frac{\sin\left(\frac{2k+1}{2}x\right) + \cancel{2}\cdot\cancel{\frac{1}{2}}\left[\sin\left((k+1)x + \frac{x}{2}\right) - \sin\left((k+1)x - \frac{x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2} \\
&= \frac{\cancel{\sin\left(\frac{2k+1}{2}x\right)} + \sin\left(\frac{2k+3}{2}x\right) - \cancel{\sin\left(\frac{2k+1}{2}x\right)}}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2} \\
&= \frac{\sin\left(\frac{2k+3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2} \\
&= RHS
\end{aligned}$$

$\therefore$ true for $n=k+1$

$\therefore$ true by the principle of mathematical induction for positive integers n.

COMMENT:

Students need to be familiar with the "new" trig. formulas on the reference sheet.

Also, we know what we are working towards so that should give us a hint as to what to do.

For example if we are to show that

$$\frac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \cos x$$

then the numerator must be $2\cos x \sin\left(\frac{x}{2}\right)$.

c) i) $Re(\alpha z)=1$

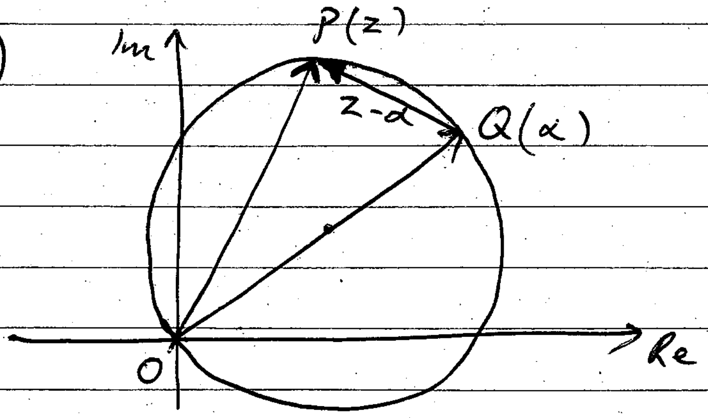
let $z=x+iy$

$Re((p+iq)(x+iy))=1$

$Re(px+pyi+qxi-qy)=1$

$px-qy=1$

ii)



iii) $\arg\left(\frac{z-\alpha}{z}\right)=\arg(z-\alpha)-\arg(z)$

$=\frac{\pi}{2}$ $\left(\angle OPQ=\frac{\pi}{2}\text{, angle in a semicircle}\right)$

OR $-\frac{\pi}{2}$ depending on the position of P.

$\therefore Re\left(\frac{z-\alpha}{z}\right)=0$

$Re\left(\frac{z}{z} - \frac{\alpha}{z}\right) = 0$

$Re\left(1 - \frac{\alpha}{z}\right) = 0$

$1 - Re\left(\frac{\alpha}{z}\right) = 0$

$Re\left(\frac{\alpha}{z}\right) = 1$

iv) $Re\left(\alpha \cdot \frac{1}{z}\right) = 1$

using part (i), since $\frac{1}{z}$ satisfies $Re\left(\alpha \cdot \frac{1}{z}\right) = 1$

then the point representing $\frac{1}{z}$ lies on $px - qy = 1$

COMMENT:

parts (i) & (ii) were generally done well.

Students should be thinking geometrically not algebraically for part (iii)

Another possible solution is:

$z - \alpha = ikz$ ($z-\alpha$ is an anticlockwise rotation by $\frac{\pi}{2}$ of z) [angle in a semicircle]

$\frac{z-\alpha}{z} = \frac{ik\cancel{z}}{\cancel{z}}$ where k is real.

$= ik$

$\therefore Re\left(\frac{z-\alpha}{z}\right) = 0$

Part (iv) is simply a consequence of part (i). Not many students noticed. It can be done algebraically but why would you?

OVERALL students did well on this Question. Even more so given it was the last question of the exam.