SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

2022

YEAR 11
TERM 4
HSC TASK 1

Maths Class:
11MX

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- All answers, unless asked, should be left in a simplified exact form.
- **In Questions 6 – 8, show ALL relevant mathematical reasoning and/or calculations**

Total Marks: 34

Section I – 5 marks (pages 2 – 4)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 29 marks (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 52 minutes for this section

Examiner:

RW

NOT TO BE REMOVED FROM EXAMINATION ROOM

Section I

5 marks

Attempt all questions.
Allow about 8 minutes for this section.
Use the multiple choice answer sheet for Questions 1 – 5.

---

1 Given $z^2 = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$, which of the following is equal to z?

A. $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

B. $-\sqrt{2} - \sqrt{2}i$

C. $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

D. $-\sqrt{2} + \sqrt{2}i$

2 Suppose that z_1 and z_2 are two complex numbers such that $z_1 z_2 = 1$.

Given that $z_1 = re^{i\theta}$, which of the following is equal to z_2?

A. $r(\cos\theta - i\sin\theta)$

B. $\frac{1}{r}(\cos\theta - i\sin\theta)$

C. $r(\cos\theta + i\sin\theta)$

D. $\frac{1}{r}(\cos\theta + i\sin\theta)$

3 Suppose a nonzero complex number z_1 is multiplied by $\dfrac{1+i}{1-i}$ to produce the complex number z_2.

Which of the following best describes z_2 in relation to z_1?

A. z_2 is a clockwise rotation of z_1 by $\dfrac{\pi}{2}$ radians about the origin.

B. z_2 is a clockwise rotation of z_1 by $\dfrac{\pi}{4}$ radians about the origin.

C. z_2 is an anticlockwise rotation of z_1 by $\dfrac{\pi}{2}$ radians about the origin.

D. z_2 is an anticlockwise rotation of z_1 by $\dfrac{\pi}{4}$ radians about the origin.

4 Given that $z^{\pi} = (-1)^{3i}$, which of the following is equal to z?

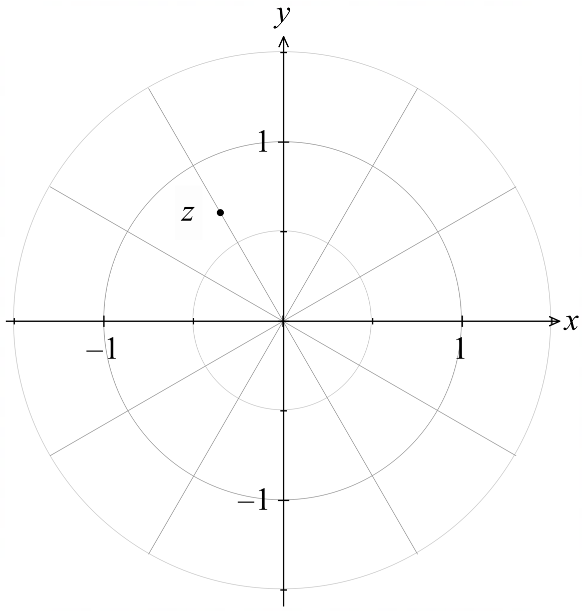
A. $\dfrac{1}{e^3}$

B. 3^e

C. $\sqrt[3]{e}$

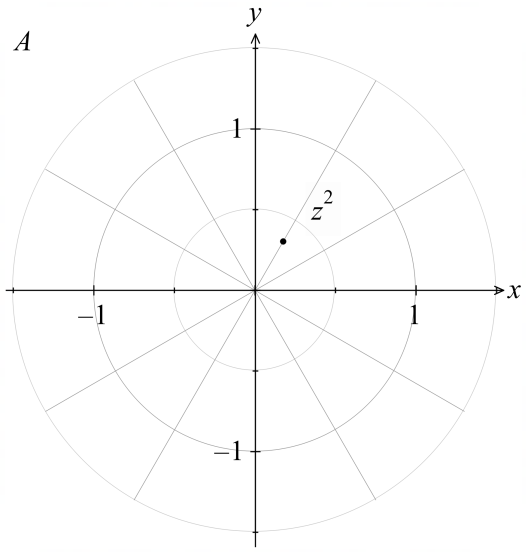
D. $\dfrac{1}{\sqrt[3]{e}}$

5 The Argand diagram below shows a complex number z below.

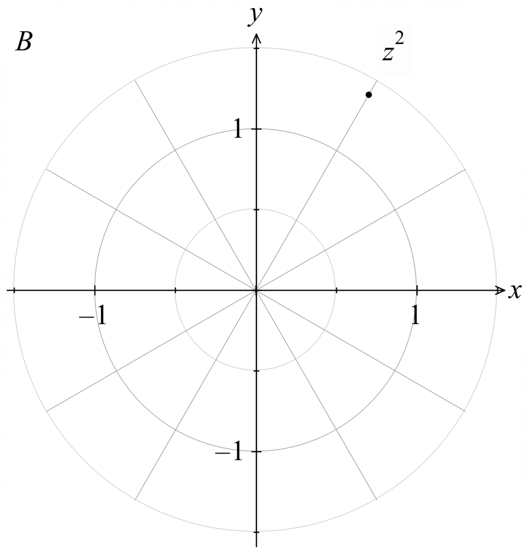


Which of the following best represents z^2 ?

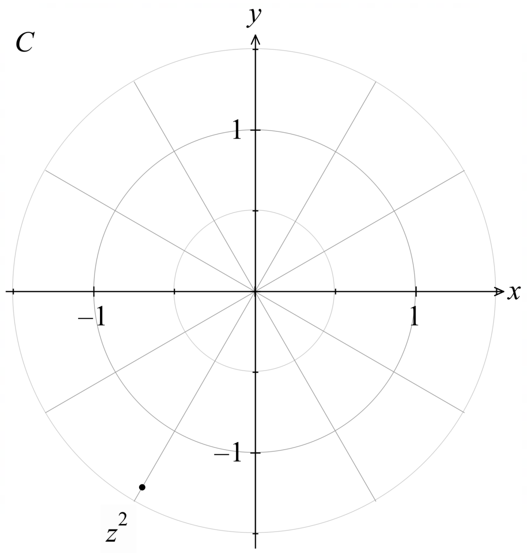
A



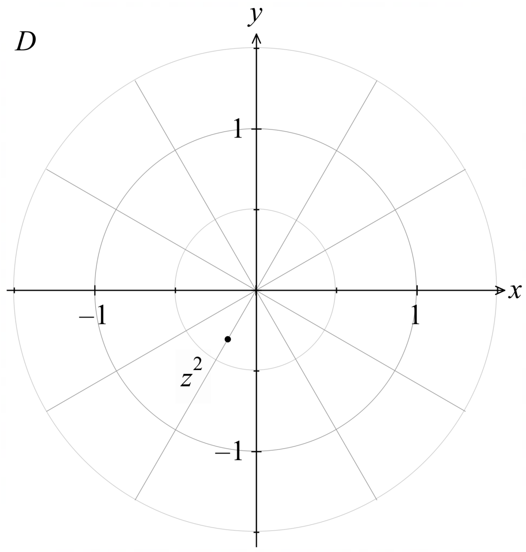
B



C



D



Section II

29 marks

Attempt all questions.
Allow about 52 minutes for this section.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks) Start a NEW Answer Booklet

(a) Let $z = 4 - 7i$.

(i) Sketch z on an Argand diagram. **1**

(ii) Find the value of $|z|$. **1**

(iii) Find the value of $\arg(z)$, where $-\pi < \arg(z) < \pi$. **1**

Express your answer in radians correct to two decimal places.

(b) Let $z = 6 + 2i$.

(i) Find $\text{Re}(z^2)$. **1**

(ii) Find $\text{Im}\left(\dfrac{z}{\overline{z}}\right)$. **2**

(c) Let $z = 5e^{\frac{3\pi}{8}i}$.

(i) Express $\dfrac{1}{z^2}$ in the form $x + yi$, where $x, y \in \mathbb{R}$. **2**

(ii) Find all rational values of k such that z^k is purely real. **2**

End of Question 6

Question 7 (11 marks) Start a NEW Answer Booklet

(a) (i) Find the square roots of $-3+4i$. **2**

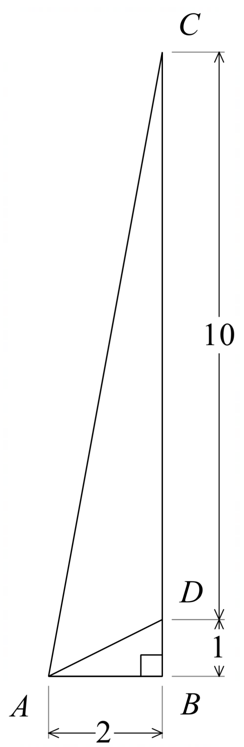
Express your answers in the form $x+yi$, where $x, y \in \mathbb{R}$

(ii) Hence, solve $z^2-(4-2i)z+(6-8i)=0$. **2**

Express your answers in the form $x+yi$, where $x, y \in \mathbb{R}$.

(b) (i) Expand $(a+b)^3$. **1**

(ii) By considering $(2+i)^3$, show that $\angle CAB = 3\angle DAB$ **3**



(c) Given $z=\frac{1}{2}+\frac{\sqrt{3}i}{2}$, show that $\sum_{n=0}^{20} z^n = 2z$. **3**

End of Question 7

Question 8 (8 marks) Start a NEW Answer Booklet

(a) Show that if $x \in \mathbb{R}$, then there are no solutions to the equation $e^{ix} = \sec x + i \operatorname{cosec} x$. **2**

(b) Let $z = (x+yi)^3 - 11i$, where x, y and z are positive integers.

Using the result in Question 7 (b) (i), or otherwise:

(i) Show that $11 = y(3x^2 - y^2)$. **1**

(ii) Hence, find z. **2**

(c) (i) Express ie^{-it} in the form $x+iy$, where x and y are real. **1**

(ii) Hence, or otherwise, find all integers n such that **2**

$$(\sin x + i\cos x)^n = \sin nx + i\cos nx$$

is true for all $x \in \mathbb{R}$.

End of paper

SYDNEY
BOYS
HIGH
SCHOOL

2022

YEAR 11
TERM 4
HSC TASK 1

Mathematics Extension 2

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.
Your responses should have included relevant mathematical reasoning and/or calculations.

Just because you have shown 'working' does not justify that your solution is worth any marks.

Do NOT suggest that you know the marking scheme for each question.

MC Answers

1 D

2 B

3 C

4 A

5 D

Section I **Multiple Choice**

---

1 Given $z^2 = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$, which of the following is equal to z?

A. $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

B. $-\sqrt{2} - \sqrt{2}i$

C. $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

D. $-\sqrt{2} + \sqrt{2}i$

$z^2 = 4e^{\frac{3\pi}{2}i}$

$z = \pm 2e^{\frac{3\pi}{4}i}$

Considering the argument, it must be A or D.

Considering the modulus it must be B or D.

2 Suppose that z_1 and z_2 are two complex numbers such that $z_1 z_2 = 1$.

Given that $z_1 = re^{i\theta}$, which of the following is equal to z_2?

A. $r(\cos\theta - i\sin\theta)$

B. $\frac{1}{r}(\cos\theta - i\sin\theta)$

C. $r(\cos\theta + i\sin\theta)$

D. $\frac{1}{r}(\cos\theta + i\sin\theta)$

$z_1 z_2 = 1$

$z_2 = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

3 Suppose a nonzero complex number z_1 is multiplied by $\dfrac{1+i}{1-i}$ to produce the complex number z_2.

Which of the following best describes z_2 in relation to z_1?

A. z_2 is a clockwise rotation of z_1 by $\dfrac{\pi}{2}$ radians about the origin.

B. z_2 is a clockwise rotation of z_1 by $\dfrac{\pi}{4}$ radians about the origin.

C. z_2 is an anticlockwise rotation of z_1 by $\dfrac{\pi}{2}$ radians about the origin.

D. z_2 is an anticlockwise rotation of z_1 by $\dfrac{\pi}{4}$ radians about the origin.

$$\frac{1+i}{1-i}=\frac{\sqrt{2}e^{\frac{\pi}{4}i}}{\sqrt{2}e^{-\frac{\pi}{4}i}}=e^{\frac{\pi}{2}i}=i$$

4 Given that $z^{\pi}=(-1)^{3i}$, which of the following is equal to z?

A. $\dfrac{1}{e^3}$

B. e^3

C. $\sqrt[3]{e}$

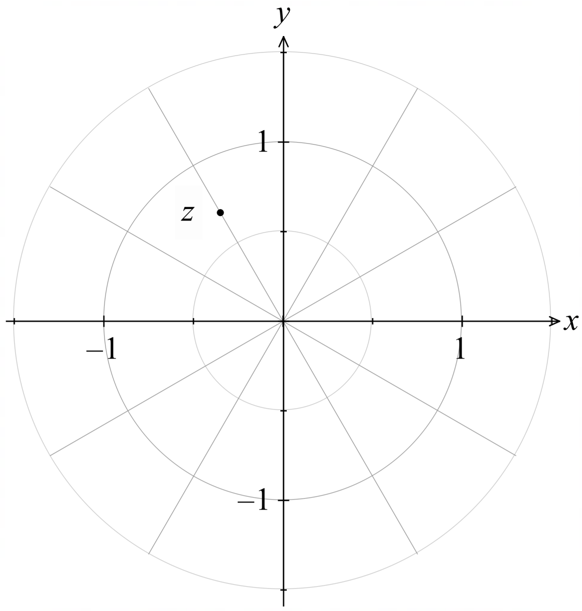
D. $\dfrac{1}{\sqrt[3]{e}}$

$-1=e^{i\pi}\Rightarrow(-1)^{3i}=\left(e^{i\pi}\right)^{3i}=e^{-3\pi}$

$\therefore\ z^{\pi}=e^{-3\pi}$

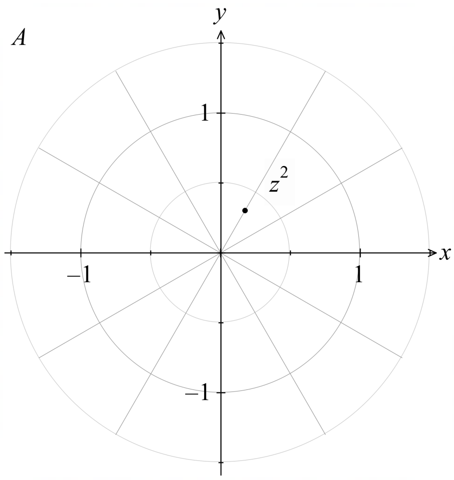
$\therefore\ z=e^{-3}$ is a solution

5 The Argand diagram below shows a complex number z below.

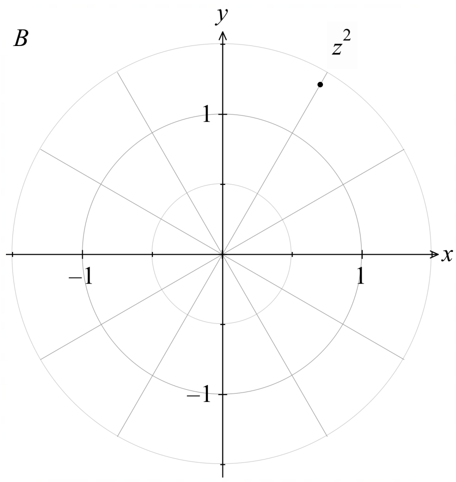


Which of the following best represents z^2 ?

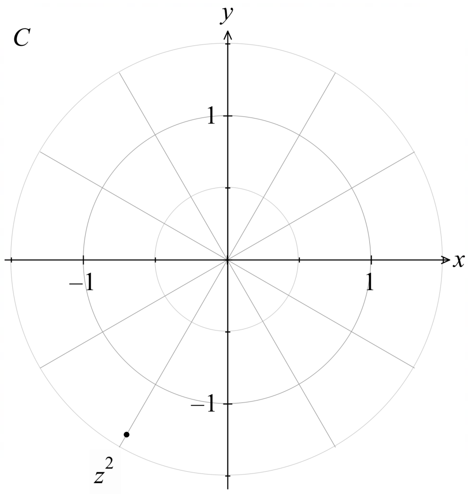
A



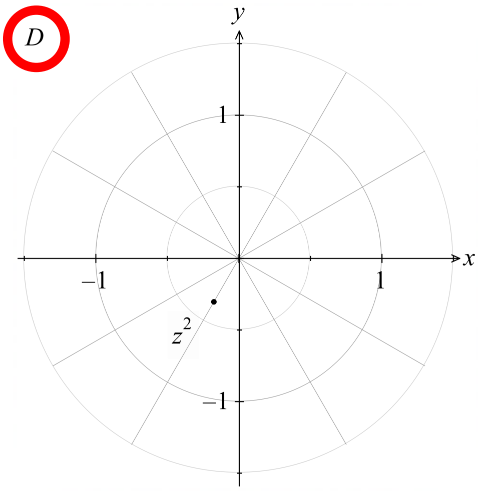
B



C



D



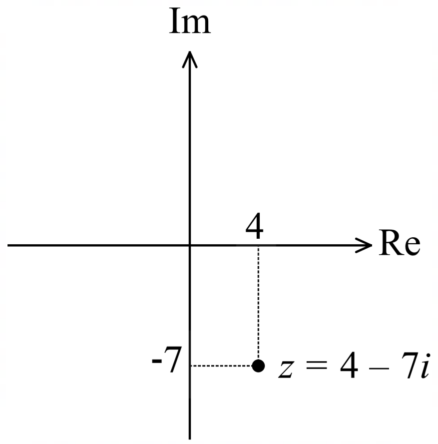
A or D as $|z| < 1 \Rightarrow |z^2| < |z| < 1$

C or D as $\arg(z^2) = 2\arg z$

Question 6 Solutions

(a) Let $z = 4 - 7i$.

(i) Sketch z on an Argand diagram. **1**



Comment: The axes must be labelled – and correctly: Im $z = y$ NOT iy

A scale also had to be shown – not just having a random point in the 4th quad.

(ii) Find the value of $|z|$. **1**

$$|z| = \sqrt{4^2 + (-7)^2}$$

$$= \sqrt{65}$$

Comment: Not many did not know the formula

(iii) Find the value of $\arg(z)$, where $-\pi < \arg(z) < \pi$. **1**

Express your answer in radians correct to two decimal places.

$$\tan(\arg z) = \frac{-7}{4}$$

$$\therefore \arg z = -1.05$$

Comment: It was very hard to get this wrong, but some persisted!

Students should have had their calculator in radians mode to get the answer very quickly.

There were several that considered there were two answers, but the sketch in part (i) should have reminded them that there was only one answer.

A few students had their answers $0 < \arg(z) < 2\pi$.

(b) Let $z = 6 + 2i$.

(i) Find $\operatorname{Re}\left(z^2\right)$. **1**

$$z = 6 + 2i$$

$$\therefore z^2 = 36 - 4 + 24i$$

$$= 32 + 24i$$

$$\therefore \operatorname{Re}\left(z^2\right) = 32$$

Comment: This was very well done.

(ii) Find $\operatorname{Im}\left(\frac{z}{\bar{z}}\right)$. **2**

$$\frac{z}{\bar{z}} = \frac{6+2i}{6-2i}$$

$$= \frac{(6+2i)^2}{36+4}$$

$$= \frac{32+24i}{40}$$

$$\therefore \operatorname{Im}\left(\frac{z}{\bar{z}}\right) = \frac{24}{40} = \frac{3}{5}$$

Note: $\frac{z}{\bar{z}} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta}$

$$\therefore \operatorname{Im}\left(\frac{z}{\bar{z}}\right) = \sin 2\theta$$

$$= 2\sin\theta\cos\theta$$

$$= 2 \times \frac{2}{\sqrt{40}} \times \frac{6}{\sqrt{40}}$$

$$= \frac{3}{5}$$

Comment: It was sensible to make use of part (i) to help with this problem.

Students were not penalised twice for not knowing the definition of Im z, i.e. if students were penalised in (a) (i) they were not penalised again.

Some students used their calculator to evaluate $\frac{z}{\bar{z}}$.
They could not get full marks for this.

(c) Let $z = 5e^{\frac{3\pi}{8}i}$.

(i) Express $\dfrac{1}{z^2}$ in the form $x + yi$, where $x, y \in \mathbb{R}$. **2**

$$z = 5e^{\frac{3\pi}{8}i}$$

$$\therefore z^2 = 25e^{\frac{3\pi}{4}i}$$

$$\therefore z^{-2} = \frac{1}{25}e^{-\frac{3\pi}{4}i}$$

$$\therefore z^{-2} = \frac{1}{25}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= \frac{1}{25}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{25\sqrt{2}}(-1 - i)$$

$$= -\frac{1}{25\sqrt{2}} - \frac{1}{25\sqrt{2}}i$$

Comment: Most students were successful with this question, but many students made this more complicated than it had to be.

(c)	(ii)	Find all rational values of k such that z^k is purely real.	**2**

$$z = 5e^{\frac{3\pi}{8}i}$$

$$\therefore z^k = 5^k e^{\frac{3\pi k}{8}i}$$

$$= 5^k\left(\cos\frac{3\pi k}{8} + i\sin\frac{3\pi k}{8}\right)$$

To be purely real: $\sin\frac{3\pi k}{8} = 0$

$$\therefore \frac{3\pi k}{8} = n\pi,\ n \in \mathbb{Z}$$

$$\therefore k = \frac{8n}{3}$$

Comment: There was some confusion as to integers vs rational numbers.
Students needed to indicate that their variable, n above, was an integer.

Students needed to show details and not just from $\frac{3\pi k}{8} = n\pi$ or equivalent.

The biggest issue with this question was the use of general solutions, in particular when $\sin x = 0$.

Question 7

General comments on the exam: It was pleasing to see so many pupils embracing $e^{i\theta}$ in their working. I hope the working in these solutions bare out that compared to $\text{cis}\,\theta$, $e^{i\theta}$ is quicker to write, looks better and is easier to work with.

a) i) $(x+iy)^2 = -3+4i$

$\therefore \quad 2xy = 4 \quad ;$ try $x=1$ & $y=2$

$x^2 - y^2 = -3$ which is true when $x=1$ & $y=2$

a solution is $x+iy = \underline{1+2i}$

$(-(1+2i))^2 = (1+2i)^2$

$\therefore$ another solution is $\underline{-1-2i}$

2

ii) $(z-(2-i))^2 - (2-i)^2 + (6-8i) = 0$

$(z-2+i)^2 = -6+8i + 4-1-4i$

$(z-2+i)^2 = -3+4i$

$\therefore \quad z-2+i = 1+2i$ & ①

$z-2+i = -1-2i$ ②

① $\Rightarrow$ $z = 3+i$ } both are

② $\Rightarrow$ $z = 1-3i$ } solutions

2

Notes on part a)

i) Rarely were there mistakes in this part. Ocassionally pupils mixed up the signs.

ii) Simple algebra let students down here. If they got it wrong it was not the method at fault.

b) i) $a^3+3a^2b+3ab^2+b^3$ **1**

ii) $(2+i)^3 = 8+12i-6-i$ **3**

$= 2+11i$

let $\text{Arg}(2+i) = \theta$

$3\theta = 3\times\text{Arg}(2+i)$

$= \text{Arg}((2+i)^3)$

$= \text{Arg}(2+11i)$

$\theta = \angle DAB \;\therefore\; 3\angle DAB = \angle CAB$

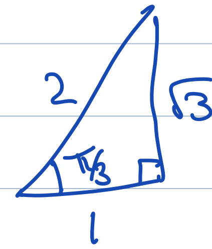
Notes on part b)

i) It was not possible to get full marks if this was not fully expanded.

ii) Pupils needed to "show" that $(2+i)^3$ is equal to $2+11i$ – not just state it is so. They also needed to state $\text{Arg}[(2+i)^3] = 3\text{Arg}(2+i)$.

Many pupils used approximations from their calculators in order to prove this result. Proofs using approximations are not acceptable, and were marked accordingly.

c) $z = e^{i\frac{\pi}{3}}$



3

Show $\displaystyle\sum_{n=0}^{20} \left(e^{i\pi/3}\right)^n = 2e^{i\pi/3}$

$$1 + e^{i\pi/3} + e^{i2\pi/3} + \dots + e^{i20\pi/3} = \frac{1-e^{i21\pi/3}}{1-e^{i\pi/3}} \left(\times \frac{e^{-i\pi/6}}{e^{-i\pi/6}}\right)$$

$$\text{RHS} = \frac{\left(1-e^{i7\pi}\right) \times e^{-i\pi/6}}{e^{-i\pi/6} - e^{i\pi/6}}$$

$$= \frac{2e^{-i\pi/6}}{-2i\sin\pi/6} \left(\times \frac{i}{i}\right)$$

$$= \frac{ie^{-i\pi/6}}{1/2} \left(\times \frac{2}{2}\right)$$

$$= 2e^{i\pi/2}e^{-i\pi/6} \qquad \left(i = e^{i\pi/2}\right)$$

$$= 2e^{i\pi/3}$$

$$= 2z$$

As required.

Alternatively

$$\sum_{n=0}^{20} z^n = z^0 + z^1 + z^2 + \dots + z^{19} + z^{20}$$

$$= e^{i0} + e^{i\pi/3} + e^{i2\pi/3} + \dots + e^{i19\pi/3} + e^{i20\pi/3}$$

$$e^{i(l\frac{\pi}{3}+2k\pi)} \equiv e^{il\pi/3}, \quad l \in \{0,1,2,3,4,5\},\ k \in \mathbb{Z}$$

& $e^{i\alpha\pi/3} = e^{-i(6-\alpha)\pi/3}$ $\quad \alpha = 4 \,\&\, 5$

$$\Rightarrow \sum_{n=0}^{20} = \left(e^{i0} + e^{i\pi/3} + e^{i2\pi/3} + e^{i\pi} + e^{-i2\pi/3} + e^{-i\pi/3}\right)\times 3 + e^{i0} + e^{i\pi/3} + e^{i2\pi/3}$$

$$= 0\times 3 + 1 + \frac{1}{2} + \frac{\sqrt{3}i}{2} - \frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$= 1 + \sqrt{3}\,i$$

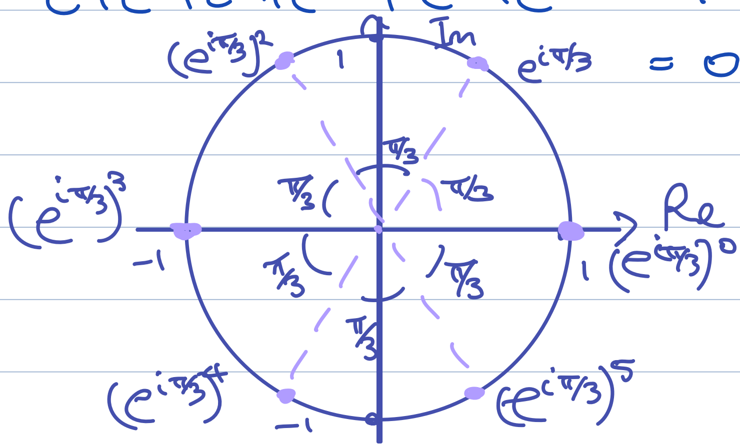
$$= 2\times\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= 2z$$

Since

$$e^{i0} + e^{i\pi} + e^{i\pi/3} + e^{-i\pi/3} + e^{i2\pi/3} + e^{-i2\pi/3} = 1 - 1 + 2\cos\pi/3 + 2\cos\frac{2\pi}{3}$$

$$= 0$$



$$e^{0i} = -e^{i\pi}$$

$$e^{i\pi/3} = -e^{-i2\pi/3}$$

$$e^{i2\pi/3} = -e^{-i\pi/3}$$

Notes on part c).

Many pupils used a geometric progression that had 20 terms in it, when it actually has 21 terms. Many pupils made their equations fit the results despite the algebra not bearing this out to be true.

Pupils could <u>NOT</u> get full marks if they made leaps in their working that were not justified with working.

Some pupils made statements like - this all cancels out - without being explicit as to what cancels out and how it does so.

Marker's Comments

Question 8

(a) A common mistake was $\frac{1}{\cos x} + \frac{i}{\sin x} = (\cos x + i\sin x)^{-1}$

Consider $\frac{1}{2} + \frac{1}{4} = \frac{3}{4} \neq \frac{1}{6} = (2+4)^{-1}$.

Also remember that $\cos^2 x = 1 \Rightarrow \cos x = \pm 1$.

(b)(i) It was very important here that you state that the imaginary part is zero since z is a positive integer. Saying that you are equating parts is not enough because we need to know equating to what.

(ii) When it says "find z", this implies we want a numerical value. Only by using that x and y are +ve integers can we use the factorisation of the prime number 11 to solve it.

(c)(i) Mostly done well.

Please simplify your expression fully to make sure it matches the required form $x + iy$.

Remember $\cos(-x) = \cos x$.

(ii) Please remember that n is an integer.

If it equals something like 3π then you are on the wrong track.

In exponential form, remember to also raise i to the power of n on the LHS, as this is the key.

Always look to use part (i) to help in such a question.

Question 8

(a) The two main ways to answer this is to equate real + imaginary parts or to compare moduli.

Try to solve $\cos x + i \sin x = \frac{1}{\cos x} + \frac{i}{\sin x}$

Equating real parts: $\cos x = \frac{1}{\cos x}$, where $\cos x \neq 0$.

$$\cos^2 x = 1 \quad ①$$

" imaginary parts: $\sin x = \frac{1}{\sin x}$, where $\sin x \neq 0$

$$\sin^2 x = 1 \quad ②$$

So $\cos^2 x + \sin^2 x = 2$ (adding solutions ①+②)

But this is impossible since $\cos^2 x + \sin^2 x = 1$, so there are no solutions.

You could also sub solutions from ① into ② to find they do not work.

$$|e^{ix}| = 1$$

$$|\sec x + i \operatorname{cosec} x| = \sqrt{\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}}$$

$$= \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\frac{1}{2}\sin(2x)}$$

We need $\frac{1}{2}\sin(2x) = 1$

$$\sin(2x) = 2$$

But this is impossible since $\sin(2x) \leq 1$.

$\therefore$ There are no solutions.

Remember in manipulating expressions, each line is an '=' followed by an expression, after the first line.

In manipulating equations, each line is an equation, i.e. expression = expression.

Question 8

(b)(i) $z = (x+yi)^3 - 11i$ where $x, y, z \in \mathbb{Z}^+$

$= x^3 + 3x^2yi + 3xy^2(-1) + y^3(-i) - 11i$

$= x^3 - 3xy^2 + i(3x^2y - y^3 - 11)$

But $z \in \mathbb{Z}^+$, so $Im(z) = 0$.

ie. $3x^2y - y^3 - 11 = 0$

ie. $y(3x^2 - y^2) = 11$ as required.

(ii) $y(3x^2 - y^2) = 11$ where $x, y \in \mathbb{Z}^+$ and 11 is a prime

So $y = 1$ or 11.

If $y = 11$, then $3x^2 - y^2 = 1$

ie. $3x^2 = 122$ which is not divisible by 3.

So $y = 1$, $3x^2 - 1^2 = 11$

$x^2 = 4$

$x = 2$

Then $z = x^3 - 3xy^2$

$= 2^3 - 3(2)(1^2)$

$= 2$.

In (i) please say why we need $Im(z) = 0$.
In (ii), we again need to use the fact that x and y are positive integers. If you see this information in the question then it will very likely be used. Also the fact that $y(3x^2 - y^2)$ is factorised is a clue, together with 11 being prime. If there does not seem to be enough information to answer the question, then read it again more closely.

Question 8

(c)(i) $ie^{-it} = e^{\frac{i\pi}{2}} e^{-it}$

$= e^{i(\frac{\pi}{2}-t)}$

$= \cos(\frac{\pi}{2}-t) + i\sin(\frac{\pi}{2}-t)$

$= \sin t + i\cos t.$

You could expand e^{-it} then multiply by i as well.

(ii) So $LHS = (\sin x + i\cos x)^n$

$= (ie^{-ix})^n$ from (i)

$= i^n e^{-inx}$

$RHS = \sin(nx) + i\cos(nx)$

$= ie^{-i(nx)}$ from (i)

Then solve $i^n e^{-inx} = ie^{-inx}$ for all $x \in \mathbb{R}$.

i.e. $i^n = i$ since $e^{-inx} \neq 0$.

$\therefore n = 4m+1$ where $m \in \mathbb{Z}$.

Again, read the question carefully to see we are solving for n, not x.

Equations are easy to manipulate in exponential form.