SYDNEY BOYS HIGH SCHOOL

Student Number:

Name: ______________________

Class: ______________________

2021

YEAR 12
TERM 1
ASSESMENT TASK 2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 53

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 43 marks (pages 6–11)

- Part A (13 marks) (page 6)
- Part B (16 marks) (pages 8–9)
- Part C (14 marks) (page 10–11)

- Attempt all Questions in Section II
- Allow about 1 hour and 15 minutes for this section

Examiner: JJ

Section I

10 marks

Attempt Questions 1–10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1. The real component of the complex numbers that satisfy $\left| z-2i \right|=3$ and $\operatorname{Im} z=1$ can be found using which of the following equations?

   A. $x^2=0$

   B. $x^2=2$

   C. $x^2=8$

   D. $x^2=9$

2. When using proof by mathematical induction to show that $\left(2-2i\right)^{4n}=\left(-64\right)^n$ where n is an integer, which of the following does the inductive step require the proof of?

   A. $(2-2i)^4=\left(-64\right)^1$

   B. $(2-2i)^{4k}=\left(-64\right)^k$

   C. $(2-2i)^{4k+1}=\left(-64\right)^{k+1}$

   D. $(2-2i)^{4k+4}=\left(-64\right)^{k+1}$

3. The polynomial equation $P\left(z\right)=0$ has real coefficients and has roots which include $z=-3+i$, $z=i$ and $z=3$. What would be the minimum degree of $P\left(z\right)$?

   A. 3

   B. 4

   C. 5

   D. 6

4. Given that $\alpha = 3e^{i\frac{\pi}{6}}$, which of the following expressions is equal to $\left(i\bar{\alpha}\right)^{-1}$?

A. $\frac{1}{3}e^{i\frac{\pi}{3}}$

B. $\frac{1}{3}e^{i\frac{\pi}{6}}$

C. $\frac{1}{3}e^{-i\frac{\pi}{3}}$

D. $\frac{1}{3}e^{-i\frac{\pi}{6}}$

5. Which of the following equations is equivalent to $\cos x = \frac{1}{4}$?

A. $2e^{2ix} + e^{ix} + 2 = 0$

B. $2e^{2ix} - 2e^{ix} - 1 = 0$

C. $2e^{2ix} - e^{ix} + 2 = 0$

D. $2e^{2ix} + 2e^{ix} - 1 = 0$

6. For a certain complex number *z,* where $\arg z = \frac{\pi}{5}$, what is the value of $\arg z^7$?

A. $-\frac{7\pi}{5}$

B. $-\frac{3\pi}{5}$

C. $\frac{2\pi}{5}$

D. $\frac{3\pi}{5}$

7. Given $P(z)$ has real coefficients and $P'(2i)=0$, which is the most correct statement?

A. $2i$ is a root of $P(z)$.

B. $2i$ is a double root of $P(z)$.

C. The sum of the multiplicities of the roots of $P(z)$ is at least 3.

D. The sum of the multiplicities of the roots of $P(z)$ is at least 4.

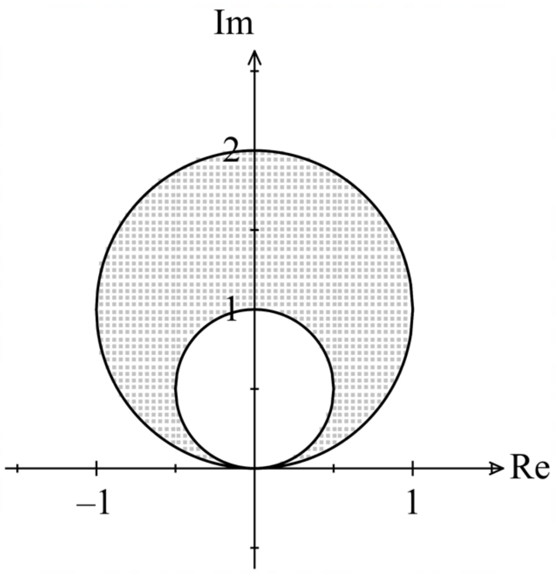
8. Which of the following inequalities describes the shaded region in the diagram below?

A. $2\left|z-i\right|\le 1\le\left|z-\frac{1}{2}i\right|$

B. $2\left|z-1\right|\le 1\le\left|z-\frac{1}{2}\right|$

C. $\left|z-i\right|\le 1\le 2\left|z-\frac{1}{2}i\right|$

D. $\left|z-1\right|\le 1\le 2\left|z-\frac{1}{2}\right|$



9. Let $P(n)$ be the statement that $\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}, n>0$.
What of the following is used during the inductive proof to go from the line (1) below to line (2)?

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \qquad (1)$$

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \qquad (2)$$

A. Base Case

B. Induction Assumption

C. A and B

D. Neither A nor B

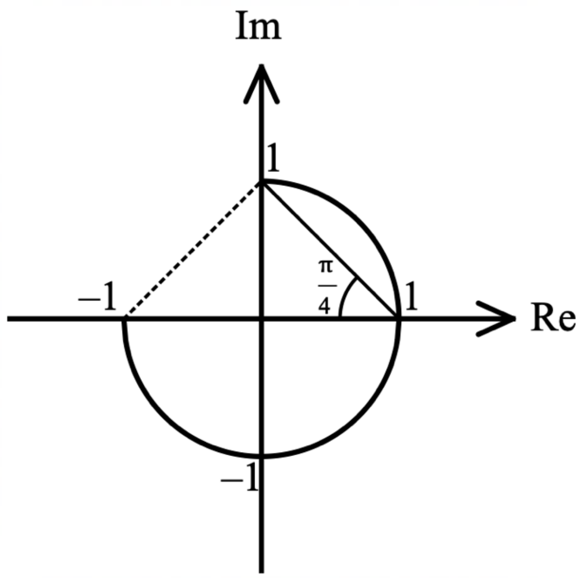
10. The arc of the unit circle in the diagram below is described by which of the following equations?

A. $\arg\left(\dfrac{z-1}{z+i}\right)=\dfrac{\pi}{4}$

B. $\arg\left(\dfrac{z+1}{z-i}\right)=\dfrac{\pi}{4}$

C. $\arg\left(\dfrac{z-i}{z+1}\right)=\dfrac{\pi}{4}$

D. $\arg\left(\dfrac{z+i}{z-1}\right)=\dfrac{\pi}{4}$



Section II

43 marks
Attempt Questions 11-13
Allow 1 hours and 15 minutes for this section

In Questions 11-13, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) Use a SEPARATE writing booklet

a) Sketch the following on separate Argand diagrams.

i. $z+\overline{z}=6$. **1**

ii. $-\dfrac{\pi}{2}\leq \arg(z-1)\leq \dfrac{\pi}{4}$. **1**

b) Consider the polynomial $P(x)=x^4-4x^3+12x^2-16x+16$, where x is a complex number.

i. By first writing $1+i\sqrt{3}$ in exponential form, write $(1+i\sqrt{3})^4$ in the form $a+ib$, where a and b are real numbers. **2**

ii. Show that $P\left(1+i\sqrt{3}\right)=0$. **2**

iii. Given that $1+i\sqrt{3}$ is a double root of $P(x)$, write $P(x)$ as a product of two quadratics with real coefficients. **2**

c) State the real and imaginary parts of e^z, where $z=a+ib$, where a and b are real numbers. **1**

d) i. Prove, using induction that $\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+...+\dfrac{1}{2n+1}<1$ for $n>1$, where n is an integer. **3**

ii. Hence show that $\displaystyle\sum_{k=n+1}^{2n}\frac{1}{k}<1$ **1**

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Question 12 (16 marks) Use a SEPARATE writing booklet

a) Consider the polynomial $x^4 + ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are all integers.

Suppose the equation has a root of the form ki, where k is real, and $k \neq 0$.

i. State why the conjugate of ki is also a root. **1**

ii. Show that $c = ak^2$. **2**

iii. Show that $c^2 + a^2 d = abc$. **2**

iv. If 2 is also a root of the equation, and $b = 0$, show that c^2 is even. **2**

b) Sketch on an Argand diagram the region described by $\mathrm{Re}\left[(3+i)z-2\right]>0$. **2**

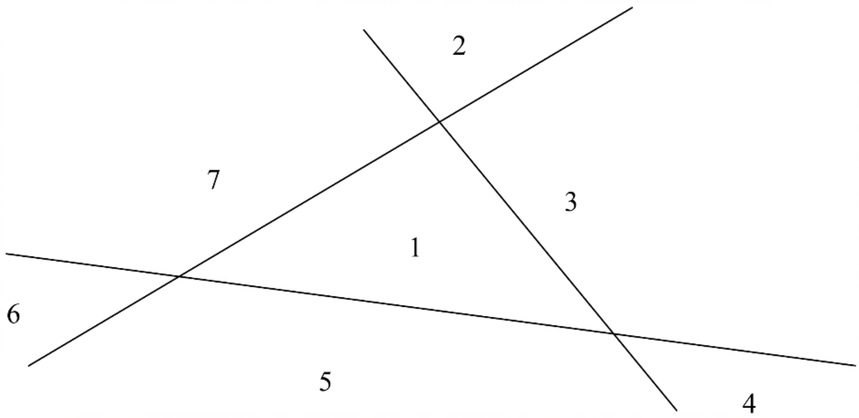
c) Sketch the locus of w where $w=-1-i+z$ and $\left|\, z-1 \,\right|=1$. **2**

Question 12 is continued on page 9

Question 12 (continued)

d) Suppose that n lines are drawn in a plane such that no three lines intersect at the same point i.e. not concurrent, and no two lines are drawn in parallel.

Let R_n be the number of regions into which these lines divide the plane.
The diagram below shows that for $n = 3$ that $R_3 = 7$.



i. By drawing diagrams, find R_1, R_2, R_4 and R_5. **2**

ii. From these results find the values of a, b and c in the recurrence relation below that describes the numbers of lines and regions, where $a, b, c \in \mathbb{Z}$. **1**

$$R_1 = a,$$

$$R_n = bR_{n-1} + cn,\ n \geq 2.$$

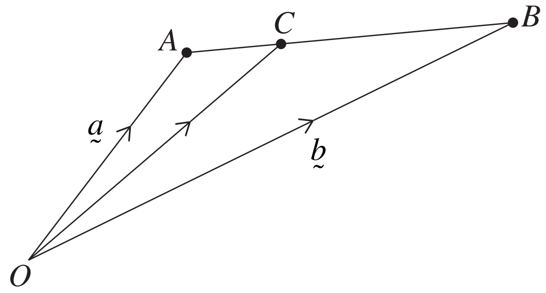
iii. Prove that $R_n = \dfrac{n^2 + n + 2}{2}$, $n \geq 1$. **2**

End of Question 12

Question 13 (14 marks) Use a SEPARATE writing booklet

a) The point C divides the interval AB so that $AC : CB = n : m$.

The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.

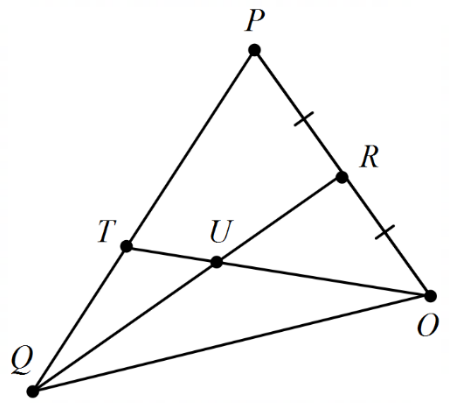


i) Prove that $\overrightarrow{OC} = \dfrac{m}{m+n}\underset{\sim}{a} + \dfrac{n}{m+n}\underset{\sim}{b}$. **2**

Consider the triangle OPQ below.

$OR = RP$ and $2QU = 3UR$.

Let the vectors $\mathbf{p}$ and $\mathbf{q}$ be the position vectors of the points P and Q respectively.



NOT TO SCALE

ii) Show that $\overrightarrow{OU} = \dfrac{1}{10}\left(3\mathbf{p} + 4\mathbf{q}\right)$ **1**

iii) Hence or otherwise, evaluate the ratio of $PT : PQ$ **2**

Question 13 is continued on page 11

Question 13 (continued)

b) Consider the equation $\left(\frac{x}{a}\right)^8 = -1$.

i. Write down the roots, leaving the roots in modulus-argument from. **2**

ii. Hence show that **2**

$$x^8 + a^8 = \left(x^2 + 2ax\cos\frac{\pi}{8} + a^2\right)\left(x^2 + 2ax\cos\frac{3\pi}{8} + a^2\right)\left(x^2 + 2ax\cos\frac{5\pi}{8} + a^2\right)\left(x^2 + 2ax\cos\frac{7\pi}{8} + a^2\right)$$

[You may assume that $(x-\alpha)(x-\bar{\alpha}) = x^2 - 2x\,\mathrm{Re}\,\alpha + |\alpha|^2$]

iii. Deduce that $(x+1)^8 + (x-1)^8$ can be written as that below **4**

$$256\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16}\left(x^2 + \tan^2\frac{\pi}{16}\right)\left(x^2 + \tan^2\frac{3\pi}{16}\right)\left(x^2 + \tan^2\frac{5\pi}{16}\right)\left(x^2 + \tan^2\frac{7\pi}{16}\right)$$

iv. Hence show that $\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16} = \frac{1}{128}$ **1**

End of paper

SYDNEY BOYS HIGH SCHOOL

2021	YEAR 12 TERM 1 ASSESSMENT TASK 2

Mathematics Extension 2

Sample Solutions

NOTE: **Some of you may be disappointed with your mark.**
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

Quick MC Answers

1. C
2. D
3. C
4. C
5. C
6. B
7. C
8. C
9. B
10. B

2021 Y12 Task 2 ME2 Multiple Choice Solutions

Mean (out of 10): 7.25

1. $|z-2i| = 3$

$\therefore |x+i(y-2)| = 3$

$\therefore x^2+(y-2)^2 = 9$

If $Im(z) = 1$, $x^2+1 = 9$

$\therefore x^2 = 8$ C

A	4
B	15
C	86
D	15

2. $S(k) \equiv (2-2i)^{4k} = (-64)^k$

$\Rightarrow S(k+1) \equiv (2-2i)^{4(k+1)} = (-64)^{k+1}$

D

A	11
B	10
C	1
D	98

3. Other roots of $P(z)$ are $-3-i$ and $-i$

$\therefore$ At least 5 roots

C

A	6
B	2
C	111
D	1

4. $\alpha = 3e^{i\frac{\pi}{6}}$

$\therefore (i\bar{\alpha})^{-1} = \left(i.3e^{-i\frac{\pi}{6}}\right)^{-1}$

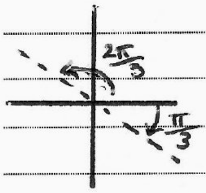
$= \frac{1}{i}\cdot\frac{1}{3}e^{i\frac{\pi}{6}}$

$= -\frac{1}{3}i\,e^{i\frac{\pi}{6}}$

$= -\frac{1}{3}e^{i(\frac{\pi}{6}+\frac{\pi}{2})}$

$= -\frac{1}{3}e^{i\frac{2\pi}{3}}$

$= \frac{1}{3}e^{-i\frac{\pi}{3}}$



C

A	15
B	12
C	82
D	11

5. $\cos x = \frac{1}{4}$

$\therefore \frac{e^{ix}+e^{-ix}}{2} = \frac{1}{4}$

$\therefore e^{ix}+e^{-ix} = \frac{1}{2}$

$\therefore e^{2ix}+1 = \frac{1}{2}e^{ix}$

$\therefore e^{2ix}-\frac{1}{2}e^{ix}+1 = 0$

$\therefore 2e^{2ix}-e^{ix}+2 = 0$ C

A	13
B	23
C	62
D	21

6. $\arg z = \frac{\pi}{5}$

$\therefore \arg z^7 = \frac{7\pi}{5} \pm 2n\pi$

$\frac{7\pi}{5} - 2\pi = -\frac{3\pi}{5}$

B

A	4
B	106
C	8
D	2

7. As $P(z)$ has real co-efficients, $P'(z)$ has real coefficients

$\therefore P'(-2i) = 0$

$\therefore P'(z)$ has at least 2 roots

$\therefore P(z)$ has at least 3 roots.

C

A	7
B	28
C	43
D	42

8. Shaded region is

$\frac{1}{2} \le |z - \frac{i}{2}| \cap |z - i| \le 1$

$\therefore 1 \le 2|z - \frac{i}{2}| \cap |z - i| \le 1$

$\therefore |z - i| \le 1 \le 2|z - \frac{i}{2}|$

C

A	17
B	1
C	102
D	0

9. The Induction Hypothesis would be

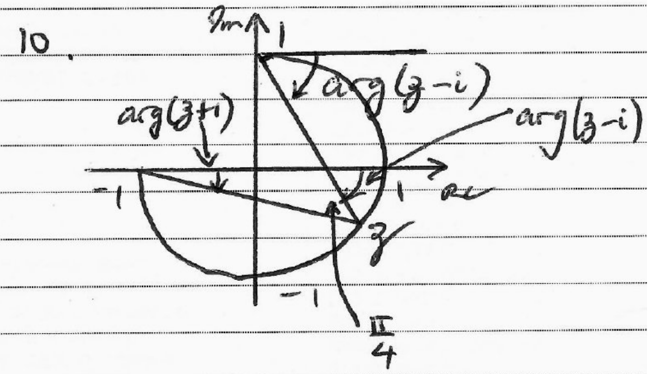
$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

which is used to go from (1) to (2).

B

A	4
B	95
C	10
D	11

10.



magnitude of $\arg(z-i)$

$=$ magnitude of $\arg(z+1) + \frac{\pi}{4}$

(exterior angle of triangle)

$\arg(z-i) - \arg(z+1)$ would be negative.

$\therefore \arg(z+1) - \arg(z-i) = \frac{\pi}{4}$

$\therefore \arg\left(\frac{z+1}{z-i}\right) = \frac{\pi}{4}$

B

A	3
B	85
C	25
D	7

Question 11

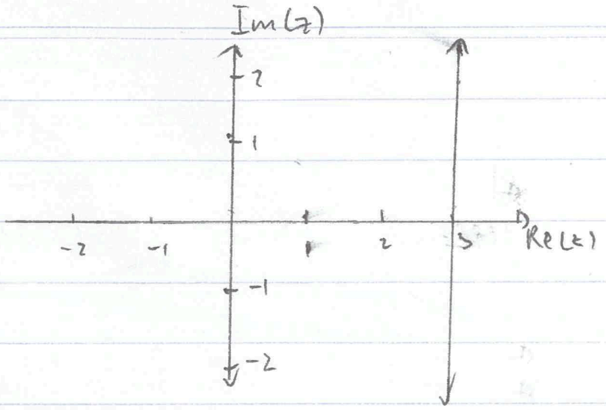
a) i) Let $z = x + iy$, where $x, y \in \mathbb{R}$

$z + \bar{z} = 6$

$x + iy + x - iy = 6$

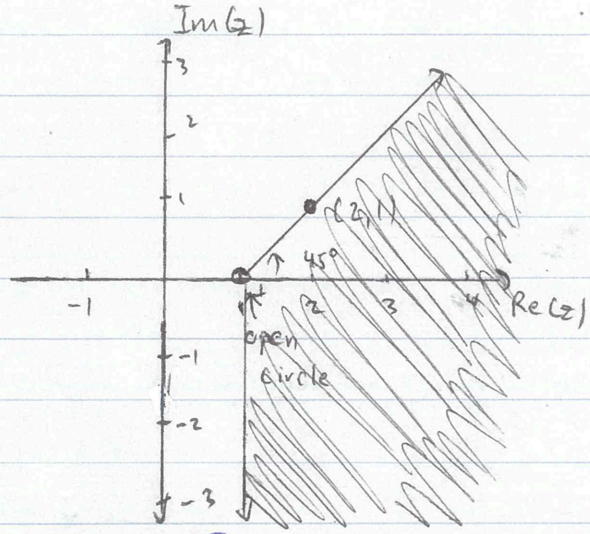
$2x = 6$

$x = 3$



Marking Guideline: 1 mark for the correct graph.
Marker's Comment: Done well by majority of candidates.

(ii) $-\frac{\pi}{2} \leq \arg(z-1) \leq \frac{\pi}{4}$

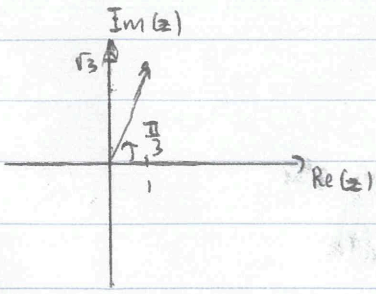


Marking Guideline: 1 mark for the correct graph.

Marker's comments: → Candidates whom did not shade the correct region were penalised as they did not answer the question correctly.

→ An open circle is required at (1,0) but no marks were penalised if candidates did not indicate in their answers.

b) i)



$1 + \sqrt{3}i = 2e^{\frac{\pi}{3}i}$

$(1+\sqrt{3}i)^4 = (2e^{\frac{\pi}{3}i})^4$

$= 2^4 e^{\frac{4\pi}{3}i}$

Question 11 continued

b) i)
$$= 16e^{-\frac{2\pi}{3}i}$$
$$= 16\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$
$$= 16\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= -8 - 8\sqrt{3}i$$

Marking Guideline: $\frac{1}{2}$ mark for writing $1+i\sqrt{3}$ in exponential form.
1 mark for correct working.
$\frac{1}{2}$ mark for correct answer.

Marker's Comments: → Significant number of candidates did not follow the question and express $1+i\sqrt{3}$ in exponential form. Candidates need to read the questions carefully.
→ Majority of the candidates were successful in finding $(1+i\sqrt{3})^4$.

(ii) $P(1+\sqrt{3}i) = P(2e^{\frac{\pi i}{3}})$
$$= (2e^{\frac{\pi i}{3}})^4 - 4(2e^{\frac{\pi}{3}i})^3 + 12(2e^{\frac{\pi}{3}i})^2 - 16(2e^{\frac{\pi}{3}i}) + 16$$
$$= -8 - 8\sqrt{3}i - 4 \times (-2)^3 + 12 \times 4 \times \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) - 16 \times 2 \times \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 16$$
$$= -8 - 8\sqrt{3}i + 32 - 24 + 24\sqrt{3}i - 16 - 16\sqrt{3}i + 16$$
$$= 0$$

Marking Guideline: 1 mark for correct substitution of $(1+\sqrt{3}i)$ into $P(x)$.
1 mark for showing All necessary steps to prove the result is true.

Marker's Comments: → Many candidates were successful in gaining full marks for this question, but candidates need to be aware that SHOW means all required steps need to be written on the page, even if it is obvious.
→ Candidates whom did not get the result should check their arithmetic and avoid fudging their answers to the result.

Question 11 continued

(iii) Since $1+i\sqrt{3}$ is a double root, its conjugate $1-\sqrt{3}i$ is also a double root, since the coefficients are real.

$$P(x) = [x-(1+\sqrt{3}i)]^2[x-(1-\sqrt{3}i)]^2$$
$$= [(x-(1+\sqrt{3}i))(x-(1-\sqrt{3}i))]^2$$
$$= [x^2 - x[1+\sqrt{3}i+1-\sqrt{3}i] + (1+\sqrt{3}i)(1-\sqrt{3}i)]^2$$
$$= (x^2-2x+4)^2$$
$$= (x^2-2x+4)(x^2-2x+4)$$

Marking Guideline: 1 mark for stating the conjugate is also a double root and rewriting $P(x)$ with the 2 double roots.

1 mark for correct answer and correct working out.

Marker's comments: → Candidates whom were successful in this question followed the steps above as this was an efficient method.

→ Some candidates were using polynomial division in which they were less likely to achieve the result required.

→ Arithmetic errors were significant in this question and candidates need to take more care.

→ Whilst no penalty was given, candidates need to follow the directive of the question and "write $P(x)$ as a PRODUCT of two quadratics" rather than leaving their answer as $(x^2-2x+4)^2$.

(c)
$$e^z = e^{a+ib}$$
$$= e^a(e^{ib})$$
$$= e^a(\cos b + i\sin b)$$
$$= e^a\cos b + ie^a\sin b$$
$$\therefore \text{Re}(e^z) = e^a\cos b$$
$$\text{Im}(e^z) = e^a\sin b$$

Marking Guideline: 1 mark for correct answer.

Question 11 continued

(c) Marker's Comments → Generally no half marks was given for this question.

→ Whilst no penalty was given, candidates should be more explicit in stating the real and imaginary part of e^z i.e. last 2 lines of the solution given.

d) i) S1: Test for $n=2$

$$LHS = \frac{1}{2+1} + \frac{1}{2+2} + \frac{1}{2+3}$$

$$= \frac{47}{60}$$

$$< 1$$

$$= RHS$$

$\therefore$ True for $n=2$

S2: Assume the result is true for $n=k$, $k \in \mathbb{Z}^+$, $k>1$

i.e. $$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} < 1$$

S3: Hence prove true for $n=k+1$

i.e. $$\frac{1}{k+2} + \frac{1}{k+3} + \frac{1}{k+4} + \dots + \frac{1}{2(k+1)+1} < 1$$

$$LHS = \frac{1}{k+2} + \frac{1}{k+3} + \frac{1}{k+4} + \dots + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$$

$$= \frac{1}{k+1} - \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$$

$$< 1 + \frac{1}{2k+2} + \frac{1}{2k+3} - \frac{1}{k+1}, \text{ by assumption.}$$

$$= 1 + \frac{1}{2(k+1)} - \frac{1}{k+1} + \frac{1}{2k+3}$$

$$= 1 + \frac{(2k+3) - 2(2k+3) + 2(k+1)}{2(k+1)(2k+3)}$$

$$= 1 - \frac{1}{(2k+2)(2k+3)}$$

Question 11 continued

$< 1 \quad \left(\text{As } k > 1, \; 0 < \frac{1}{(2k+2)(2k+3)} < 1 \right)$

$= RHS$

$\therefore$ True for $n = k+1$

S4: Hence by Mathematical Induction, the statement is proven true for $n > 1$.

Marking Guideline: 1 mark for correct solution for the base case i.e. $n = 2$.

1 mark for correct statement for $n = k+1$ case and showing some correct steps in using the assumption.

1 mark for correct solution in proving true for $n = k+1$ case and writing a brief conclusion.

Note: No marks awarded for writing a conclusion.

Marker's Comments: → This was done poorly by majority of the candidates.

→ Common errors included:

* Using $n = 1$ as the base case.
* Not understanding the summing of 3 terms for $n = 2$ in the base case.
* In proving the $n = k+1$ case, still having an extra $\frac{1}{k+1}$ in the RTP statement and hence using in the proof when clearly it starts with $\frac{1}{k+2}$.
* Missing the $\frac{1}{2k+2}$ term as the last term is $\frac{1}{2k+3}$.

→ Many candidates had a poor way of presenting their solution for an inequality and need to take more care.

→ No half marks awarded (or full mark) for starting with $n=1$ in the base case.

(ii) $LHS = \sum_{k=n+1}^{2n} \frac{1}{k}$

$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$

$< \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1}$ $\left(\text{As } \frac{1}{2n+1} > 0 \text{ for } n > 1\right)$

< 1 (from (i))

$\therefore \sum_{k=n+1}^{2n} \frac{1}{k} < 1$

Marking Guideline: 1 mark for correct solution.

Marker's Comments: → This was done poorly by many candidates.

→ Candidates were penalised if they:

* Stated $\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1}$

(LHS ≠ RHS.)

* Rewrote the question and stated this is true from (i) with no statement of how. A show question means required steps need to be written on the page even if it appears obvious.

* Not using the previous question when the key word in the question is HENCE.

→ Majority of the candidates need to take more care in how they set out their proofs. The 3rd line in the solution above is quite crucial including the reasoning part as well.

a) i)

- "For polynomials with real coefficients, complex roots occur in conjugate pairs. Hence, since $a, b, c, d \in \mathbb{R}$, and ki is a root, so is $\overline{ki} = -ki$."

- "Since coefficients are real, by the complex conjugate root theorem, the conjugate of ki is also a root."

* If students do not specify real coefficients, $\frac{1}{2}$ is deducted.

- When $x = ki$,

$$x^4 + ax^3 + bx^2 + cx + d = 0$$
$$k^4 - i.ak^3 - bk^2 + i.ck + d = 0$$
$$(k^4 - bk^2 + d) - i(ak^3 - ck) = 0.$$

$\therefore$ Equating real & imaginary parts

$$k^4 - bk^2 + d = 0$$
$$ak^3 - ck = 0$$

When $x = -ki$,

$$x^4 + ax^3 + bx^2 + cx + d = k^4 + i.ak^3 - bk^2 - i.ck + d$$
$$= (k^4 - bk^2 + d) + i.(ak^3 - ck)$$
$$= 0 + 0i$$

$\therefore \quad x = -ki$ is also a root.

a) ii) Method 1, let roots be $\alpha, \beta, ki, -ki$

$\sum \alpha = \alpha + \beta + ki + (-ki) = -a$

① $a = -(\alpha+\beta)$ $\left(\frac{1}{2}\right)$

$\sum \alpha\beta\gamma = \alpha\beta ki - \alpha\beta ki - \alpha k^2 i^2 - \beta k^2 i^2 = -c$

② $c = -k^2(\alpha+\beta)$ $\left(\frac{1}{2}\right)$

① → ② $\quad c = ak^2$ as needed

iii) $\sum \alpha\beta = \alpha\beta + \alpha ki - \alpha ki + \beta ki - \beta ki - k^2 i^2$

$b = \alpha\beta + k^2$

③ $\alpha\beta = b - k^2$ $\left(\frac{1}{2}\right)$

$\sum \alpha\beta\gamma\delta = -\alpha\beta k^2 i^2$

④ $d = \alpha\beta k^2$ $\left(\frac{1}{2}\right)$

③ → ④ $\quad d = k^2(b-k^2)$

$k^2 = \frac{c}{a}$, from (ii)

$d = \frac{c}{a}\left(b - \frac{c}{a}\right)$

$d = \frac{bc}{a} - \frac{c^2}{a^2}$

$a^2 d = abc - c^2$

$a^2 d + c^2 = abc$ as needed.

Method 2

ii) $P(ki) = k^4 - i.ak^3 - bk^2 + i.ck + d$

$= (k^4 - bk^2 + d) - i(ak^3 - ck)$

$= 0 + 0i$

Note $a, b, c, d \in \mathbb{R}$

Equating imaginary part

$ak^3 - ck = 0$

$ak^3 = ck$

$ak^2 = c \quad (k \neq 0)$

iii) Equating real part

$k^4 - bk^2 + d = 0$

$k^2 = \frac{c}{a}$

$\left(\frac{c}{a}\right)^2 - b\left(\frac{c}{a}\right) + d = 0$

$c^2 - abc + a^2 d = 0$

$a^2 d + c^2 = abc$ as needed

* Done well. However, some students didn't <u>show</u> the required statements, but instead started by assuming true.
* Students who used method 2, or similar, made fewer mistakes.

---

iv) Since $P(x)$ is a monic polynomial, all real roots must be factors of the constant term.

$\therefore \; d = 2n$, where $n \in \mathbb{Z}$

$P(2) = 2^4 + a \times 2^3 + 0 \times 2^2 + c \times 2 + d = 0$

$16 + 8a + 2c + d = 0$

$d = -2(c + 4a + 8)$

$= 2n$, where $n = -c - 4a - 8 \in \mathbb{Z}$

Substituting into (iii), with $b = 0$ gives

$c^2 + 2n \times a^2 = 0$

$c^2 = -2a^2 n$

$= 2p$, where $p = -a^2 n$ is an integer.

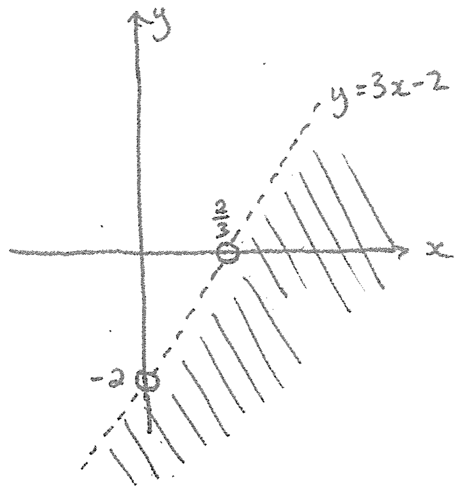
$\therefore \; c^2$ is even.

* Question was done poorly. Many students assumed that k or k^2, or root β, are integers, without proving it. Other students found an expression for c as a sum of terms including $\frac{d}{2}$, and assumed this was an integer (and even to boot).

b) Let $z = x+iy$, where $x, y \in \mathbb{R}$

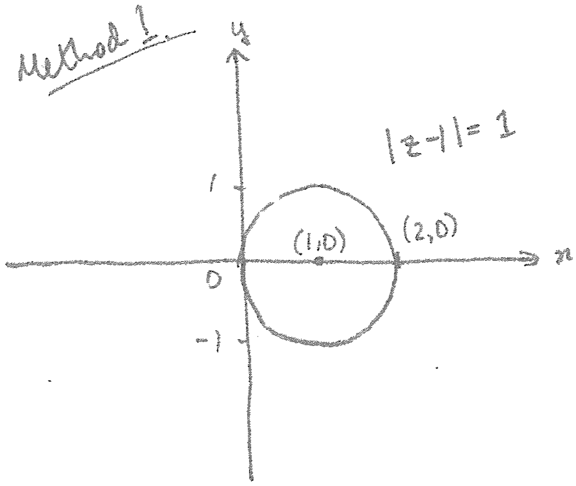
$$\begin{aligned}(3+i)z-2 &= (3+i)(x+iy)-2 \\ &= 3x+3iy+ix-y-2 \\ &= (3x-y-2)+i(3y+x)\end{aligned}$$

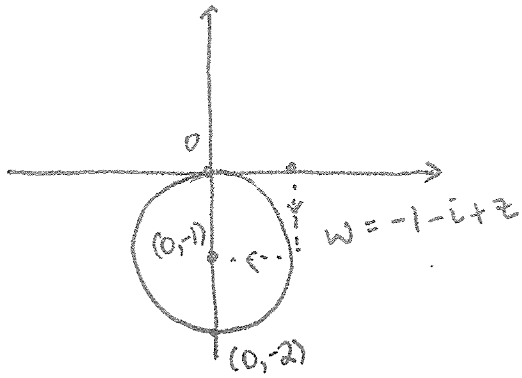
$$\begin{aligned}\mathrm{Re}[(3+i)z-2] &> 0 \\ 3x-y-2 &> 0 \\ y &< 3x-2\end{aligned}$$



* Many students missed the dotted line, or shaded the incorrect region.
* Others sketched $y < 3x$ or $y^2 < 3x-2$ instead.

c) Method 1.





Method 2. let $z = x+iy$

$|z-1| = 1$

$|(x-1)+iy| = 1$

$(x-1)^2+y^2 = 1$

$w = -1-i+z$

$= -1-i+x+iy$

$= (x-1)+i(y-1)$

$w+i = (x-1)+iy$

$|w+i| = |(x-1)+iy| = |z-1| = 1$

$\therefore$ $|w+i|=1$ is locus, with centre $(0,-1)$ and radius 1.

Method 3

$w=-1-i+z$

$z = w+1+i$

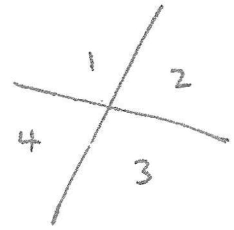
$|z-1|=1$

$|w+1+i-1|=1$

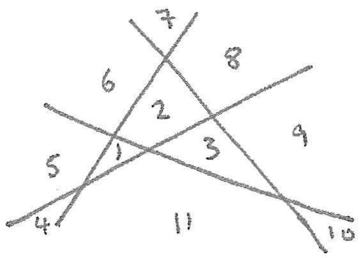
$|w+i|=1$

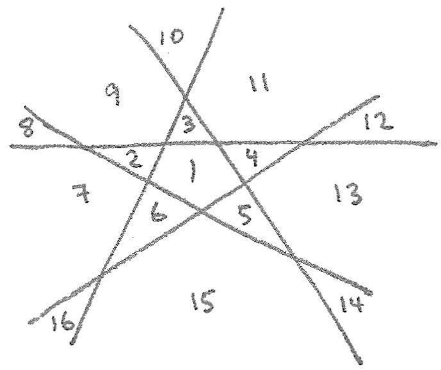
* Poorly done. Transformation not well completed.
* Students who sketched circle of radius 1 anywhere received ①.
* Some students did not mark intercepts or the centre, giving no scale to their diagram.

d) i. $R_1 = 2$ $\qquad$ $R_2 = 4$



$R_4 = 11$ $\qquad$ $R_5 = 16$





* Overall, well done.
* ½ mark for each correct diagram / number combo.
* Students usually undercounted R_4 and/or R_5 if they drew 2 parallel lines, or did not continue them till they intersected.

ii)

$$R_1 = 2 \quad \xrightarrow{+2}$$
$$R_2 = 4 \quad \xrightarrow{+3}$$
$$R_3 = 7 \quad \xrightarrow{+4}$$
$$R_4 = 11 \quad \xrightarrow{+5}$$
$$R_5 = 16$$

$$R_2 = R_1 + 2$$
$$R_3 = R_2 + 3$$
$$R_4 = R_3 + 4$$
$$R_5 = R_4 + 5$$

$$R_1 = a = 2$$
$$R_n = R_{n-1} + n$$
$$b = 1, \; c = n$$

* Poorly done. Most students attempted to find a constant value for c.
* No half marks for this question.

d) iii) Method 1

$$R_n = R_{n-1} + n$$
$$= n + R_{n-2} + (n-1)$$
$$= n + (n-1) + R_{n-3} + (n-2) \text{, and so forth}$$
$$= n + (n-1) + (n-2) + \dots + 2 + R_1$$
$$= n + (n-1) + (n-2) + \dots + 2 + 1 + 1 \text{, since } R_1 = 1+1$$
$$= \frac{n(n+1)}{2} + 1 \quad \text{as the sum of the first } n \text{ natural numbers is } \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)+2}{2}$$
$$= \frac{n^2+n+2}{2} \quad \text{as required.}$$

Method 2 - Induction

Step 1 - Show for $n=1$

$LHS = R_1 = 2$

$RHS = \dfrac{1^2+1+2}{2} = 2$

$\therefore$ true for $n=1$

Step 2 - Assume for some $n=k$

i.e. $R_k = \dfrac{k^2+k+2}{2}$

(1) for getting to this point.

Step 3 - Show for $n=k+1$

RTP: $R_{k+1} = \dfrac{(k+1)^2+(k+1)+2}{2}$

$LHS = R_{k+1}$

$= R_k + (k+1)$, using recurrence in (iii)

$= \dfrac{k^2+k+2}{2} + \dfrac{2(k+1)}{2}$, by inductive step

$= \dfrac{k^2+2k+1+k+1+2}{2}$

$= \dfrac{(k+1)^2+(k+1)+2}{2} = RHS$

$\therefore$ True for $n=k+1$ if true for $n=k$

Step 4 - Hence true by Mathematical Induction

* Due to issues completing (ii), most students were limited to at most (1).

13) a) i) $AC : CB = n : m$

$AC : AB = n : n+m$

$$\vec{AC} = \frac{n}{n+m}\vec{AB}$$

$$= \frac{n}{n+m}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$\vec{OC} = \vec{OA} + \vec{AC}$$

$$= \underset{\sim}{a} + \frac{n}{n+m}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \cancel{\frac{\underset{\sim}{a}n + \underset{\sim}{a}m + \underset{\sim}{b}n - \underset{\sim}{a}n}{n+m}}$$

$$= \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}$$

ii) $QU : UR = 3 : 2$

$$\vec{OU} = \frac{2}{2+3}\underset{\sim}{q} + \frac{3}{2+3}\left(\frac{1}{2}\underset{\sim}{p}\right) \qquad \text{using (i)}$$

$$= \frac{2}{5}\underset{\sim}{q} + \frac{3}{10}\underset{\sim}{p}$$

$$= \frac{1}{10}(3\underset{\sim}{p} + 4\underset{\sim}{q})$$

iii) Let $PT : TQ = n : m$

$PT : PQ = n : n+m$

$$\vec{OT} = k\vec{OU} \qquad \text{where } k \text{ is a constant}$$

$$= \frac{k}{10}(3\underset{\sim}{p} + 4\underset{\sim}{q})$$

$$= \frac{3k}{10}\underset{\sim}{p} + \frac{4k}{10}\underset{\sim}{q}$$

$$\vec{OT} = \frac{m}{m+n}\underset{\sim}{p} + \frac{n}{m+n}\underset{\sim}{q}$$

$$\frac{m}{m+n} = \frac{3k}{10} \quad \text{——— } (1)$$

$$\frac{n}{m+n} = \frac{4k}{10} \quad \text{——— } (2)$$

$(1) + (2)$

$$\frac{m+n}{m+n} = \frac{7k}{10}$$

$$k = \frac{10}{7}$$

sub into (2)

$$\frac{n}{m+n} = \frac{4\left(\frac{10}{7}\right)}{10}$$

$$= \frac{4}{7}$$

$\therefore \; PT : PQ = 4 : 7$

COMMENTS

The result in (i) is the general result for dividing an interval in a ratio and can be applied to parts (ii) & (iii).
Most students repeated their working for part (ii) by not using part (i).
Not many students were able to answer (iii).
It is another example of writing something in two different ways and equating.

13)b)i) let $z = \frac{x}{a}$

$$z^8 = -1$$

$$(re^{i\theta})^8 = 1e^{i\pi}$$

$$r^8 e^{i8\theta} = 1e^{i\pi}$$

equate modulus and argument.

$r^8 = 1$ $\quad 8\theta = \pi + 2k\pi \quad k = 0, \pm 1, \pm 2, \pm 3, -4$

$r = 1$ $\quad \theta = \frac{\pi + 2k\pi}{8}$

$$z = e^{i\frac{\pi}{8}}, e^{i\frac{3\pi}{8}}, e^{-i\frac{\pi}{8}}, e^{i\frac{5\pi}{8}}, e^{-i\frac{3\pi}{8}}, e^{i\frac{7\pi}{8}}, e^{-i\frac{5\pi}{8}}, e^{-i\frac{7\pi}{8}}$$

since $z = \frac{x}{a}$

$$x = a\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$
$$a\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$$
$$a\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$
$$a\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right)$$
$$a\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$$
$$a\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$$
$$a\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$$
$$a\left(\cos\left(-\frac{7\pi}{8}\right) + i\sin\left(-\frac{7\pi}{8}\right)\right)$$

ii) $\left(\frac{x}{a}\right)^8 = -1$

$$\frac{x^8}{a^8} = -1$$

$$x^8 = -a^8$$

$$x^8 + a^8 = 0$$

we have found the roots of this equation in (i).

The zeros of x^8+a^8 correspond to these also.

$$x^8+a^8 = \left(x-a\,\text{cis}\tfrac{\pi}{8}\right)\left(x-a\,\text{cis}\left(-\tfrac{\pi}{8}\right)\right)\left(x-a\,\text{cis}\tfrac{3\pi}{8}\right)\left(x-a\,\text{cis}\left(-\tfrac{3\pi}{8}\right)\right)\left(x-a\,\text{cis}\tfrac{5\pi}{8}\right)\left(x-a\,\text{cis}\left(-\tfrac{5\pi}{8}\right)\right)\left(x-a\,\text{cis}\tfrac{7\pi}{8}\right)\left(x-a\,\text{cis}\left(-\tfrac{7\pi}{8}\right)\right)$$

$$= \left(x-a\,\text{cis}\tfrac{\pi}{8}\right)\left(x-\overline{a\,\text{cis}\tfrac{\pi}{8}}\right)\left(x-a\,\text{cis}\tfrac{3\pi}{8}\right)\left(x-\overline{a\,\text{cis}\tfrac{3\pi}{8}}\right)\left(x-a\,\text{cis}\tfrac{5\pi}{8}\right)\left(x-\overline{a\,\text{cis}\tfrac{5\pi}{8}}\right)\left(x-a\,\text{cis}\tfrac{7\pi}{8}\right)\left(x-\overline{a\,\text{cis}\tfrac{7\pi}{8}}\right)$$

$$= \left(x^2-2\left(a\cos\tfrac{\pi}{8}\right)x+a^2\right)\left(x^2-2\left(a\cos\tfrac{3\pi}{8}\right)x+a^2\right)\left(x^2-2\left(a\cos\tfrac{5\pi}{8}\right)x+a^2\right)\left(x^2-2\left(a\cos\tfrac{7\pi}{8}\right)x+a^2\right)$$

$$= \left(x^2+2a\cos\left(\pi-\tfrac{\pi}{8}\right)x+a^2\right)\left(x^2+2a\cos\left(\pi-\tfrac{3\pi}{8}\right)x+a^2\right)\left(x^2+2a\cos\left(\pi-\tfrac{5\pi}{8}\right)x+a^2\right)\left(x^2+2a\cos\left(\pi-\tfrac{7\pi}{8}\right)x+a^2\right)$$

$$= \left(x^2+2ax\cos\tfrac{7\pi}{8}+a^2\right)\left(x^2+2ax\cos\tfrac{5\pi}{8}+a^2\right)\left(x^2+2ax\cos\tfrac{3\pi}{8}+a^2\right)\left(x^2+2ax\cos\tfrac{\pi}{8}+a^2\right)$$

iii) Consider $x^2+2ax\cos\theta+a^2$

Let $x = X+1$ & $a = X-1$

$$x^2+2ax\cos\theta+a^2 = (X+1)^2+2(X-1)(X+1)\cos\theta+(X-1)^2$$

$$= X^2+\cancel{2X}+1+2(X^2-1)\cos\theta+X^2-\cancel{2X}+1$$

$$= 2\left(X^2+1+(X^2-1)\cos\theta\right)$$

$$= 2\left(X^2+1+(X^2-1)\left(2\cos^2\tfrac{\theta}{2}-1\right)\right)$$

$$= 2\left(\cancel{X^2}+1+2X^2\cos^2\tfrac{\theta}{2}-\cancel{X^2}-2\cos^2\tfrac{\theta}{2}+1\right)$$

$$= 2\left(2+2\cos^2\tfrac{\theta}{2}(X^2-1)\right)$$

$$= 4\left(1+\cos^2\tfrac{\theta}{2}(X^2-1)\right)$$

$$= 4\cos^2\tfrac{\theta}{2}\left(\sec^2\tfrac{\theta}{2}+X^2-1\right)$$

$$= 4\cos^2\tfrac{\theta}{2}\left(\tan^2\tfrac{\theta}{2}+X^2\right)$$

$$\therefore x^8+a^8 = 4\cos^2\tfrac{\pi}{16}\left(X^2+\tan^2\tfrac{\pi}{16}\right)\cdot 4\cos^2\tfrac{3\pi}{16}\left(X^2+\tan^2\tfrac{3\pi}{16}\right)\cdot 4\cos^2\tfrac{5\pi}{16}\left(X^2+\tan^2\tfrac{5\pi}{16}\right)\cdot 4\cos^2\tfrac{7\pi}{16}\left(X^2+\tan^2\tfrac{7\pi}{16}\right)$$

$$= 256\cos^2\tfrac{\pi}{16}\cos^2\tfrac{3\pi}{16}\cos^2\tfrac{5\pi}{16}\cos^2\tfrac{7\pi}{16}\left(X^2+\tan^2\tfrac{\pi}{16}\right)\left(X^2+\tan^2\tfrac{3\pi}{16}\right)\left(X^2+\tan^2\tfrac{5\pi}{16}\right)\left(X^2+\tan^2\tfrac{7\pi}{16}\right)$$

iv) Equate coefficients of x^8 from part (iii)

$$2 = 256\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16}$$

$$\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16} = \frac{2}{256}$$

$$= \frac{1}{128}$$

COMMENTS

Part (i) was done well by students.

Most students only gained 1 mark in part (ii)

$$(x-\alpha)(x-\bar{\alpha}) = x^2 - 2x\,Re(\alpha) + |\alpha|^2$$
$$\neq x^2 + 2x\,Re(\alpha) + |\alpha|^2$$

as was given in the question.

$\cos(\pi-\theta) = -\cos\theta$ was needed

Not many students gained 4 marks in part (iii).

Some students made the substitution but did not follow it through.

Part (iv) should have been an easy mark for all to get. Even though it is the very last question it is one of the easiest marks to gain in the paper.