SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:
12Ma

2021

YEAR 12
TERM 2
HSC Task #3

Mathematics Extension 2

General Instructions

- Working Time – 65 minutes
- Write using black pen
- NESA approved calculators may be used
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 1–3, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 35

Section I – 8 marks (Personal Reference sheet)

Leave your Reference Sheet inside the booklet for Question 1

Section II – 27 marks (page 3–14)

- Attempt Questions 1–3

Examiner:

PSP

Marks	
Section I	**/8**
Section II	**/27**
Total	**/35**

NESA Number									

Q1

YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Question 1

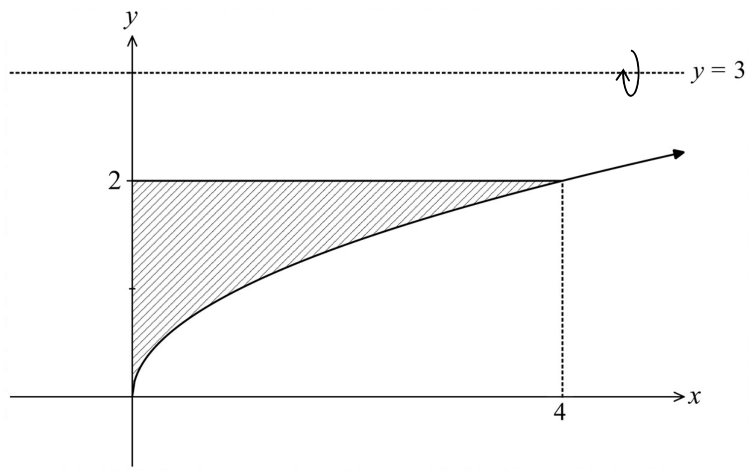
Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (11 marks)

(a) The area bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$ is rotated about the line $y = 3$. The volume is to be calculated by taking slices perpendicular to the axis of rotation. Which integral gives the volume of the solid formed? **1**
Circle the correct answer.



A. $\pi\displaystyle\int_0^2 \left((3-y)^2-1\right)dy$

B. $\pi\displaystyle\int_0^4 \left(x-6\sqrt{x}+8\right)dx$

C. $\pi\displaystyle\int_0^4 \left(2-2\sqrt{x}+x\right)dx$

D. $\pi\displaystyle\int_0^4 \left((3-x)^2-1\right)dx$

(b) Find $\displaystyle\int \frac{1}{2x\sqrt{\ln x}}\,dx$. **2**

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(c) (i) Find real constants a, b and c such that **2**

$$\frac{5}{(1-2x)(1+x^2)} = \frac{a}{1-2x} + \frac{bx+c}{1+x^2}$$

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(ii) Hence, or otherwise show that $\displaystyle\int_{-\frac{\pi}{4}}^{0} \frac{5}{1-2\tan\theta}\,d\theta = \frac{\pi}{4} + \ln\left(\frac{9}{2}\right)$ **3**

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(d) Find $\int 3x^2 \arccos x \, dx$. **3**

Use this space to re-write any question. Clearly indicate that you have done so.

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Q2 [ ]

YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Question 2

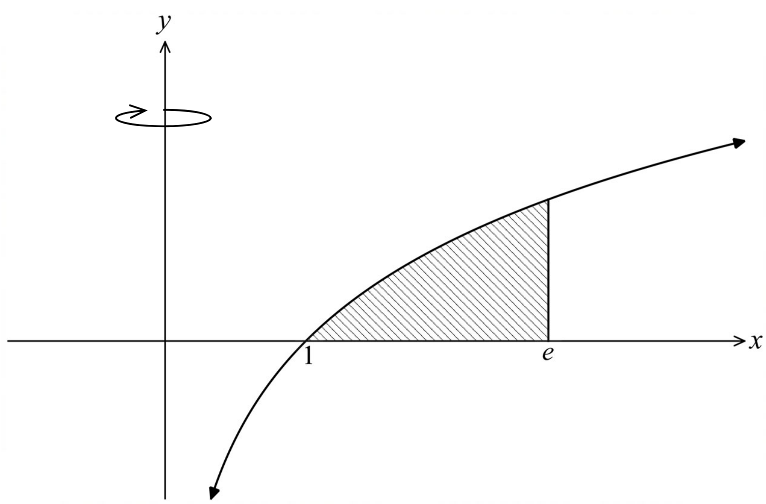
Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 2 (9 marks)

(a) The diagram below shows the region bounded by the curve $y = \log_e x$, the x-axis and the line $x = e$. The region is rotated about the y-axis to form a solid. **4**



Find the volume of the solid by the method of cylindrical shells.

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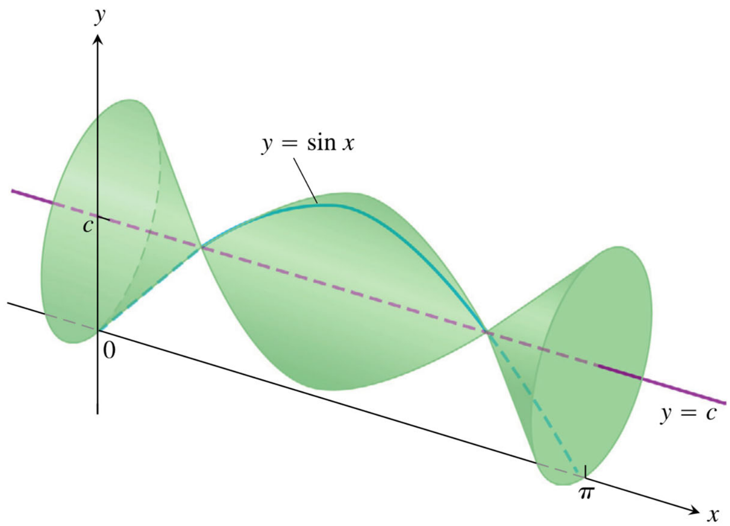
(b) The graph of $y = \sin x$ for $0 \le x \le \pi$ is revolved around the line $y = c$ to generate the solid shown below. Find the value of c that minimises the volume. **5**



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NESA Number									

Q3

YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Question 3

Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 3 (7 marks)

Let $I_n = \int_0^1 x^n e^{-x}\, dx$, where n is a non-negative integer.

(i) Show that $I_0 = 1 - \frac{1}{e}$ **1**

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(ii) Show that $I_n = nI_{n-1} - \frac{1}{e}$, for $n \geq 1$. **2**

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(b) (iv) Deduce by the principle of mathematical induction that for all $n \geq 0$, **3**

$$I_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}.$$

Note: $n! = n \times (n-1)!$ for $n \in \mathbb{Z}^+$, where $0! = 1$.

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(v) Conclude that $e = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!}$. **1**

You may assume that $I_n \to 0$ as $n \to \infty$ (Do NOT prove)

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End of paper

Use this space to re-write any question. Clearly indicate that you have done so.

SYDNEY
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SCHOOL

2021

Year 12
Term 2
HSC Task 3

Mathematics Extension 2

Sample Solutions

NOTE: **Some of you may be disappointed with your mark.**
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

SOLUTIONS
+
COMMENTS

VERITATE ET VIRTUTE

NESA Number											Q1	

YEAR 12 Mathematics Extension 2

TERM 2 HSC TASK #3

Question 1

Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (11 marks)

(a) The area bounded by the curve $y=\sqrt{x}$, the y-axis and the line $y=2$ is rotated about the line $y=3$. The volume is to be calculated by taking slices perpendicular to the axis of rotation. Which integral gives the volume of the solid formed? **1**

Circle the correct answer.

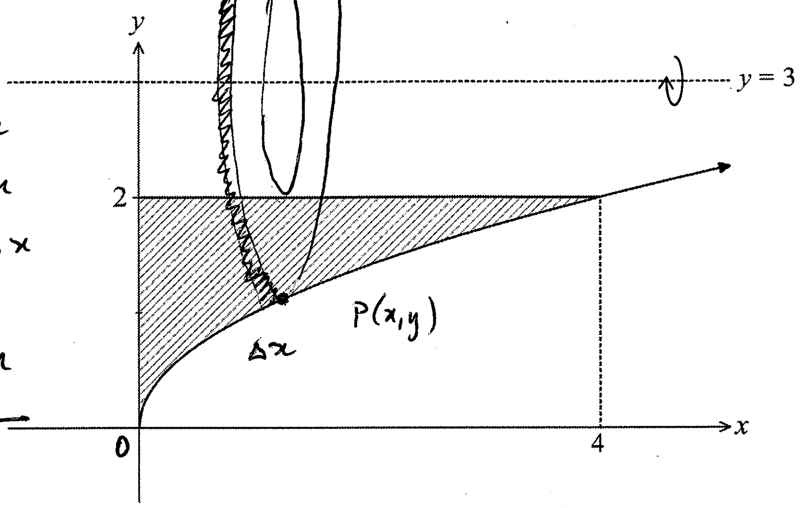
$\Delta V = \pi\left((3-y)^2 - 1^2\right)\Delta x$

$\Delta V = \pi\left((3-\sqrt{x})^2 - 1\right)\Delta x$

$\Delta V = \pi\left(9 - 6\sqrt{x} + x - 1\right)\Delta x$

$\Delta V = \pi\left(x - 6\sqrt{x} + 8\right)\Delta x$

$V = \pi\int_0^4 \left(x - 6\sqrt{x} + 8\right)dx$



A. $\pi\int_0^2 \left((3-y)^2 - 1\right)dy$

B. $\pi\int_0^4 \left(x - 6\sqrt{x} + 8\right)dx$

C. $\pi\int_0^4 \left(2 - 2\sqrt{x} + x\right)dx$

D. $\pi\int_0^4 \left((3-x)^2 - 1\right)dx$

(b) Find $\int \frac{1}{2x\sqrt{\ln x}}\,dx$. **2**

$= \frac{1}{2}\int \frac{1}{x}\cdot(\ln x)^{-\frac{1}{2}}\,dx$ Note: $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$= \cancel{\frac{1}{2}\cdot\frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}}} + C$

$= \sqrt{\ln x} + C$

(c) (i) Find real constants a, b and c such that **2**

$$\frac{5}{(1-2x)(1+x^2)} = \frac{a}{1-2x} + \frac{bx+c}{1+x^2}$$

$5 = a(1+x^2) + (bx+c)(1-2x)$

let $x = \frac{1}{2}$ — equate coefficients of

$5 = a(1+(\frac{1}{2})^2)$ — x^2: $0 = 4-2b$ — x^0: $5 = 4+c$

$\frac{5}{4}a = 5$ — $2b = 4$ — $\underline{c = 1}$

$\underline{a = 4}$ — $\underline{b = 2}$

(ii) Hence, or otherwise show that $\displaystyle\int_{-\frac{\pi}{4}}^{0} \frac{5}{1-2\tan\theta}\,d\theta = \frac{\pi}{4} + \ln\left(\frac{9}{2}\right)$ **3**

let $x = \tan\theta$ — when $\theta = 0$, $x = 0$ — $\theta = -\frac{\pi}{4}$, $x = -1$

$\frac{dx}{d\theta} = \sec^2\theta$

$= 1 + \tan^2\theta$

$= 1 + x^2$

$d\theta = \frac{dx}{1+x^2}$

$I = \int_{-1}^{0} \frac{5}{1-2x} \cdot \frac{dx}{1+x^2}$

$= \int_{-1}^{0} \left(\frac{4}{1-2x} + \frac{2x+1}{1+x^2}\right) dx$ using (i)

$= \int_{-1}^{0} \left(-2\frac{-2}{1-2x} + \frac{2x}{1+x^2} + \frac{1}{1+x^2}\right) dx$

$= \left[-2\ln|1-2x| + \ln|1+x^2| + \tan^{-1}x\right]_{-1}^{0}$

$= \cancel{-2\ln|1-2(0)| + \ln|1+(0)^2|} + \tan^{-1}(0) - \left(-2\ln|1-2(-1)| + \ln|1+(-1)^2| + \tan^{-1}(-1)\right)$

$= -\left(-2\ln 3 + \ln 2 + (-\frac{\pi}{4})\right)$

$= 2\ln 3 - \ln 2 + \frac{\pi}{4}$

$= \ln 3^2 - \ln 2 + \frac{\pi}{4}$

$= \ln(\frac{9}{2}) + \frac{\pi}{4}$

(d) Find $\int 3x^2 \arccos x \, dx$. **3**

$$u = \cos^{-1} x \qquad v' = 3x^2$$
$$u' = -\frac{1}{\sqrt{1-x^2}} \qquad v = x^3$$

$$I = x^3 \cos^{-1} x + \int \frac{x^3}{\sqrt{1-x^2}} \, dx$$

$$= x^3 \cos^{-1} x + \int \frac{x^3 - x + x}{\sqrt{1-x^2}} \, dx$$

$$= x^3 \cos^{-1} x + \int \left[\frac{-x(1-x^2)}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right] dx$$

$$= x^3 \cos^{-1} x + \int \left(\frac{1}{2} \cdot -2x(1-x^2)^{\frac{1}{2}} - \frac{1}{2} \cdot -2x(1-x^2)^{-\frac{1}{2}} \right) dx$$

$\underline{\text{Note}}$: $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$$= x^3 \cos^{-1} x + \frac{1}{2} \frac{(1-x^2)^{3/2}}{3/2} - \cancel{\frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}} + C$$

$$= x^3 \cos^{-1} x + \frac{\sqrt{(1-x^2)^3}}{3} - \sqrt{1-x^2} + C$$

Use this space to re-write any question. Clearly indicate that you have done so.

COMMENTS:

students performed well on Q1.a, b, c.i

A lot of students failed to see the link with c.ii.

Using the substitution $t = \tan\frac{\theta}{2}$ did not produce the desired result

ALTERNATIVELY

$$\frac{5}{(1-2\tan\theta)(1+\tan^2\theta)} = \frac{4}{1-2\tan\theta} + \frac{2\tan\theta+1}{1+\tan^2\theta}$$

$$\frac{5}{1-2\tan\theta} = \frac{4(1+\tan^2\theta)}{1-2\tan\theta} + 2\tan\theta + 1$$

$$\therefore \int_{-\frac{\pi}{4}}^{0} \frac{5}{1-2\tan\theta}\,d\theta = \int_{-\frac{\pi}{4}}^{0}\left(-2\cdot\frac{-2\sec^2\theta}{1-2\tan\theta} - 2\cdot\frac{-\sin\theta}{\cos\theta} + 1\right)d\theta$$

$$= \Big[-2\ln|1-2\tan\theta| - 2\ln|\cos\theta| + \theta\Big]_{-\frac{\pi}{4}}^{0}$$

$$= \cancel{-2\ln|1-2\tan(0)|} \cancel{-2\ln|\cos(0)|} + \cancel{(0)} - \left(-2\ln\left|1-2\tan\left(-\tfrac{\pi}{4}\right)\right| - 2\ln\left|\cos\left(-\tfrac{\pi}{4}\right)\right| + \left(-\tfrac{\pi}{4}\right)\right)$$

$$= 2\ln|1-2(-1)| + 2\ln\left|\frac{1}{\sqrt{2}}\right| + \frac{\pi}{4}$$

$$= 2\ln 3 + \ln\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi}{4}$$

$$= \ln 3^2 + \ln\left(\frac{1}{2}\right) + \frac{\pi}{4}$$

$$= \ln\left(\frac{9}{2}\right) + \frac{\pi}{4}$$

Most students recognised that Q1.d. required integration by parts.

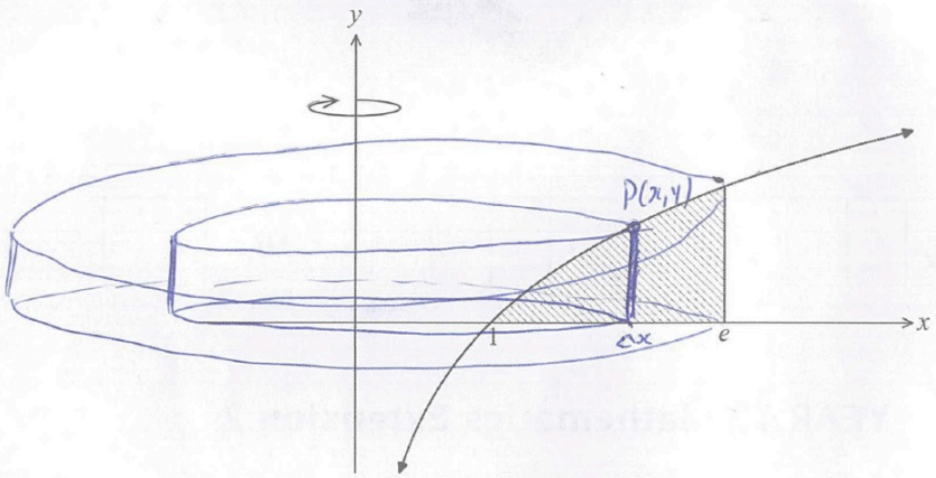
A trig. substitution $x = \sin\theta$ could have been used for $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$.

The substitution $u = 1 - x^2$ would also work.

Many students made careless mistakes in their working out. Usually to do with the sign.

Question 2 (9 marks)

(a) The diagram below shows the region bounded by the curve $y = \log_e x$, the x-axis and the line $x = e$. The region is rotated about the y-axis to form a solid. **4**



Find the volume of the solid by the method of cylindrical shells.

$\Delta V = 2\pi r h \Delta x$

$= 2\pi y x \Delta x$

$\therefore V = 2\pi \int_1^e x \ln x \, dx$

$u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$V = 2\pi \left[\left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x^2}{2} \times \frac{1}{x} \, dx \right.$

$= 2\pi \left[\left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x}{2} \, dx \right]$

$= 2\pi \left[\left[\frac{x^2}{2} \ln x \right]_1^e - \left[\frac{x^2}{4} \right]_1^e \right]$

$= 2\pi \left[\left(\frac{e^2}{2} - 0 \right) - \left(\frac{e^2}{4} - \frac{1}{4} \right) \right]$

$= 2\pi \left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \right]$

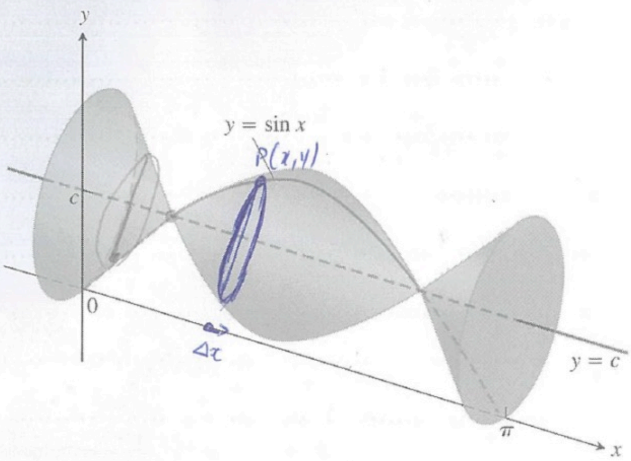
$= \frac{\pi}{2} \left[e^2 + 1 \right]$ units^3

Marker's comments:

Many candidates were able to gain full marks in this section. Well done!

Those whom didn't gain full marks mainly made silly error in substitution.

(b) The graph of $y = \sin x$ for $0 \le x \le \pi$ is revolved around the line $y = c$ to generate the solid shown below. Find the value of c that minimises the volume. **5**



$\Delta V = \pi r^2 \Delta x$

$V = \pi \int_0^{\pi} (y-c)^2 \, dx$

$V = \pi \int_0^{\pi} (\sin x - c)^2 \, dx$

$= \pi \int_0^{\pi} \sin^2 x - 2c \sin x + c^2 \, dx$

$= \pi \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x - 2c \sin x + c^2 \, dx$

$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x + 2c\cos x + c^2 x \right]_0^{\pi}$

$= \pi \left[\left(\frac{1}{2}\pi - \frac{1}{4}\sin 2\pi + 2c\cos\pi + c^2\pi\right) - \left(0 - \frac{1}{4}\sin 0 + 2c\cos 0 + c^2 \times 0\right) \right]$

$= \pi \left[\frac{\pi}{2} - 2c + \pi c^2 - 2c \right]$

$\therefore V = \pi^2 c^2 - 4\pi c + \frac{\pi^2}{2}$

This is a concave up $(\pi^2 > 0)$ parabola in c, minimum V occurring at axis of symmetry.

$\therefore c = -\frac{B}{2A}$

$c = -\frac{(-4\pi)}{2(\pi^2)} = \frac{2}{\pi}$

Marker's comments:

Significant number of candidates used the Volume by Shell method for this question, which **none** were successful in getting the correct answer. A maximum of 2 marks were gained depending on the amount of "relevant" working was shown. The Volume by slicing method was an efficient method to be used for this question and majority of the candidates whom used this method were able to gain full marks.

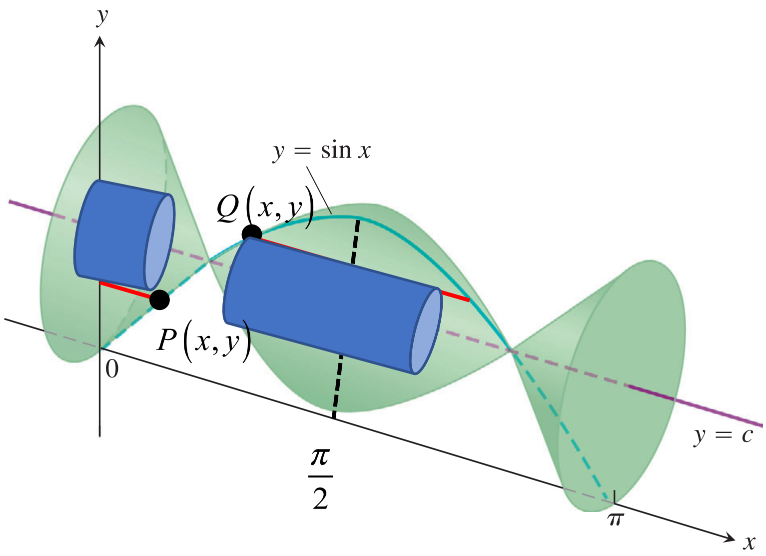
Candidates whom used the Volume by slicing method but lost marks due to mainly did not state wh*c* would minimise the volume (either by solution above or use of second derivative – this was a skill learnt in 2 unit Advanced Mathematics) or silly errors.

Alternative Solution (Not recommended)

$c = \sin x \Rightarrow x = \arcsin c$ or $\pi - \arcsin c$

Section 1 is defined for $0 \le x \le \arcsin c$
Section 2 is defined for $\arcsin c \le x \le \pi - \arcsin c$
Section 3 is defined for $\pi - \arcsin c \le x \le \pi$

Note Sections 1 and 3 have the same volume



Section 1
Radius = $c - y$
Height = $x = \arcsin y$

Volume Sections 1 & 3

$$V = 2\int_0^c 2\pi(c-y)(x)\,dy$$

$$= 4\pi\int_0^c (c-y)(2\arcsin y)\,dy$$

$$= \pi^2c^2 - 4\pi c + \frac{\pi^2}{2}$$

Section 2
Radius = $y - c$

Height = $2\left(\frac{\pi}{2} - x\right) = \pi - 2x = \pi - 2\arcsin y$

$$V = \int_c^1 2\pi(y-c)(\pi - 2x)\,dy$$

$$= 2\pi\int_c^1 (y-c)(\pi - 2\arcsin y)\,dy$$

$$= \frac{\pi^2}{2} - \pi^2c^2 - \pi\arcsin c$$

$$V = 2\pi^2 c^2 - 4\pi c + \pi \arcsin c + \frac{\pi^2}{2} - \pi^2 c^2 - \pi \arcsin c$$

$$= \pi^2 c^2 - 4\pi c + \frac{\pi^2}{2}$$

Solution now follows the standard solution

Question 3

(i) $I_0 = \int_0^1 x^0 e^{-x}\,dx$

$= \int_0^1 e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^1$

$= -e^{-1} - (-e^0)$

$= 1 - \frac{1}{e}$

* Most students did this well.

(ii) Let $u = x^n$ $\qquad v = -e^{-x}$

$u' = nx^{n-1}$ $\qquad v' = e^{-x}$

$I_n = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1} e^{-x}\,dx$, by Integration by Parts

$= \left[-1 \times e^{-1} - (-0.e^0)\right] + nI_{n-1}$

$= nI_{n-1} - \frac{1}{e}$

* Most students did this well.

(iv) Step 1 - Show for $n=0$

$LHS = I_0 = 1-\frac{1}{e}$, from (i)

$RHS = 0! - \frac{0!}{e}\left(\frac{1}{0!}\right)$

$= 1 - \frac{1}{e}$

$LHS = RHS$, so true for $n=0$

Step 2 - Assume true for some $n=k$

i.e.

$$I_k = k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$$

Step 3 - Show for $n=k+1$

$$\text{RTP: } I_{k+1} = (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$$

$LHS = I_{k+1}$

$= (k+1)I_k - \frac{1}{e}$, from (ii)

$= (k+1)\left[k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right] - \frac{1}{e}$, by inductive step

$= (k+1)! - \left[\frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} + \frac{(k+1)!}{e}\frac{1}{(k+1)!}\right]$

$= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!} = RHS$

* Most students did this well.
* Some students were penalised for using base case $n=1$
* Some students were penalised for not specifying why $\frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} + \frac{1}{e} = \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$
* Many students successfully moved from RHS to LHS.
* Some poor attempts worked directly on the statement needing proving.
* Many students wasted time re-proving results from (i) and/or (ii) in their proof.

(v)

$$I_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

$$\lim_{n\to\infty}(I_n) = \lim_{n\to\infty}\left(n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}\right)$$

$$0 = \lim_{n\to\infty}(n!)\lim_{n\to\infty}\left(1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}\right)$$

Since $\lim_{n\to\infty}(n!) \neq 0$

$$\lim_{n\to\infty}\left(1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}\right) = 0$$

$$1 - \frac{1}{e}\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} = 0$$

$$1 = \frac{1}{e}\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$$

$$e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$$

* Many students did this well.
* Some students substituted $n!$ with ∞, and then incorrectly tried to simplify $\frac{\infty}{\infty} = 1$.